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LAMBEK'S OPERATIONAL CATEGORIES

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An operational category is a category of models for an equational theory where the interpretation of some operations is predetermined. Examples include the equational and co-equational categories of Linton, categories of functors preserving some class of limits, and algebras for a prop as defined by MacLane. The chief result is a characterisation of the operational categories and functors in terms of their internal structure.

0. Introduction

The class of operational categories includes those of Lawvere's algebraic categories [*8*] and the equational categories of Linton [*9*],[*10*], which include the tripleable categories. Since the dual of an operational category is one too, coequational and cotripleable categories are operational. The definition of operational category used here is that introduced by J. Lambek in his lecture to the Midwest Category Theory Seminar at Waterloo University in 1968, as distinct from that of O. Wyler in [*15*].

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This paper is primarily a summary of [6], though there are new examples and the notation has been altered. Other generalisations of equational categories may be found in [3],[4],[14] and [15].

Our chief purpose here is to characterise the operational categories over some fixed category A . This done by constructing a tower of triples S_n based on Cat/A with a limit S_*-Alg whose image in Cat/A consists of exactly the operational categories and operational functors. In the process it is proved that there is a fixed, finite theory, from which every category of operational algebras can be constructed. This allows it to be shown that categories of algebras for props (MacLane [11]) are also operational.

I would like to thank Professor J. Lambek for introducing me to operational categories and the problem of their characterisation, and also Professor G. M. Kelly for showing me how to present mathematics clearly, yet concisely.

1. Operational categories

Let U be a Grothendieck universe. All categories used here will have their set of morphisms in U and the category of these is denoted Cat . Fix a category A . In some examples there is an element V of U which is also a Grothendieck universe with A being the category of V-small sets.

A *presentation*, (θ,H) , consists of a *theory* $\theta : B \to T$, by which is meant a functor that is bijective on objects, and a functor $H : A \times B \to C$ called the *base functor*. A *morphism of presentations* $(j,k) : (\theta,H) \to (\theta',H')$ consists of $j = (j_1,j_2)$ where $j_1 : B' \to B$ and $j_2 : T' \to T$ are functors such that $\theta j_1 = j_2\theta'$, and a functor $k : C \to C'$ such that $kH(1 \times j_1) = H'$. Together they form a category Pres(A) .

We define as follows a functor Op(A) : Pres(A) $\to$ Cat/A: it sends a presentation (θ,H) to $U : \mathcal{D} \to A$ given by the pullback (1.1) where H^* corresponds to H , and sends a morphism of presentations to the induced functor between the pullbacks. An object $U : \mathcal{D} \to A$ of Cat/A , identified loosely with the category $\mathcal{D}$, is called a *category of operational algebras* if it is the image under Op(A) of some presentation,

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\ \Phi\ } & [\mathcal{T},\mathcal{C}] \\ \downarrow & & \downarrow{\scriptstyle [\theta,1]} \\ \mathcal{A} & \xrightarrow[H^*]{\qquad} & [\mathcal{B},\mathcal{C}] \end{array} \tag{1.1}$$

and it is called an *operational category* if it is equivalent in $\mathrm{Cat}/\mathcal{A}$ to some category of operational algebras. A morphism in $\mathrm{Cat}/\mathcal{A}$ is an *operational functor* if it is equivalent to a functor in the image of $\mathrm{Op}(\mathcal{A})$. Note that $\mathrm{Op}(\mathcal{A})$ is neither full nor faithful.

$\mathcal{B}$ may be thought of as a language, the objects being sorts and the morphisms operations, into which new operations and equations may be added via θ . Then H^* maps each object of $\mathcal{A}$ to an interpretation of $\mathcal{B}$ in $\mathcal{C}$, and the algebras extend these interpretations of the language to models of the theory.

For notational convenience, if ω is a morphism of $\mathcal{T}$, that is an operation, then $\Phi_D\omega$ may be written as ω_D .

