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PSEUDOCOMPLEMENTED DISTRIBUTIVE LATTICES WITH SMALL ENDOMORPHISM MONOIDS

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By a result of K.B. Lee, the lattice of varieties of pseudocomplemented distributive lattices is the $\omega + 1$ chain

$$B_{-1} \subset B_0 \subset B_1 \subset \ldots \subset B_n \subset \ldots \subset B_\omega$$

where B_{-1}, B_0, B_1 are the varieties formed by all trivial, Boolean, and Stone algebras, respectively. General theorems on relative universality proved in the present paper imply that there is a proper class of non-isomorphic algebras in B_3 with finite endomorphism monoids, while every infinite algebra from B_2 has infinitely many endomorphisms. The variety B_4 contains a proper class of non-isomorphic algebras with endomorphism monoids consisting of the identity and finitely many right zeros; on the other hand, any algebra in B_3 with a finite endomorphism monoid of this type must be finite.

1. Introduction

For an arbitrary algebra L , let $\mathrm{End}(L)$ denote the monoid of all endomorphisms of L .

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A *pseudocomplemented distributive lattice* is an algebra $(L; \vee, \wedge, ^*, 0, 1)$ of type $(2, 2, 1, 0, 0)$ where $(L; \vee, \wedge, 0, 1)$ is a distributive $(0, 1)$-lattice and the unary operation * of pseudocomplementation is defined by $y \leq x^*$ if and only if $x \wedge y = 0$ in $(L; \vee, \wedge, 0, 1)$. Pseudocomplemented distributive lattices form a variety B_ω, Ribenboim [18], and the lattice of its subvarieties is an $\omega + 1$ chain $B_{-1} \subset B_0 \subset B_1 \subset B_2 \subset \ldots \subset B_\omega$, Lee [11]. Of these varieties, two have already been studied extensively: the variety B_0 of Boolean algebras and the variety B_1 of Stone algebras $\left(B_{-1}\right.$ is the trivial variety$\left.\right)$. Further information and references can be found in, for example, Balbes and Dwinger [3] or Grätzer [5], [6].

Independently, Magill [12] and Schein [19] have shown that a Boolean algebra is uniquely determined by its endomorphism monoid: for $K, L \in B_0$, $\mathrm{End}(K) \cong \mathrm{End}(L)$ only if $K \cong L$. Investigations of other varieties of pseudocomplemented distributive lattices [1] have extended this result to the variety B_1 of Stone algebras. For B_2 the situation is somewhat different. If $\mathrm{End}(K) \cong \mathrm{End}(L)$ for nonisomorphic $K, L \in B_2$, then there is a uniquely defined algebra $L^+ \cong K$; that is, an algebra in B_2 is determined by its endomorphism monoid up to one of two algebras. For B_3 and larger varieties a radical change occurs: for any infinite cardinal κ there exists a family $\left\{L_i \in B_3 : i < 2^\kappa\right\}$ of pair-wise nonisomorphic algebras with $|L_i| = \kappa$ and $\mathrm{End}(L_i) \cong \mathrm{End}(L_j)$ for all $i, j \in 2^\kappa$. This fact is an immediate consequence of the more general result [1] that B_3 is an almost universal variety. A category C is *almost universal* if the category G of (undirected) graphs and all their compatible maps is isomorphic to a subcategory of C formed by all nonconstant morphisms between objects from a suitably chosen subclass D of C; note that this definition requires that the nonconstant morphisms between members of D be closed under composition.

For any minimal prime ideal $I \subseteq K \in B_\omega$, the mapping $H : K \to \{0, 1\}$ given by $h^{-1}\{0\} = I$ is a homomorphism. A homomorphism $g : K \to L$ in B_ω

is *constant* if and only if $g(K) = \{0, 1\}$, that is if $g(K)$ is the set of constants of L. Since every pseudocomplemented distributive lattice has as least one minimal prime ideal, there exist constant homomorphisms between any two such algebras. The constant endomorphisms are precisely the right zeros in $\mathrm{End}(K)$ and thus every nontrivial $K \in B_\omega$ has at least one right zero in its endomorphism monoid [1].

