J. Austral. Math. Soc. (*Series A*) **47** (1989), 430–437

SYMPLECTIC COMPLEX BUNDLES OVER REAL ALGEBRAIC FOUR-FOLDS

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(Received 28 April, 1988)

Communicated by J. H. Rubinstein

Abstract

Let X be a compact affine real algebraic variety of dimension 4. We compute the Witt group of symplectic bilinear forms over the ring of regular functions from X to $\mathbb{C}$. The Witt group is expressed in terms of some subgroups of the cohomology groups $H^{2k}(X, \mathbb{Z})$ for $k = 1, 2$.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 57 R 22, 19 G 12.

1. Introduction

Let X be an affine real algebraic variety, that is, X is biregularly isomorphic to an algebraic subset of $\mathbb{R}^n$ for some n (for definitions and notions of real algebraic geometry we refer to [3]). Denote by $\mathscr{R}(X, \mathbb{C})$ the ring of regular $\mathbb{C}$-valued functions on X (cf. [3, page 279]). Thus if X is an algebraic subset of $\mathbb{R}^n$ and $X_\mathbb{C}$ is its Zariski closure in $\mathbb{C}^n$, then $\mathscr{R}(X, \mathbb{C})$ is canonically isomorphic to the localization of the affine ring $A(X_\mathbb{C})$ of $X_\mathbb{C}$ with respect to the multiplicatively closed subset

$$S = \{f \in A(X_\mathbb{C}) | f(X) \subset \mathbb{C} \setminus \{0\}\}.$$

In this note we study symplectic (that is, skew-symmetric) nonsingular bilinear forms over $\mathscr{R}(X, \mathbb{C})$. More precisely, let $W^{-1}(\mathscr{R}(X, \mathbb{C}))$ denote the Witt group of symplectic bilinear forms over $\mathscr{R}(X, \mathbb{C})$ (cf. Section 2 or [1, 2, 11]).

https://doi.org/10.1017/S1446788700033152 Published online by Cambridge University Press

In [4, 6] (cf. also Section 2) we have defined the graded subring

$$H^{\text{even}}_{\mathbb{C}\text{-alg}}(X, \mathbb{Z}) = \bigoplus_{k \geq 0} H^{2k}_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$$

of the cohomology ring $H^{\text{even}}(X, \mathbb{Z})$. Assuming that X is compact, nonsingular, $\dim X = 4$, we compute the group $W^{-1}(\mathscr{R}(X, \mathbb{C})) \otimes \mathbb{Z}/2$ and, in some cases, also the group $W^{-1}(\mathscr{R}(X, \mathbb{C}))$ in terms of the groups $H^{2k}_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$, $k = 1, 2$. Combining this result with [4], we obtain that for "most" algebraic hypersurfaces X of the real projective space $\mathbb{R}P^5$ of sufficiently high degree, the group $W^{-1}(\mathscr{R}(X, \mathbb{C}))$ is zero (the precise meaning of "most" is explained in Section 2). We also give examples of "exceptional" algebraic hypersurfaces X in $\mathbb{R}P^5$ of arbitrarily high degree with $W^{-1}(\mathscr{R}(X, \mathbb{C})) \neq 0$.

Let us recall that the real projective space $\mathbb{R}P^n$ with its usual structure of an abstract real algebraic variety is in fact an affine variety [3, Theorem 3.4.4]. Hence every algebraic subvariety of $\mathbb{R}P^n$ is also affine.