EXAMPLE 1.1. Let N be a skeleton of the category of finite sets so that N^{op} is the language of (one-sorted) finite products and let $H : \mathrm{Set} \times \mathrm{N}^{op} \to \mathrm{Set}$ be the restriction of the homfunctor. Consider a category $\mathcal{T}$ with finite products and a theory θ lying in $[\mathrm{N}^{op},\mathrm{T}]_p$ (the category of product-preserving functors from N^{op} to $\mathcal{T}$) . Then the resulting operational category is the category of algebras for the Lawvere theory, namely $[\mathcal{T},\mathrm{Set}]_p$.

EXAMPLE 1.2. Let $\mathcal{S}$ be the category of V-small sets for some universe V in U . If $\mathcal{A}$ is an $\mathcal{S}$-category, with homfunctor $H : \mathcal{A} \times \mathcal{A}^{op} \to \mathcal{S}$ and $\theta \in [\mathcal{A}^{op},\mathcal{T}]_p$ then the resulting category of algebras is equational in Linton's sense.

EXAMPLE 1.3. Given any natural transformation $\alpha : F \Rightarrow G : \mathcal{A} \to \mathcal{B}$, let $\mathrm{Inv}(\alpha) \to \mathcal{A}$ be the full subcategory of $\mathcal{A}$ of objects A for which α_A is an isomorphism. Then $\mathrm{Inv}(\alpha) \to \mathcal{A}$ is operational. The theory is the inclusion of the arrow category $\mathbf{2}$ into the chaotic category on two

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elements. The base functor is $H : A \times 2 \to B$ with $H^{*}A = \alpha_A$. For example, if α is the unit for the reflection from presheaves to sheaves on some site then $\mathrm{Inv}(\alpha)$ is just the sheaves. Also, given a limit (or a family of limits) in K and a complete category L then the functors from K to L which preserve the limit (or limits) form a full operational subcategory of $[K,L]$.

LEMMA 1.4. *If* $U : D \to A$ *is a category of operational algebras then* U *is faithful, reflects isomorphisms and creates coequalizers of* U*-split pairs.*

Proof. Let D have a presentation (θ, H) . Since θ is bijective on objects, $[\theta, 1]$ has the desired properties, and these are preserved by pullbacks. □

LEMMA 1.5. *Let* $U : D \to A$ *be a category of operational algebras. Then* $U^{op} : D^{op} \to A^{op}$ *and* $[Y,U] : [Y,D] \to [Y,A]$ *as well as the pullback of* D *by any* $A' \to A$ *are all categories of operational algebras.*

Proof. θ^{op} and $Y \times \theta$ as well as θ are all bijective on objects. □

Lemma 1.4 together with Beck's Tripleability Theorem ([12]) yields the following result.

THEOREM 1.6 (Lambek). $U : D \to A$ *is tripleable if and only if* U *is operational and has a left adjoint.* □

Conversely, if D has and U preserves pullbacks then U is operational if it creates coequalizers of U-split coequalisers (see [6]).

2. Operational retracts

Let (θ, H) be a presentation and $U : D \to A$ the resulting category of operational algebras. Consider a diagram in A of the following type:

$$A \underset{y}{\overset{x}{\rightleftarrows}} UD \ . \tag{2.1}$$

For each morphism ω in T define $\phi\omega$ by

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(2.2) $$\phi\omega = H(y,1)\omega_D H(x,1) \ .$$

LEMMA 2.1. *ϕ is a functor from T to C if and only if $H(yx,1) = 1$ and for each composite $\omega'\omega$ of morphisms in T*

(2.3) $$H(y,1).\omega'_D.H(xy,1).\omega_D.H(x,1) = H(y,1)(\omega'\omega)_D H(x,1) \ .$$

Further, if ϕ is a functor then (A,ϕ) is an algebra.