All infinite algebras of cardinality κ constructed to prove the almost universality of B_3 in [1] have 2^κ right zero endomorphisms; therefore no light is shed on the question of whether an infinite pseudocomplemented distributive lattice necessarily has an infinite endomorphism monoid. Although this is always the case in B_2, a surprising reversal occurs in larger varieties; Corollary 1.3 below claims the existence of arbitrarily large algebras in B_3 that have only finitely many endomorphisms. Furthermore, it will be seen that the construction presented in [1] was no accident: if $K \in B_3$ is infinite then either K has infinitely many constant endomorphisms or there exists a non-constant nontrivial endomorphism of K (see Theorem 1.1). By contrast, B_4 contains arbitrarily large algebras whose endomorphism monoids consist of the identity mapping and finitely many right zeros (Corollary 1.5). The results proved here will be more general, however, and additional concepts are needed for their formulation.

A variety V is *W-universal* if W is a subvariety of V and if the category of all compatible maps between (undirected) graphs is isomorphic to a subcategory C of V consisting of all those homomorphisms $h : C_1 \to C_2$ between objects of C for which $h(C_j) \in V \backslash W$. If, in addition, there are less than κ homomorphisms $h : C_1 \to C_2$ with $h(C_1) \in W$ for any pair of objects in C, the variety V is *(κ, W)-universal*. In this terminology, since $\{0, 1\}$ is a Boolean algebra, the almost universality of B_3 shown in [1] implies that B_3 is B_0-universal. By the preceding discussion, however, [1] does not imply (κ, B_0)-universality for any cardinal κ. The results presented here are as follows.

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THEOREM 1.1. *B_3 is neither (ω, B_0)-universal nor (ω, B_1)-universal.*

THEOREM 1.2. *B_3 is (ω, B_2)-universal.*

In fact, a stronger result is shown: B_3 is (m, B_2)-universal for some integer m.

In connection with the immediate consequence below of Theorem 1.2 (see, for instance, Pultr and Trnková [*17*]) it is also interesting to recall that every infinite Boolean algebra has an uncountable endomorphism monoid.

COROLLARY 1.3. *In B_3, there exists a proper class of pairwise non-isomorphic algebras whose endomorphism monoids are finite.*

THEOREM 1.4. *B_n is (ω, B_0)-universal for every $n \geq 4$.*

As before, it is actually shown that B_4 is (m, B_0)-universal for some finite m. Furthermore, if Boolean, the image of an endomorphism is always the subalgebra $\{0, 1\}$; the statement below is an immediate consequence of this fact.

COROLLARY 1.5. *In B_4, there exists a proper class of pairwise non-isomorphic algebras whose endomorphism monoids consist of the identity and finitely many right zeros.*

2. Preliminaries

In Priestley [*13*], a topological duality was introduced for the category of distributive $(0, 1)$-lattices. A brief outline follows; for further information see, for instance, Davey and Duffus [*4*] or Priestley [*16*].

A mapping $\psi : P_1 \to P_2$ between partially ordered sets P_1, P_2 is *order preserving* if $x \leq y$ implies $\psi(x) \leq \psi(y)$. Let $\mathrm{Min}(P)$ denote the set of all minimal elements of a poset P. Further, for $S \subseteq P$, set $[S) = \{x : \exists s \in S, x \geq s\}$, $(S] = \{x : \exists s \in S, x \leq s\}$, and $\mathrm{Min}(S) = \mathrm{Min}(P) \cap (S]$ (if $S = \{x\}$ then $\mathrm{Min}(S)$ is written as $\mathrm{Min}(x)$). A subset S of P is *increasing* if $[S) = S$, *decreasing* if $S = (S]$. A

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poset P equipped with a topology is *totally order disconnected* if, for $x, y \in P$, $x \not\leq y$ implies the existence of a clopen decreasing set $S \subseteq P$ such that $x \in S$ and $y \notin S$.