2. Results

Let A be a commutative ring with an identity element. A *symplectic space* over A is a pair (P, s), where P is a finitely generated projective A-module and $s \colon P \times P \to A$ is a bilinear nonsingular symplectic form (recall that s is said to be nonsingular if the homomorphism $P \to P^* = \operatorname{Hom}(P, A)$, $x \to s(x, \cdot)$ is bijective). Every finitely generated projective A-module Q gives rise to a symplectic space $H(Q) = (Q \oplus Q^*, h)$, where $h((x, x^*), (y, y^*)) = x^*(y) - y^*(x)$ for x, y in Q and x^*, y^* in Q^*. An *isometry* of symplectic spaces is an isomorphism of the underlying modules preserving the forms. The *orthogonal sum* of two symplectic space (P_1, s_1) and (P_2, s_2), denoted by $(P_1, s_1) \perp (P_2, s_2)$, is the symplectic space $(P_1 \oplus P_2, s)$, where $s((x_1, x_2), (y_1, y_2)) = s_1(x_1, y_1) + s_2(x_2, y_2)$ for x_1, y_1 in P_1 and x_2, y_2 in P_2. Two symplectic spaces (P_1, s_1) and (P_2, s_2) are said to be *equivalent* if there exist finitely generated projective A-modules Q_1 and Q_2 such that the symplectic spaces $(P_1, s_1) \perp H(Q_1)$ and $(P_2, s_2) \perp H(Q_2)$ are isometric. The set $W^{-1}(A)$ of equivalence classes of symplectic spaces over A forms an abelian group with operation induced by orthogonal sum (we shall use additive notation). The equivalence class of (P, s) in $W^{-1}(A)$ will be denoted by $[P, s]$. The group $W^{-1}(A)$, called the Witt group of symplectic bilinear forms over A, is an interesting invariant of A (cf. [1, 2, 11]).

Now we need to recall some notions introduced in [4, 6].

Let V be a quasi-projective nonsingular n-dimensional complex algebraic variety. One defines the natural ring homomorphism

$$\mathrm{cl}\colon A^*(V) \to H^*(V, \mathbf{Z}),$$

where $A^*(V) = \bigoplus_{k\geq 0} A^k(V)$ is the Chow ring of V and $H^*(V, \mathbf{Z})$ is the Čech cohomology of V, as follows. Let $Y \subset V$ be a closed irreducible subvariety of dimension k and let $\{Y\}$ be the elements of $A^{n-k}(V)$ represented by Y. Denote by $[Y]$ the fundamental class of Y in the Borel-Moore homology group $H_{2k}^{BM}(Y, \mathbf{Z})$ (cf. [5] or [7, Chapter 19]). Then $\mathrm{cl}(\{Y\})$ is the element of $H^{2n-2k}(V, \mathbf{Z})$ which corresponds, via Poincaré duality, to the image of $[Y]$ in $H_{2k}^{BM}(V, \mathbf{Z})$ under the homomorphism $H_{2k}^{BM}(Y, \mathbf{Z}) \to H_{2k}^{BM}(V, \mathbf{Z})$ induced by the inclusion $Y \subset V$. Extending by linearity, cl defines a natural homomorphism $\mathrm{cl}\colon A^*(V) \to H^*(V, \mathbf{Z})$. We set

$$H_{\mathrm{alg}}^{2k}(V, \mathbf{Z}) = \mathrm{cl}(A^k(V)).$$

Now let X be an affine nonsingular real algebraic variety and suppose for a moment that X is embedded in $\mathbf{R}P^n$ as a locally closed subvariety. We shall consider $\mathbf{R}P^n$ as a subset of the complex projective space $\mathbf{C}P^n$. Let $X_{\mathbf{C}}$ be the Zariski (complex) closure of X in $\mathbf{C}P^n$ and let U be a Zariski neighborhood of X in the set of nonsingular points of $X_{\mathbf{C}}$. We set

$$H_{\mathbf{C}\text{-alg}}^{2k}(X, \mathbf{Z}) = H^*(i_U)(H_{\mathrm{alg}}^{2k}(U, \mathbf{Z})),$$
$$H_{\mathbf{C}\text{-alg}}^{\mathrm{even}}(X, \mathbf{Z}) = \bigoplus_{k\geq 0} H_{\mathbf{C}\text{-alg}}^{2k}(X, \mathbf{Z}),$$

where $H^*(i_U)$ is the homomorphism induced by the inclusion mapping $i_U\colon X \to U$. One easily sees that $H_{\mathbf{C}\text{-alg}}^{\mathrm{even}}(X, \mathbf{Z})$ does not depend on the choice of U (cf. [4] and [6]).