Proof. $H(yx,1) = 1$ if and only if ϕ preserves identities and ϕ satisfies (2.3) if and only if it preserves composites. For the second statement note that if $H(yx,1) = 1$ then $\phi\theta = H(A,-)$. □

In the light of the above proposition we define a diagram (A,y,D,x) as in (2.1) to be an *operational retract* if $H(yx,1) = 1$ and (2.3) holds. Then (2.2) defines an algebra structure for A .

EXAMPLE 2.2. Let $U : \mathcal{D} \to \mathcal{A}$ be a category of operational algebras. Given a U-split coequalizer diagram in $\mathcal{A}$

$$\begin{array}{ccccc} & \xrightarrow{Uf} & & \xrightarrow{y} & \\ UC & \xrightarrow[Ug]{} & UD & \xleftarrow[x]{} & A \\ & \xleftarrow[t]{} & & & \end{array}$$

with

(2.4) $$\begin{aligned} y.Uf &= y.Ug \\ yx &= 1 \\ Uf.t &= 1 \\ Ug.t &= xy \end{aligned}$$

then Lemma 1.4 implies that (A,y,D,x) is an operational retract. (It is instructive to prove (2.3) directly from (2.4).)

LEMMA 2.3. *Let $U : \mathcal{D} \to \mathcal{A}$ be a category of operational algebras and (A,y,D,x) be an operational retract determining the algebra (A,ϕ) . Then*

(i) *If $y = y'.Uf$ for some $f : D \to D'$ then (A,y',D',x') (with $x' = Uf.x$) is another operational retract determining (A,ϕ) .*

(ii) *If D is the algebra determined by some operational retract (A',y',D',x') then $(A,yy',D',x'x)$ is another operational retract determining (A,ϕ) .*

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Proof. For (i) note that the components of the natural transformation $\Phi_{\mathcal{D}} f$ are given by $H(Uf,1)$ and so

$$H(y,1).\omega_{\mathcal{D}}.H(x,1) = H(y',1).\Phi_{\mathcal{D}} f.\Phi_{\mathcal{D}}\omega.H(x,1)$$
$$= H(y',1).\omega_{\mathcal{D}'}.H(x',1) .$$

The proof of (ii) is trivial. □

Given an arbitrary functor $U : \mathcal{D} \to A$, then there are some retracts, namely the shuffle retracts defined below, which are forced to be operational retracts if $\mathcal{D}$ is an operational category.

Let $U : \mathcal{D} \to A$ be an object of Cat/A and let $\equiv$ be the smallest equivalence relation on diagrams in A of the form

$$A \xrightarrow{x} UD \xrightarrow{y} A'$$

such that $(yUf,D,x) \equiv (y,D',Uf.x)$ where $f : D \to D'$ in $\mathcal{D}$. This is called a *right shuffle* of f over D . (The equivalence classes may be thought of as the connected components of an appropriate category of functors.) Similarly let $\equiv$ be the smallest equivalence relation on diagrams in A of the form

$$A \xrightarrow{x} UC \xrightarrow{z} UD \xrightarrow{y} A'$$

($z \neq Uz'$ in general) such that if $(z,D,x) \equiv (z',D',x')$ then $(y,C,z,D,x) \equiv (y,C,z',D',x')$ and if $(y,C,z) \equiv (y',C',z')$ then $(y,C,z,D,x) \equiv (y',C',z',D,x)$ (here shuffling occurs over C and D) . The (1-)*shuffle retracts* are those retracts (y,D,x) such that $(y,D,xy,D,x) \equiv (y,D,1,D,x)$. The domain of x is called the *domain* of the retract. A (1-) *shuffle morphism* between shuffle retracts from (y,D,x) to (y',D',x') is a morphism $f : \text{dom } x \to \text{dom } x'$ such that $(fy,D,x) \equiv (y',D',x'f)$.