PROPOSITION 2.1 (Priestley [13]). *The category $D_{0,1}$ of all $(0, 1)$-homomorphisms of distributive $(0, 1)$-lattices is dually isomorphic to the category T of all continuous order preserving maps of compact totally order disconnected spaces.*

Under this duality the poset associated with a distributive $(0, 1)$-lattice is the inclusion ordered poset of its prime ideals, and lattice elements correspond to clopen decreasing subsets of the representing totally order disconnected space. Furthermore, if L_j is a distributive $(0, 1)$-lattice associated with the space (P_j, τ_j) for $j = 1, 2$, and if $h : L_1 \to L_2$ is a $(0, 1)$-homomorphism represented by the continuous order preserving map $\psi : P_2 \to P_1$, then, for the clopen decreasing set $X \subseteq P_1$ corresponding to $x \in L_1$, the element $h(x)$ of L_2 corresponds to the clopen decreasing set $\psi^{-1}(X) \subseteq P_1$.

PROPOSITION 2.2 [2]. *The category G of graphs is dually isomorphic to a full subcategory of the category T_5 of all compact totally order disconnected spaces with five distinguished elements and all continuous order preserving maps that also preserve these five elements.*

An object of T_5 is thus a compact totally order disconnected space (P, τ) together with five distinct elements a_i of P for $i < 5$; as given in [2], these five elements are minimal in P and the partial order P is connected. Inverting the order and retaining the topology of every such space gives rise to a full subcategory of T_5 isomorphic to the original one; thus it can be assumed that P is connected and that all five distinguished elements are maximal in the order of P.

Since every pseudocomplemented distributive lattice has 0 and 1, Priestley's duality restricts to a topological duality for B_ω. A totally order disconnected space (P, τ) has the *p-property* if $[S)$ is clopen for every clopen decreasing set $S \subseteq P$. An order preserving map

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$\psi : P_1 \to P_2$ is a *p-map* if $\psi(\mathrm{Min}(x)) = \mathrm{Min}(\psi(x))$ for every $x \in P_1$.

The category B_ω is dually isomorphic to the category T^p of all compact totally order disconnected spaces with the p-property and all continuous p-maps between such spaces. For more extensive background information see Priestley [15]. Note that any constant homomorphism of pseudocomplemented distributive lattices corresponds in this duality to a constant p-map whose value is a minimal element.

Finally, for $n < \omega$, we need to recognize spaces with the p-property that represent algebras in B_n.

PROPOSITION 2.3 (Lee [11]). *For $L \in B_\omega$ and $n \geq 1$, the algebra L belongs to B_n if and only if every prime ideal of L contains at most n minimal prime ideals.*

3. Proof of Theorem 1.1

In this section, let $L \in B_3$ be an infinite algebra with only a finite number of constant endomorphisms; if (P, τ) represents such L then $\mathrm{Min}(P)$ must be finite. Denote $P(M) = \{x \in P : \mathrm{Min}(x) = M\}$ for every $M \subseteq \mathrm{Min}(P)$.

LEMMA 3.1. *$P(M)$ is clopen for every $M \subseteq \mathrm{Min}(P)$.*

Proof. Since P is totally order disconnected, for any $m \in \mathrm{Min}(P)$ there exists a clopen decreasing set $Q_m \subseteq P$ such that $\mathrm{Min}(Q_m) = \{m\}$. By the p-property, $[Q_m)$ is clopen, so that

$$P(M) = \cap\{[Q_m) : m \in M\} \backslash \cup\{[Q_m) : m \in \mathrm{Min}(P)\backslash M\}$$

is clopen as well.

LEMMA 3.2. *For every $p \in P(M)\backslash\mathrm{Min}(P)$ with $|M| < 3$ there exists a non-trivial $h \in \mathrm{End}(L)$ whose image is represented by $\{p\} \cup \mathrm{Min}(p)$.*

Proof. Suppose $M = \{m\}$ and $p \in P(M)\backslash\mathrm{Min}(P)$. There exists a clopen decreasing $Q \subseteq P$ not containing p and such that $Q \supseteq \mathrm{Min}(P)$; set $\psi(x) = m$ for $x \in Q$ and $\psi(x) = p$ for $x \in P\backslash Q$. It is clear that ψ is a p-map and that $\psi(P) = \{p\} \cup \mathrm{Min}(p)$.