Given a continuous complex vector bundle ξ on X, let $c_k(\xi)$ denote its kth Chern class (cf. [10]). We shall consider $\mathscr{R}(X, \mathbf{C})$ as a subring of the ring $\mathscr{C}(X, \mathbf{C})$ of continuous $\mathbf{C}$-valued functions on X (note that $\mathscr{R}(X, \mathbf{C})$ is dense in $\mathscr{C}(X, \mathbf{C})$ in the C^0 topology). If P is a finitely generated projective $\mathscr{R}(X, \mathbf{C})$-module, then $\mathscr{C}(X, \mathbf{C}) \otimes P$ is a finitely generated projective $\mathscr{C}(X, \mathbf{C})$-module. We shall denote by ξ_P the continuous complex vector bundle on X associated with $\mathscr{C}(X, \mathbf{C}) \otimes P$ in the usual way (cf. [12]).

LEMMA 1. *Let X be an affine nonsingular real algebraic variety.*

(i) *If P is a finitely generated projective $\mathscr{R}(X, \mathbf{C})$-module, then $c_k(\xi_P)$ belongs to $H_{\mathbf{C}\text{-alg}}^{2k}(X, \mathbf{Z})$ for $k \geq 0$.*

(ii) *If v is in $H_{\mathbf{C}\text{-alg}}^{2}(X, \mathbf{Z})$, then there exists an invertible $\mathscr{R}(X, \mathbf{C})$-module L with $c_1(\xi_L) = v$.*

Proof. Both (i) and (ii) are quite straightforward consequences of the definition of $H^{2k}_{\mathbf{C}\text{-alg}}(X,\mathbf{Z})$; (i) is proved in [4, Theorem 5.3] (cf. also [6]), while (ii) follows from [4, Proposition 5.1, Remark 5.4] (cf. also the proof of Lemma 2 below).

Lemma 2. *Let X be a compact affine nonsingular real algebraic variety of dimension* 4.

(i) *For every element u in $H^4_{\mathbf{C}\text{-alg}}(X,\mathbf{Z})$, there exists a symplectic space (P,s) over $\mathscr{R}(X,\mathbf{C})$ with $c_2(\xi_P)=u$.*

(ii) *If (P,s) is a symplectic space over $\mathscr{R}(X,\mathbf{C})$ and $c_2(\xi_P)=0$, then (P,s) is isometric to $H(\mathscr{R}(X,\mathbf{C})^n)$, where $2n=\operatorname{rank}P$.*

Proof. First observe that every finitely generated projective $\mathscr{R}(X,\mathbf{C})$-module M with $\operatorname{rank}M\geq 3$ has a unimodular element. Indeed, since $\dim X=4$, the complex vector bundle ξ_M admits a nowhere zero continuous section (cf. [9, Chapter 8, Proposition 1.1]). This implies, from [13, Theorem 2.2(a)], that M has a unimodular element.

In the proof of (i) we may assume that X is a locally closed subvariety of $\mathbf{R}P^n$. Let U be a Zariski neighborhood of X in the set of nonsingular points of the Zariski (complex) closure of X in $\mathbf{C}P^n$. By definition of $H^4_{\mathbf{C}\text{-alg}}(X,\mathbf{Z})$, there exists an element v in $A^2(U)$ such that $H^*(i)(\mathrm{cl}(v))=u$, where

$$H^*(i)\colon H^4(U,\mathbf{Z})\to H^4(X,\mathbf{Z})$$

is the homomorphism induced by the inclusion mapping $i\colon X\to U$. Clearly, we may assume that U is an affine variety (cf. for example the proof of [4, Proposition 5.1]). Now it follows from [7, Example 15.3.6] that there exists an algebraic (complex) vector bundle η on U with $C_1(\eta)=0$ and $C_2(\eta)=v$, where $C_k(\cdot)$ stands for the kth Chern class with values in the Chow ring. Since $\mathrm{cl}\circ C_k=c_k$ (cf. [5, (4.13)], where this relation is proved for $k=1$; by a standard argument, $\mathrm{cl}\circ C_k=c_k$ must be true for all k), we obtain $c_1(\eta|X)=0$ and $c_2(\eta|X)=u$, where the restriction $\eta|X$ is considered as a continuous complex vector bundle on X. It easily follows (cf. [4, Proposition 5.1]) that $\eta|X$ is topologically isomorphic to a vector bundle of the form ξ_Q for some finitely generated projective $\mathscr{R}(X,\mathbf{C})$-module Q. By the remark at the beginning of the proof, $Q=P\oplus F$, where F is free and $\operatorname{rank}P=2$. In particular,