Define $S_1 : \text{Cat}/A \to \text{Cat}/A$ as follows: $S_1\mathcal{D}$ is the category of shuffle retracts and shuffle morphisms with $S_1U : S_1\mathcal{D} \to A$ sending shuffle retracts to their domains and shuffle morphisms to themselves as morphisms of A . Given $G : \mathcal{D} \to \mathcal{D}'$ in Cat/A , $S_1G(y,D,x) = (y,GD,x)$ and $S_1Gf = f$. S_1 underlies a triple (S_1,η,μ) with unit $\eta_{\mathcal{D}} : \mathcal{D} \to S_1\mathcal{D}$ and multiplication $\mu_{\mathcal{D}} : S_1^2\mathcal{D} \to S_1\mathcal{D}$ given by

$$D \longmapsto (1,D,1)$$
$$f \longmapsto Uf$$

and

$$(y',(y,D,x),x') \longmapsto (y'y,D,xx')$$
$$f \longmapsto f$$

respectively. The proof that the triple is well-defined may be found in [6]. A typical S_1-algebra will be denoted $(\mathcal{D},d_0,d_1)$ where $d_0 = U : \mathcal{D} \to A$ and $d_1 : S_1\mathcal{D} \to \mathcal{D}$ is the structure map.

If $\mathcal{D}$ is operational then the shuffle retracts are operational retracts and $\mathcal{D}$ is an S_1-algebra. Not all S_1-algebras are operational, however: let $(\mathcal{D},d_0,d_1)$ be an S_1-algebra such that (y,D,x) and (y',D',x') are shuffle retracts with $d_1(y',D',x') = D$. Then in general it does not follow that $(yy',D',x'x)$ is a shuffle retract, and so by Lemma 2.3(ii) not all operational retracts are 1-shuffle retracts. In an effort to obtain this property we enlarge our equivalence relations.

Given an S_1-algebra $(\mathcal{D},d_0,d_1)$, let $\equiv_2$ be the smallest equivalence relation containing $\equiv$ such that if (y,D,x) is a shuffle retract with $d_1(y',D',x') = D$ then $(yy',D',x'x) \equiv_2 (y,D,x)$. The definition of $\equiv_2$ is the same as that of $\equiv$ with $\equiv$ replaced by $\equiv_2$. By replacing $\equiv$ and $\equiv$ by $\equiv_2$ and $\equiv_2$ respectively, we obtain definitions of 2-*shuffle retract*, 2-*shuffle morphism* and of a triple S_2 on S_1-Alg. The S_2-algebras are not all operational either since the closure problem revealed by Lemma 2.3(ii) remains. Iteration of this process yields for each n a triple S_{n+1} on S_n-Alg for all finite n .

Let S_*-Alg be the limit of the S_n-Alg with their forgetful functors $U_n : S_n\text{-Alg} \to S_{n-1}\text{-Alg}$ (and $S_0\text{-Alg} = \text{Cat}/A$) . Then the objects of S_*-Alg are $(\mathcal{D},\{d_n\})$ where, for each n , $(\mathcal{D},d_0,d_1,\ldots,d_n)$ is an S_n-algebra and the morphisms are those functors which are S_n-homomorphisms for each n . The induced forgetful functor

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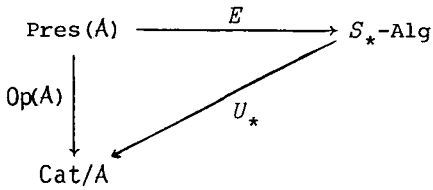
$U_* : S_*\text{-Alg} \to \mathrm{Cat}/A$ has a left adjoint F_* with $U_* F_* = S_1$. Since S_1-Alg differs from S_*-Alg it follows that U_* is not tripleable.