Let $M = \{m, n\}$ and $p \in P(M)\setminus\mathrm{Min}(P)$. Define $\psi : P \to P$ by $\psi(x) = n$ if $\mathrm{Min}(x) = \{n\}$, $\psi(x) = m$ for $n \notin \mathrm{Min}(x)$, and $\psi(x) = p$ otherwise. Then ψ is a p-map whose continuity follows from Lemma 3.1, and again $\psi(P) = \{p\} \cup \mathrm{Min}(p)$.

PROPOSITION 3.3. $\mathrm{End}(L)$ *is infinite for arbitrary infinite* $L \in B_2$.

Proof. Recall that $\mathrm{Min}(P)$ is finite; Proposition 2.3 implies that $P(M)$ must be infinite for some $M \subseteq \mathrm{Min}(P)$ with $|M| < 3$. Lemma 3.2 concludes the proof.

LEMMA 3.4. *If* $P(M) = \emptyset$ *for all two-element* $M \subseteq \mathrm{Min}(P)$ *then* $\mathrm{End}(L)$ *is infinite.*

Proof. Since $\mathrm{Min}(P)$ is finite, for some $M \subseteq \mathrm{Min}(P)$ the clopen set $P(M)$ must be infinite. By the hypothesis and by Lemma 3.2 it is enough to assume that $|M| = 3$; since $L \in B_3$, Proposition 2.3 implies that $P(M)$ is an increasing set. For each $p \in P(M)$ now define ψ_p by $\psi_p(x) = p$ for $x \in P(M)$, $\psi(x) = m$ whenever $\mathrm{Min}(x) = \{m\}$ for some $m \in M$, and $\psi_p(x) = x$ otherwise. From Lemma 3.1 it is clear that ψ_p is a p-map for every $p \in P(M)$. Thus $\mathrm{End}(L)$ is infinite.

There exist arbitrarily large graphs with no non-trivial endomorphisms [17]. If B_3 were (ω, B_j)-universal for some $j < 2$ there would certainly exist an infinite algebra L in B_3 with finite $\mathrm{End}(L)$ and such that $h(L) \in B_2\setminus B_j$ for no $h \in \mathrm{End}(L)$. The two preceding lemmata show that this is impossible.

4. Proof of Theorem 1.2

A *graph* is a pair (X, R) where R is a set of two-element subsets of X . For graphs (X_1, R_1), (X_2, R_2) a mapping $\varphi : X_1 \to X_2$ is *compatible* if $\{\varphi(x), \varphi(y)\} \in R_2$ for every $\{x, y\} \in R_1$.

By Proposition 2.2, there is a full and faithful contravariant functor Φ from the category G of all graphs and their compatible maps to the category T_5 of all compact totally order disconnected spaces with five distinguished elements and continuous order preserving maps that preserve these elements: the existence of Φ shows that G is dually isomorphic to a full subcategory of T_5 . The image $(P, \tau) = \Phi(X, R)$ of any $(X, R) \in G$ is a compact totally order disconnected space in which P is order connected and the five constants a_i for $i < 5$ are maximal elements of P . To show that B_3 is (ω, B_2)-universal, a faithful contravariant functor $\Psi : G \to T^P$ containing Φ will be constructed in such a way that $\Psi(X, R)$ is the space of an algebra in B_3 and, if $\psi : \Psi(X_1, R_1) \to \Psi(X_2, R_2)$ is a morphism in T^P then either $\psi = \Psi(\varphi)$ for some compatible $\varphi : (X_2, R_2) \to (X_1, R_1)$ or ψ is one of finitely many mappings with $\psi(\Psi(X_1, R_1))$ representing an algebra in B_2 .

A graph G is *rigid* if the only compatible map of G into itself is the identity. Once and for all select a finite rigid graph (V, E) with $|V| > 7$; such graphs exist according to Hedrlín and Pultr [9] (see also [17]). Set $V = \{v_i : i < |V|\}$, and let T denote the collection of all three-element subsets of V . For convenience, denote $t_i = \{v_i, v_6, v_7\}$ for all $i < 6$, and let $C = \{c_i : i < 5\}$.

The functor Ψ is defined as follows.