$$c_1(\xi_P)=c_1(\xi_Q)=0,\qquad c_2(\xi_P)=c_2(\xi_Q)=u.$$

Let $L=\det P$. Since $c_1(\xi_L)=c_1(\xi_P)=0$, the bundle ξ_L is topologically trivial (cf. [9, Chapter 16, Theorem 3.4]) and, by virtue of [13, Theorem 2.2(a)], L is free.

In order to finish the proof of (i) it suffices to show that there exists a symplectic nonsingular bilinear form on P. This however is obvious because $\det P$ is free and $\operatorname{rank} P = 2$.

Now we turn to the proof of (ii). First suppose that $\operatorname{rank} P > 2$. Then P has a unimodular element and, by [2, (4.11.2)], (P, s) is isometric to a symplectic space of the form $(Q, t) \perp H(\mathscr{R}(X, \mathbb{C}))$. Since, obviously, $c_2(\xi_Q) = 0$, using induction with respect to $\operatorname{rank} P$, one reduces the proof to the case $\operatorname{rank} P = 2$. In that case, $c_2(\xi_P) = 0$ implies that ξ_P has a nowhere zero continuous section (cf. [10, page 171, Problem 14-C]). Thus, by [13, Theorem 2.2(a)], P has a unimodular element and, finally, by [2, (4.11.2)], (P, s) is isometric to $H(\mathscr{R}(X, \mathbb{C}))$.

Let X be an affine nonsingular real algebraic variety. Observe that

$$G(X) = \{2u + v^2 | u \in H^4_{\mathbb{C}\text{-alg}}(X, \mathbb{Z}), v \in H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})\}$$

is a subgroup of $H^4_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$. Indeed, if u_i are in $H^4_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ and v_i are in $H^2_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})$ for $i = 1, 2$, then

$$(2u_1 + v_1^2) - (2u_2 + v_2^2) = 2(u_1 - u_2 + v_1 v_2 - v_2^2) + (v_1 - v_2)^2$$

is in $G(X)$.

For every finitely generated projective $\mathscr{R}(X, \mathbb{C})$-module Q, we have

$$\begin{aligned} c_2(\xi_{Q \oplus Q^*}) &= c_2(\xi_Q \oplus \xi_{Q^*}) \\ &= c_2(\xi_Q) + c_2(\xi_{Q^*}) + c_1(\xi_Q) c_1(\xi_{Q^*}) \\ &= c_2(\xi_Q) + c_2((\xi_Q)^*) + c_1(\xi_Q) c_1((\xi_Q)^*) \\ &= 2c_2(\xi_Q) - c_1(\xi_Q)^2 \end{aligned}$$

and hence, by Lemma 1(i), $c_2(\xi_{Q \oplus Q^*})$ is in $G(X)$. It easily follows (again from Lemma 1(i)) that

$$\begin{aligned} &\varphi_X : W^{-1}(\mathscr{R}(X, \mathbb{C})) \to H^4_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})/G(X) \\ &\varphi_X([P, s]) = c_2(\xi_P) + G(X) \end{aligned}$$

is a well-defined group homomorphism.

THEOREM 3. *Let X be a compact affine nonsingular real algebraic variety of dimension* 4. *Then the homomorphism*

$$\varphi_X : W^{-1}(\mathscr{R}(X, \mathbb{C})) \to H^4_{\mathbb{C}\text{-alg}}(X, \mathbb{Z})/G(X)$$

is surjective and

$$\ker \varphi_X = 2W^{-1}(\mathscr{R}(X, \mathbb{C})).$$

In particular,

$$W^{-1}(\mathscr{R}(X, \mathbb{C}))/2W^{-1}(\mathscr{R}(X, \mathbb{C})) \cong W^{-1}(\mathscr{R}(X, \mathbb{C})) \otimes \mathbb{Z}/2$$

is canonically isomorphic to $H^4_{\mathbf{C}\text{-alg}}(X,\mathbf{Z})/G(X)$. *Moreover, if* $2H^4_{\mathbf{C}\text{-alg}}(X,\mathbf{Z}) = 0$, *then* φ_X *is bijective.*

PROOF. It follows from Lemma 2(i) that φ_X is surjective.