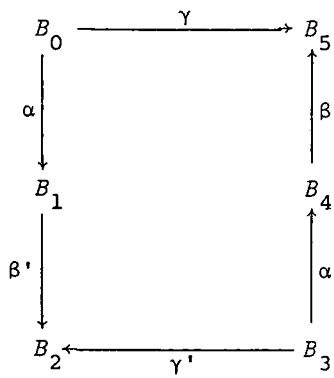
Clearly, all operational categories are S_*-algebras and there is a commuting triangle



In turn, it will be shown that S_*-algebras "are" operational categories by constructing a right inverse for E . The presentations thus constructed will all employ the same theory, θ_0 .

3. The characterisation theorem

DEFINITION 3.1. The *standard theory* $\theta_0 : B_0 \to T_0$ is given as follows: B_0 is freely generated by the graph



and T_0 is generated by

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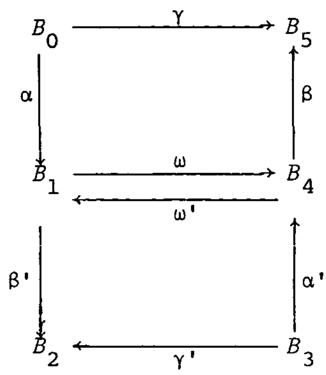


and the equations

$$\beta\omega\alpha = \gamma$$

$$\beta'\omega'\alpha' = \gamma'$$

$$\omega'\omega = 1$$

$$\omega\omega' = 1 .$$

θ_0 is the inclusion. (See [2] for a construction of a category from a sketch). $\mathrm{Pres}_0(A)$ is the subcategory of $\mathrm{Pres}(A)$ with the objects of the form (θ_0, H) and morphisms (j,k) where j is the identity.

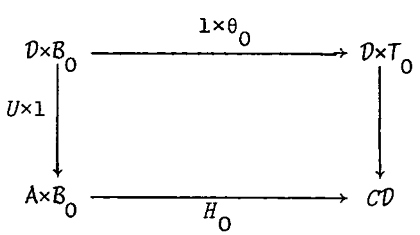
THEOREM 3.2. *There is a right inverse* M *for* E *factorising through* $\mathrm{Pres}_0(A)$ *such that* $\mathrm{Op}(A).M = U_*$. *Consequently, the following are equivalent:*

(i) $U : \mathcal{D} \to A$ *is operational,*

(ii) $U : \mathcal{D} \to A$ *underlies an* S_**-algebra,*

(iii) $U : \mathcal{D} \to A$ *is operational with respect to* θ_0 .

Proof. For each S_*-algebra a base functor will be constructed. Other details may be found in [6]. Firstly, for any $U : \mathcal{D} \to A$ in Cat/A, define $C\mathcal{D}$ by the pushout (in Cat)



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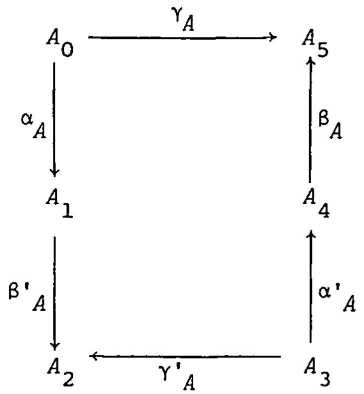
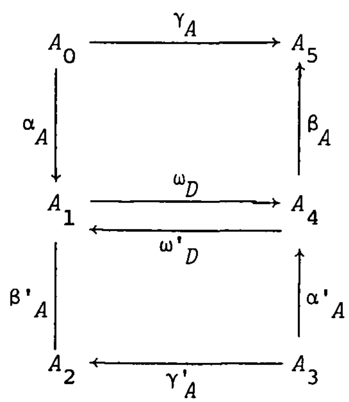
(Note in passing that $\mathrm{Op}_0(\mathcal{A})$ has a left adjoint, whose value at $\mathcal{D}$ is (θ_0, H_0) , though $\mathrm{Op}(\mathcal{A})$ never does.)