For $(X, R) \in G$, let

$$Q = V \cup E \cup C \cup \{b\} \cup (T\backslash\{t_5\}) \cup P ,$$

where $(P, \tau) = \Phi(X, R)$ and the union is disjoint; intuitively, the triple t_5 will be replaced by a copy of P . Define a partial order on Q by

(i) for $v \in V$ and $e \in E$, $v \le e$ if and only if $v \in e$;

(ii) $b \le e$ for all $e \in E$;

(iii) for $v \in V$ and $t \in T\backslash\{t_5\}$, $v \le t$ if and only if $v \in t$;

(iv) $v_5, v_6, v_7 \le x$ for all $x \in P$;

(v) if $i < 5$ then $v_6, v_7 \leq c_i \leq a_i, t_i$ for $c_i \in C$ and for the distinguished maximal element a_i of P ;

(vi) for $x, y \in P$, $x \leq y$ in Q if and only if $x \leq y$ in P .

Let σ denote the topology on Q obtained as the union of τ on P with the discrete topology on the finite set $Q \backslash P$; the space $(Q, \sigma) = \Psi(X, R)$ clearly is compact and totally disconnected.

For any morphism $\varphi : (X_2, R_2) \to (X_1, R_1)$ in G , let $\Psi(\varphi) : \Psi(X_1, R_1) \to \Psi(X_2, R_2)$ be defined as the extension of $\Phi(\varphi)$ by the identity mapping on $Q_1 \backslash P_1 = Q_2 \backslash P_2$. It is routine to verify that $\Psi(\varphi)$ is a continuous order preserving mapping. Since $\Phi(X, R)$ is a totally order disconnected space, it follows easily that (Q, σ) also is such a space. Clearly $[S) = [\mathrm{Min}(S))$ for any clopen decreasing set $S \subseteq Q$; hence as $\mathrm{Min}(Q) = V \cup \{b\}$, the set $[S)$ is either a finite subset of $Q \backslash P$ or $[S)$ contains P . In either case $[S)$ is clopen, so that $\Psi(X, R)$ has the p-property. Since $|\mathrm{Min}(q)| \leq 3$ for all $q \in Q$, Proposition 2.3 shows that $\Psi(X, R)$ represents an algebra in B_3 . The mapping $\Psi(\psi)$ is the identity on $Q \backslash P \supseteq \mathrm{Min}(Q)$, and it is readily apparent that $\Psi(\varphi)$ is a p-map. This concludes the proof of the claim below.

LEMMA 4.1. *$\Psi : G \to T^p$ is a well defined contravariant faithful functor.*

Observe that, for $j = 1, 2$, if $L_j \in B_\omega$ is represented by (Q_j, σ_j) and $\psi : Q_1 \to Q_2$ is a morphism in T^p representing $h : L_2 \to L_1$, then $h(L_2) \in B_n$ if and only if the subspace $\psi(Q_1) \subseteq Q_2$ with the order induced by Q_2 represents an algebra in B_n .

Thus it must be shown that, for $(X_j, R_j) \in G$, there are only finitely many continuous order preserving p-maps $\psi : \Psi(X_1, R_1) \to \Psi(X_2, R_2)$ with $\psi(Q_1)$ representing algebras from B_2 , and that any other ψ has the form $\psi = \Psi(\varphi)$ for some G-morphism $\varphi : (X_2, R_2) \to (X_1, R_1)$.

For the remainder of this section let $\psi : \Psi(X_1, R_1) \to \Psi(X_2, R_2)$ be a continuous order preserving p-map with $(X_j, R_j) \in G$.

LEMMA 4.2. *If* $\psi \upharpoonright (V \cup \{b\})$ *is not one-to-one then* $\psi(Q_1) \subseteq V \cup \{b\} \cup C$, *and* $\psi \upharpoonright P_1$ *is constant.*

Proof. Recall that $V \cup \{b\} = \mathrm{Min}(Q)$; furthermore, $|\mathrm{Min}(c)| = 2$ for $c \in C$, and $|\mathrm{Min}(x)| = 3$ for all $q \in Q \backslash (\mathrm{Min}(Q) \cup C)$.

Suppose that $\psi(u) = \psi(v)$ for distinct $u, v \in V$; then $\psi(u) = m$ is minimal since ψ is a p-map.

