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Coloring Four-uniform Hypergraphs on Nine Vertices

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Abstract. Every 4-uniform hypergraph on 9 vertices with at most 25 edges has property B. This gives the answer $m_9(4) = 26$ to a question raised by Erdős in 1968.

1 Introduction

In [3, p. 447] Erdős defines $m_n(k)$ as the smallest integer m for which there exists a k-uniform hypergraph with n vertices and m hyperedges without property B, which means that for every 2-coloring of its vertices some hyperedge is monochromatic. In [4, p. 416] Erdős admits "I cannot compute $m_{2k+1}(k)$ and in fact do not know the value of $m_9(4)$" (compare [2, p. 155]). In 1980 Abbott and Liu ([1]) proved that $24 \leq m_9(4) \leq 26$. In particular, they gave an example of a 4-uniform hypergraph with 26 hyperedges without property B.

Our main result is contained in the following theorem.

Theorem 1.1 *Any family of four-element subsets of a given nine-element set without property B must have at least* 26 *elements.*

The proof is partly based on computer calculations, which provide us with many non-isomorphic 4-uniform hypergraphs with 26 edges without property B. In the following sample family, any two elements intersect:

$$\{1234\}\{1235\}\{1237\}\{1456\}\{1467\}\{1489\}\{1567\}\{1589\}\{1789\}\{2369\}$$
$$\{2459\}\{2468\}\{2479\}\{2568\}\{2578\}\{2579\}\{2678\}\{3457\}\{3458\}\{3478\}$$
$$\{3569\}\{3578\}\{3679\}\{3689\}\{4567\}\{4579\}.$$

Thus it is not permutation conjugate to the family given by Abbott and Liu, which has 3 disjoint pairs.

2 Separating triplets in an eight-set

Consider a 3-uniform hypergraph (V_8, S) on eight nodes. We say that a quadruple $Q \subset V_8$ *separates* S if and only if both Q and its complement $\overline{Q}$ in V_8 contain an edge. Let $T(S)$ denote the family of all quadruples separating S. The maximal size of $T(S)$

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for S having n elements will be denoted by $\omega(n)$. These numbers (for $n \le 11$) will be crucial in the proof of Theorem 1.1 in Section 3. They are listed in Proposition 2.1.

Proposition 2.1 *The values $\omega(n)$ for $n \le 11$ are given in the table*

$$\begin{bmatrix} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \omega(n) & 0 & 0 & 4 & 8 & 12 & 16 & 24 & 26 & 34 & 36 & 38 & 40 \end{bmatrix}.$$

All the values have been calculated with a computer. Total CPU time on one core of the processor AMD Athlon 64 X2 1.79 GHz is about 9 minutes for $\omega(10)$ and less than 50 minutes for $\omega(11)$.

The algorithm for calculating $\omega(n)$ is very simple. We consider n-element families S of triplets in V_8 that contain at least two disjoint triplets. Up to a permutation of V_8 we may assume these two triplets are fixed. The number of the remaining triplets is $\binom{8}{3} - 2 = 54$. We consider all $n-2$-element subsets of this set, which, together with the two fixed triplets, form S. For each S we consider $\binom{n}{2}$ pairs (A, B) in S. If the pair is disjoint, then four quadruples separating S are created. They are two pairs of complementary sets, represented by $A \cup \{x\}$ for $x \in V_8 \setminus (A \cup B)$. All members in $T(S)$ are obtained in this way. Thus, $|T(S)|$ is twice the number of such pairs. The total number of such operations is

$$\binom{54}{n-2}\binom{n}{2} \cdot 2 \le \binom{54}{9}\binom{11}{2} \cdot 2 \approx 5.8 \cdot 10^{11}.$$

3 Proof of Theorem 1.1

Suppose (V_9, E) is a hypergraph on 9 nodes with $|E| \le 25$. We have to find a coloring of V_9 for which no edge is monochromatic. As $\sum\{|A| : A \in E\} \le 25 \cdot 4 = 100 < 108 = 12 \cdot |V_9|$, we can find $P \in V_9$ such that the family $E_1 = \{A \in E : P \in A\}$ has at most 11 elements. Let

$$V_8 = V_9 \setminus \{P\}, \qquad E_0 = \{A \in E : P \notin A\},$$
$$S = \{A \setminus \{P\} : A \in E_1\}, \qquad n = |S| = |E_1| \le 11.$$

Let $T(S)$ be the family of all quadruples in V_8 separating S. Proposition 2.1 applied to the 3-uniform hypergraph (V_8, S) gives the maximal possible values of $|T(S)|$. In particular $|T(S)| \le 18 + 2n$. Let

$$R = \big\{ B \subset V_8 : (|B| = 4) \wedge ((B \in E_0) \vee (V_8 \setminus B \in E_0)\big\}.$$

We have

$$|R| \le 2 \cdot |E_0| \le 2 \cdot (25 - |E_1|) = 50 - 2n.$$

The number of quadruples in V_8 is $\binom{8}{4} = 70$. As $(18 + 2n) + (50 - 2n) = 68 < 70$ we can find a quadruple K in V_8 not belonging to $T(S) \cup R$. Clearly $\overline{K} = V_8 \setminus K$ cannot belong to $T(S) \cup R$ either. Let

$$S_0 = \{C \in S : (C \subset K) \vee (C \subset \overline{K})\}, \quad S_1 = S \setminus S_0.$$

As $K \notin T(S)$, K does not separate S. Therefore, all $C \in S_0$ are contained either in K or in $\overline{K}$. Replacing K with $\overline{K}$ if necessary, we may assume that no $C \in S$ is

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contained in $\overline{K}$. We shall easily verify now that K is a coloring for V_9 that has no monochromatic set belonging to E. Indeed, if $A \in E$ is monochromatic, then either $A \subset K$ or $A \subset \overline{K} \cup \{P\}$. If $A \subset K$, then $K = A \in E_0 \subset R$, which is impossible, since $K \notin T(S) \cup R$. If $A \subset \overline{K} \cup \{P\}$ and $P \notin A$, then $\overline{K} = A \in E_0$ and again $K \in R$. If $A \subset \overline{K} \cup \{P\}$ and $P \in A$ then $C = A \setminus \{P\} \in S, C \subset \overline{K}$, which contradicts our choice of K. This completes the proof of Theorem 1.1.

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