$\mathcal{CD}$ may be generated by the following sketch: each object A in $\mathcal{A}$ yields a copy of $\mathcal{B}_0$ in $\mathcal{A} \times \mathcal{B}_0$ whose image in $\mathcal{CD}$ is written



Since $1 \times \theta$ is bijective on objects, H_0 is and so all objects of $\mathcal{CD}$ are of the form A_i . Each object D of $\mathcal{D}$ yields a pair of inverses ω and ω' in $\mathcal{CD}$ pictured as follows:



where $A = UD$. They also satisfy the equations

$$\beta_A \omega_D \alpha_A = \gamma_A$$

$$\beta'_A \omega'_D \alpha'_A = \gamma'_A \ .$$

Given $f : A \to A'$ in $\mathcal{A}$ we have $f_i = H_0(f, B_i) : A_i \to A'_i$ for each i .

The f_i's compose as they do in A and commute with the morphisms from B_0, for example $f_1\alpha_A = \alpha_{A'}f_0$. Given $f : D \to D'$ in $\mathcal{D}$ then Uf has the additional property that $(Uf)_4\, \omega_D = \omega_{D'}(Uf)_1$ and similarly for ω'.

$S_1\mathcal{D}$ is given by the pullback

$$\begin{array}{ccc} S_1\mathcal{D} & \longrightarrow & [T_0, C\mathcal{D}] \\ \downarrow & & \downarrow{\scriptstyle [\theta_0,1]} \\ A & \underset{H_0^*}{\longrightarrow} & [B_0, C\mathcal{D}] \end{array}$$

Now when $(\mathcal{D}, \{d_n\})$ is an S_*-algebra we define $q_1 : C\mathcal{D} \to C_1\mathcal{D}$ to be the quotient of $C\mathcal{D}$ given by imposing the relations

$$y_4\omega_D x_1 = \omega_{D'}$$
$$y_1\omega'_D x_4 = \omega'_{D'}$$

whenever $D' = d_n(y,D,x)$ for some n-shuffle retract. Then

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & [T_0, C_1\mathcal{D}] \\ \downarrow & & \downarrow{\scriptstyle [\theta_0,1]} \\ A & \underset{H_1^*}{\longrightarrow} & [B_0, C_1\mathcal{D}] \end{array}$$

is a pullback (where $H_1 = q_1H_0$) as required. □

Note that M is not left adjoint to E and S_*-Alg is not equivalent to $\text{Pres}_0(A)$.

4. Props

We may ask that our presentations have some extra structure, say, a monoidal structure or that of finite products, and perform our constructions, not in Cat, but in some other appropriate category, say, the category of symmetric monoidal categories and strong symmetric monoidal functors, denoted StM. Below we examine the monoidal case

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(see [5] for terminology), in which props (MacLane [11]) become examples of theories. In this paper all monoidal categories and functors are assumed symmetric.

Denote by $\mathrm{StM}(\mathcal{B},\mathcal{C})$ the monoidal category of strong monoidal functors from $\mathcal{B}$ to $\mathcal{C}$ and let $(\bar{\ }) : \mathrm{Cat} \to \mathrm{Stm}$ be the left bi-adjoint ([13]) to the forgetful functor, with unit $\eta : \mathcal{C} \to \bar{\mathcal{C}}$.

A *monoidal presentation* (θ, H) consists of a *prop* θ , by which is meant a strong monoidal functor $\theta = (\theta,\tilde{\theta},\theta^{o}) : \mathcal{B} \to \mathcal{T}$ in which θ is bijective on objects, and a (strong monoidal) functor $H = (h,\tilde{h},h^{o}) : \mathrm{A} \to \mathrm{Stm}(\mathcal{B},\mathcal{C})$.