Consider the case of $\psi(u) \notin \{v_6, v_7\}$ first. For every $w \in V\backslash\{u, v\}$ there exists $t \in T$ with $\mathrm{Min}(t) = \{u, v, w\}$, and $\mathrm{Min}(\psi(t)) = \psi(\mathrm{Min}(t))$ implies that $\psi(t) = \psi(w) = m$. For every three-element subset U of $V \subseteq \mathrm{Min}(Q)$ there exists $t \in T$ with $\mathrm{Min}(t) = U$, so that both $\psi(V) = \{m\}$ and $\psi((T\backslash\{t_5\}) \cup P_1 \cup C) = \{m\}$ easily follow. If $\{u, v\} \in E$ then $\psi(\{b, u, v\}) = \{m\}$ is similarly obtained from $\psi(V) = \{m\}$, and hence ψ is a constant p-map.

Assume next that $m \in \{v_6, v_7\}$; if $\mathrm{Min}(t) = \{u, v, w\}$ then either $\psi(t) = \psi(w) = m$, or $\psi(t) \in C$ and $m \neq \psi(w) \in \{v_6, v_7\}$. If $\psi(V) = \{m\}$ then $\psi(P_1 \cup (T\backslash\{t_5\})) = \{m\}$ and $\psi(E) \subseteq C \cup V$ finish the proof. If $\psi(V) = \{v_6, v_7\}$ then either $\psi(\{v_5, v_6, v_7\}) = m$ or $\{v_6, v_7\}$; in the first case $\psi(P_1) = \{m\}$, in the second $\psi(P_1) \subseteq C$; since P_1 is order connected while C is an antichain, $\psi \upharpoonright P_1$ must be constant in either case. The rigid graph (V, E) cannot be bipartite, so that $\psi(x) = \psi(y)$ for some $\{x, y\} \in E$; therefore $\psi(b) \in \{v_6, v_7\}$ and, consequently, $\psi(\mathrm{Min}(Q_1)) = \{v_6, v_7\}$ in this case. It is now easily seen that $\psi(Q_1) \subseteq C \cup \{v_6, v_7\}$.

The remaining possibility is that $\psi(b) = \psi(v)$ for some $v \in V$. For any $\{u, v\} \in E$ then either $\psi(b) = \psi(u) = \psi(v)$ and the previous arguments apply, or else $\mathrm{Min}(\psi(\{b, u, v\})) = \{v_6, v_7\}$. The rigidity of (V, E) now implies that $\{v, w\} \in E$ for some $w \in V\backslash\{u\}$; if $\psi(w) \neq \psi(v)$ then $\psi(w) = \psi(u)$ must hold, reducing the argument to the

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previously considered case.

LEMMA 4.3. *If* ψ *is one-to-one on* $V \cup \{b\}$ *then* $\psi = \Psi(\varphi)$ *for some* G*-morphism* $\varphi : (X_2, R_2) \to (X_1, R_1)$.

Proof. If $\psi(x) = b$ for some $x \in V$ then also $\psi(b) \in V$. Since (V, E) is a rigid graph with more than seven vertices, there exists $\{u, v\}$ not in E and such that $\psi(b) \neq u, v$; by finiteness, $u = \psi(y)$ and $v = \psi(z)$ for some $y, z \in V$. By definition, there is a $q \in Q_1$ with $\mathrm{Min}(q) = \{x, y, z\}$, so that $\mathrm{Min}(\psi(q)) = \psi(\mathrm{Min}(q)) = \{b, u, v\}$ which contradicts the choice of $\{u, v\}$. Therefore $\psi(b) = b$ and $\psi(V) = V$.

Since $p \geq b$ and $\mathrm{Min}(p) \cap V \neq \emptyset$ if and only if $p \in E$, it follows that $\psi(E) \subseteq E$; hence ψ is an endomorphism of the rigid graph (V, E) , that is, ψ is the identity on $V \cup E$. Consequently, ψ is also the identity on $C \cup (T \backslash \{t_5\})$, and $\psi(P_1) \subseteq P_2$. Since, for $i < 5$, a_i is the only element of P_j above $c_i \in C$, from $\psi(c_i) = c_i$ we obtain $\psi(a_i) = a_i$. By Proposition 2.2 there exists a G-morphism $\varphi : (X_2, R_2) \to (X_1, R_1)$ with $\Phi(\varphi) = \psi \restriction P_1$. Altogether, $\psi = \Psi(\varphi)$ as was to be shown.