Now we turn to the proof of $\ker\varphi_X = 2W^{-1}(\mathscr{R}(X,\mathbf{C}))$.

Let $[P,s]$ be in $W^{-1}(\mathscr{R}(X,\mathbf{C}))$. Then

$$\begin{aligned}\varphi_X(2[P,s]) &= c_2(\xi_{P\oplus P}) + G(X)\\ &= c_2(\xi_P \oplus \xi_P) + G(X)\\ &= 2c_2(\xi_P) + c_1(\xi_P)^2 + G(X) = 0.\end{aligned}$$

This shows that $2W^{-1}(\mathscr{R}(X,\mathbf{C}))$ is contained in $\ker\varphi_X$.

Suppose that $[P,s]$ is in $\ker\varphi_X$. Then $c_2(\xi_P) = 2u + v^2$, where u is in $H^4_{\mathbf{C}\text{-alg}}(X,\mathbf{Z})$ and v is in $H^2_{\mathbf{C}\text{-alg}}(X,\mathbf{Z})$. By Lemma 2(i), there exists a symplectic space (Q,t) over $\mathscr{R}(X,\mathbf{C})$ such that $c_2(\xi_Q) = -u$. Also, by Lemma 1(ii), one can find an invertible $\mathscr{R}(X,\mathbf{C})$-module L with $c_1(\xi_L) = v$. Let

$$(P',s') = (P,s) \perp (Q,t) \perp (Q,t) \perp H(L).$$

Then one obtains

$$\begin{aligned}c_2(\xi_{P'}) &= c_2(\xi_P) + 2c_2(\xi_Q) - c_1(\xi_L)^2\\ &= (2u+v^2) - 2u - v^2 = 0.\end{aligned}$$

By Lemma 2(ii), $[P',s'] = 0$ and hence $[P,s] = -2[Q,t]$. Thus $[P,s]$ is in $2W^{-1}(\mathscr{R}(X,\mathbf{C}))$, which shows that $\ker\varphi_X$ is contained in $2W^{-1}(\mathscr{R}(X,\mathbf{C}))$.

To finish the proof of the theorem, we note that if $2H^4_{\mathbf{C}\text{-alg}}(X,\mathbf{Z}) = 0$, then, by Lemma 2(ii), $2W^{-1}(\mathscr{R}(X,\mathbf{C})) = 0$ and hence φ_X is an isomorphism.

Theorem 3 immediately implies the following

COROLLARY 4. *Let* X *be a compact affine nonsingular real algebraic variety of dimension* 4. *Assume that each connected component of* X *is nonorientable as a* C^∞ *manifold. Then the groups* $W^{-1}(\mathscr{R}(X,\mathbf{C}))$ *and* $H^4_{\mathbf{C}\text{-alg}}(X,\mathbf{Z})/G(X)$ *are canonically isomorphic.*

PROOF. Let M be a connected component of X. Since M is nonorientable, $H^4(M,\mathbf{Z}) \cong \mathbf{Z}/2$ (cf. [8, (23.28), (22.28), (26.18)]). It follows that $2H^4(X,\mathbf{Z}) = 0$ and hence $2H^4_{\mathbf{C}\text{-alg}}(X,\mathbf{Z}) = 0$. Now it suffices to apply Theorem 3.

Our next result says that for a "generic" hypersurface X of $\mathbf{R}P^5$ of sufficiently high degree, one has $W^{-1}(\mathscr{R}(X,\mathbf{C})) = 0$.