PROPOSITION 4.1. *Let* (θ,H) *be a monoidal presentation. Then there is a pullback*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\quad P \quad} & \mathrm{StM}(\mathcal{T},\mathcal{C}) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle \mathrm{StM}(\theta,1)} \\ \mathrm{A} & \xrightarrow{\qquad} & \mathrm{StM}(\mathcal{B},\mathcal{C}) \end{array}$$

in StM *and* $U = (u,u,u^{o}) : \mathcal{D} \to \mathrm{A}$ *is strict.*

Proof. It is well known that StM has all pseudo-limits (as defined in [7]). The pullback $u : \mathcal{D} \to \mathrm{A}$ of $\mathrm{StM}(\theta,1)$ by h in Cat is equivalent to a subcategory of the pseudo-pullback determined by the presentation. The result follows. □

A strict monoidal functor $U : \mathcal{D} \to \mathrm{A}$ arising from a monoidal presentation as above is called a *category of prop algebras*. Any category equivalent to such a category of algebras is called *proppable*.

EXAMPLE 4.2. Let P be a skeleton of $\bar{1}$. Its objects may be identified with the natural numbers and

$$P(m,n) = \begin{cases} \phi & m \neq n \\ P_n & m = n \end{cases}$$

where P_n is the permutation monoid on n elements. On objects the tensor is given by addition. Let $\theta : P \to T$ be a strict prop. Then T is a prop in MacLane's sense ([11]). Let A be a monoidal category. Define $H = (h, \tilde{h}, h^o) : A \to \mathrm{StM}(P, A)$ by $(hA)n = A^n = A \otimes A \otimes \ldots \otimes A$ (n times) and $(hA)\sigma$ is the composite of symmetries which permutes the A's in $(hA)n$ according to σ, together with the obvious choices of $\tilde{h}$ and h^o. Then $\mathcal{D} = \mathrm{StM}[T, A]$ is proppable in Barr's sense ([1]).

Note that the following square (whose horizontal maps are the inclusions)

$$\begin{array}{ccc} \mathrm{StM}(T,C) & \longrightarrow & [T,C] \\ {\scriptstyle \mathrm{StM}(\theta,1)}\downarrow & & \downarrow{\scriptstyle [\theta,1]} \\ \mathrm{StM}(B,C) & \longrightarrow & [B,C] \end{array}$$

is not a pullback. Thus, it is not immediate that proppable categories are operational.

THEOREM 4.3. *The following are equivalent:*

(i) $U : \mathcal{D} \to A$ *is proppable,*

(ii) U *is strict monoidal and operational,*

(iii) U *is proppable with respect to* $\overline{\theta_0}$.

Proof. (iii) ⇒ (i) is trivial.

(i) ⇒ (ii): U is strict monoidal by Proposition 4.1 and straightforward verification shows that $\mathcal{D}$ is an S_*-algebra. Now use Theorem 3.2.

(ii) ⇒ (iii): Without loss of generality assume that $\mathcal{D}$ is a category of algebras with presentation (θ_0, H_0) as in the proof of (ii) ⇒ (iii) in Theorem 3.2. H_0 does not underly a monoidal functor. This problem is overcome by choosing a $C_2\mathcal{D}$ and $Q_2 = (q_2, \tilde{q}_2, q_2^o) : C_1\mathcal{D} \to C_2\mathcal{D}$ with the following property: there is a strong monoidal functor $H_2 = (h_2, \tilde{h}_2, h_2^o)$ such that $h_2 = qH_0$ and for any monoidal category V , $\mathrm{StM}(C_2\mathcal{D}, V)$ is equivalent to the full subcategory of $\mathrm{StM}(C_1\mathcal{D}, V)$ of monoidal functors $R = (r, \tilde{r}, r^o)$ such that rH_0

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underlies a strong monoidal functor. Then, as in Theorem 3.2, we have the pullback

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathrm{StM}(\bar{T}_0, C_2\mathcal{D}) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle \mathrm{StM}(\bar{\theta}_0,1)} \\ A & \underset{H_2}{\longrightarrow} & \mathrm{StM}(\bar{B}_0, C_2\mathcal{D}) \end{array}$$

showing that $\mathcal{D}$ is proppable over A . □

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