To complete the proof of Theorem 1.2 it suffices to note that $\Psi(\varphi)(Q_1)$ always represents an algebra in $B_3 \backslash B_2$ while $V \cup C$ with its induced order and with the discrete topology corresponds to a finite algebra in B_2 .

5. Proof of Theorem 1.4

The construction used and also the proof are analogous to those of the last section. Let (V, E) and the set T of triples be as before. Define a contravariant functor $\Psi : G \to T^p$ as follows.

For $(X, R) \in G$ set

$$Q = V \cup E \cup \{b\} \cup (T \backslash \{t_5\}) \cup P ,$$

where the union is disjoint and $\Phi(X, R) = (P, \tau)$. The topology σ on Q again is the union of τ and the discrete topology on the finite set

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$Q \backslash P$. Conditions (i), (ii), (iii), (iv), (vi) of Section 4 together with

(v') $v_i \leq a_i$ for $i < 5$

define the partial order on Q. Set $(Q, \sigma) = \Psi(X, R)$.

For $(X_i, R_i) \in G$ and for a morphism $\varphi : (X_2, R_2) \to (X_1, R_1)$ of G let $\Psi(\varphi) : \Psi(X_1, R_1) \to \Psi(X_2, R_2)$ be defined as the extension of $\Phi(\varphi)$ by the identity on $Q_1 \backslash P_1 = Q_2 \backslash P_2$, exactly as in Section 4.

Again it is routine to verify that Ψ is a well defined faithful contravariant functor. Note that $\mathrm{Min}(Q) = V \cup \{b\}$ once more, while $|\mathrm{Min}(a_i)| = 4$ for $i < 5$ and $|\mathrm{Min}(q)| = 3$ for all other elements of $Q \backslash \mathrm{Min}(Q)$; the space $\Psi(X, R)$ thus represents an algebra in $B_4 \backslash B_3$.

Let $\psi : \Psi(X_1, R_1) \to \Psi(X_2, R_2)$ be a continuous order preserving p-map.

The proof of Lemma 5.1 below reads as that of Lemma 4.2 with the following modifications. First, if $\psi(u) = m = \psi(v)$ for some distinct $u, v \in V$ and if $q \in Q$ satisfies $\mathrm{Min}(q) = \{u, v, w\}$, then it is always the case that $\psi(q) = \psi(w) = m \in \mathrm{Min}(Q_2)$, that is, ψ is constant on V. From $b \leq \{u, v\} \in E$ it directly follows that $\psi(b) = m$ as well. Thus $\psi \restriction \mathrm{Min}(Q_1)$ is constant and, consequently, ψ is a constant map whose value lies in $V \cup \{b\}$. If, on the other hand, $\psi(b) = \psi(v) = m$ for some $v \in V$ then, since $|\mathrm{Min}(q)| \geq 3$ for all non-minimal $q \in Q$, $\psi(u) = m$ whenever $\{u, v\} \in E$, so that the previous case applies.

LEMMA 5.1. *If ψ is not one-to-one on $V \cup \{b\}$ then ψ is a constant with a value in $V \cup \{b\}$.*

The next lemma follows along the lines of the proof of Lemma 4.3. Once it is shown that ψ is the identity on $Q_1 \backslash P_1 = Q_2 \backslash P_2$, and that $\psi(P_1) \subseteq P_2$, it suffices to note that a_i is the only element of P_j above v_i in order to deduce that $\psi(a_i) = a_i$ for all $i < 5$. Hence $\psi \restriction P_1 = \Phi(\varphi)$ as in Lemma 4.3.

LEMMA 5.2. *If ψ is one-to-one on $V \cup \{b\}$ then $\psi = \Psi(\varphi)$ for* *some compatible* $\varphi : (X_2, R_2) \to (X_1, R_1)$.

The space $\Psi(\varphi)(Q_1)$ represents an algebra in $B_4 \backslash B_3$ for every compatible φ , while any other map ψ is constant; thus its image represents one of the finitely many maps whose image represents a two-element Boolean subalgebra of $\Psi(X_1, R_1)$. This concludes the proof of Theorem 1.4.

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