More precisely, let n and k be positive integers. Denote by $P(n,k)$ the projective space associated with the vector space of all homogeneous polynomials in $\mathbf{R}[x_0,\ldots,x_n]$ of degree k. If an element H in $P(n,k)$ is represented

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by a polynomial G, then $V(H)$ will denote the subvariety of $\mathbf{R}P^n$ defined by G.

THEOREM 5. *There exists a nonnegative integer* k_0 *such that, for every integer* k *greater than* k_0, *one can find a subset* Σ_k *of* $P(5,k)$ *which is a countable union of proper Zariski closed algebraic subvarieties of* $P(5,k)$ *and has the property that for every* H *in* $P(5,k)\backslash\Sigma_k$, *the set* $V(H)$ *is empty or* $V(H)$ *is nonsingular,* $\dim V(H) = 4$, *and* $W^{-1}(\mathscr{R}(V(H),\mathbf{C})) = 0$.

PROOF. Let n be an integer, $n \geq 3$. It is proved in [4, Theorem 4.10] (cf. also [6]) that there exists a positive integer k_0 such that for every integer k greater than k_0, one can find a subset Σ_k of $P(n,k)$ which is a countable union of proper Zariski closed algebraic subvarieties of $P(n,k)$ and has the property that for every H in $P(n,k)\backslash\Sigma_k$, the set $V(H)$ is empty or $V(H)$ is nonsingular, $\dim V(H) = n-1$, and $H^{\text{even}}_{\mathbf{C}\text{-alg}}(V(H),\mathbf{Z})$ is equal to the image of the homomorphism

$$H^{\text{even}}(\mathbf{R}P^n,\mathbf{Z}) \to H^{\text{even}}(V(H),\mathbf{Z})$$

induced by the inclusion $V(H) \subset \mathbf{R}P^n$.

Recall that $H^{2k}(\mathbf{R}P^n,\mathbf{Z}) \cong \mathbf{Z}/2$ for $0 < 2k \leq n$. Moreover, if $n \geq 4$, then the nonzero element u of $H^4(\mathbf{R}P^n,\mathbf{Z})$ is of the form $u = v^2$, where v is the nonzero element of $H^2(\mathbf{R}P^n,\mathbf{Z})$. Hence $2H^{2k}_{\mathbf{C}\text{-alg}}(V(H),\mathbf{Z}) = 0$ for $0 < 2k \leq n$ and $H^4_{\mathbf{C}\text{-alg}}(V(H),\mathbf{Z}) = G(V(H))$ for H in $P(n,k)\backslash\Sigma_k$.

With $n = 5$, the conclusion follows from Theorem 3.

REMARK 6. Theorem 5 cannot be much improved. More precisely, for every positive integer k_0 there exists an integer k greater than k_0 and an element H_{2k} in $P(5,2k)$ such that $V(H_{2k})$ is a nonsingular algebraic hypersurface of $\mathbf{R}P^5$ and $W^{-1}(\mathscr{R}(V(H_{2k}),\mathbf{C})) \neq 0$. Let H_{2k} be the element of $P(5,2k)$ represented by the polynomial $x_0^{2k} - \sum_{i=1}^{5} x_i^{2k}$. Clearly, $V(H_{2k})$ is a nonsingular algebraic hypersurface of $\mathbf{R}P^5$ diffeomorphic to the 4-dimensional sphere S^4. Moreover, by [4, Proposition 4.8],

$$H^4_{\mathbf{C}\text{-alg}}(V(H_{2k}),\mathbf{Z}) = H^4(V(H_{2k}),\mathbf{Z}) \cong \mathbf{Z}.$$

Since $H^2(V(H_{2k}),\mathbf{Z}) \cong H^2(S^4,\mathbf{Z}) = 0$, one obtains

$$G(V(H_{2k})) = 2H^4_{\mathbf{C}\text{-alg}}(V(H_{2k}),\mathbf{Z}).$$

Hence, by Theorem 3, $W^{-1}(\mathscr{R}(V(H_{2k}),\mathbf{C})) \otimes \mathbf{Z}/2$ is isomorphic to $\mathbf{Z}/2$, and $W^{-1}(\mathscr{R}(V(H_{2k}),\mathbf{C})) \neq 0$.

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