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C^*-CROSSED PRODUCTS BY PARTIAL ACTIONS AND ACTIONS OF INVERSE SEMIGROUPS

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Abstract

The recently developed theory of partial actions of discrete groups on C^*-algebras is extended. A related concept of actions of inverse semigroups on C^*-algebras is defined, including covariant representations and crossed products. The main result is that every partial crossed product is a crossed product by a semigroup action.

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1. Introduction

The theory of C^*-crossed products by group actions is very well developed. In [6, 1, 2], Duncan and Paterson investigate C^*-algebras of inverse semigroups as a generalization of C^*-algebras of discrete groups. In this paper we show that the theory of crossed products also can be generalized to inverse semigroups. In Section 3 we define an inverse semigroup action as a homomorphism from the inverse semigroup into the inverse semigroup of partial automorphisms of a C^*-algebra. A partial automorphism is an isomorphism between two closed ideals of a C^*-algebra. In Section 4 we define C^*-crossed products by actions of inverse semigroups.

Our development is based upon another generalization of group actions, the notion of partial actions of discrete groups, defined by McClanahan [5] as a generalization of Exel's definition in [3] . In the definition of partial actions we also use the inverse semigroup of partial automorphisms instead of the automorphism group of the C^*-algebra. Of course we cannot talk about a homomorphism from a group into an inverse

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semigroup; a partial action is an appropriate generalization. In Section 1 we give a detailed discussion of partial actions.

It turns out that there is a close connection between partial crossed products and crossed products by inverse semigroup actions. In Section 5 we explore this connection, showing that every partial crossed product is isomorphic to a crossed product by an inverse semigroup action.

In [8] Renault investigates the connection between a locally compact groupoid and its ample inverse semigroup. Paterson [7] shows that there is a strong connection between the C^*-algebras of locally compact groupoids and inverse semigroups. This connection promises a connection between the groupoid crossed products of [9] and inverse semigroup crossed products.

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2. Partial actions

In this section we discuss the notion of *partial actions* defined by McClanahan in [5] which is a generalization of Exel's definition in [3] . The major new results are Theorems 2.6, 2.9 and Corollary 2.11.

DEFINITION 2.1. Let A be a C^*-algebra. A *partial automorphism* of A is a triple (α, I, J) where I and J are closed ideals in A and $\alpha : I \to J$ is a *-isomorphism. We are going to write α instead of (α, I, J) if the domain and range of α are not important.

If (α, I, J) and (β, K, L) are partial automorphisms of A then the product $\alpha\beta$ is defined as the composition of α and β with the largest possible domain, that is, $\alpha\beta : \beta^{-1}(I) \to A$, $\alpha\beta(a) = \alpha(\beta(a))$. It is clear that $\beta^{-1}(I)$ is a closed ideal of K. Since a closed ideal of a closed ideal of A is also a closed ideal of A, the product $(\alpha\beta, \beta^{-1}(I), \alpha\beta(\beta^{-1}(I)))$ is a partial automorphism too.

A semigroup S is an *inverse semigroup* if for every $s \in S$ there exists a unique element s^* of S so that $ss^*s = s$ and $s^*ss^* = s^*$. The map $s \mapsto s^*$ is an involution. An element $f \in S$ satisfying $f^2 = f$ is called an *idempotent* of S. The set of idempotents of an inverse semigroup is a semilattice. Our general reference on semigroups is [4] . It is easy to see that the set $\mathrm{PAut}(A)$ of partial automorphisms of A is a unital inverse semigroup with identity (ι, A, A), where ι is the identity map on A, and $(\alpha, I, J)^* = (\alpha^{-1}, J, I)$.

DEFINITION 2.2. Let A be a C^*-algebra and G be a discrete group with identity e. A *partial action* of G on A is a collection $\{(\alpha_s, D_{s^{-1}}, D_s) : s \in G\}$ of partial automorphisms (denoted by α or by (A, G, α)) such that

(i) $D_e = A$;
(ii) α_{st} extends $\alpha_s\alpha_t$, that is, $\alpha_{st}|\alpha_t^{-1}(D_{s^{-1}}) = \alpha_s\alpha_t$ for all $s, t \in G$.

This is equivalent to McClanahan's definition in [5] , and we believe Proposition 2.3 and Lemma 2.4 below give evidence that our definition is an improvement of McClanahan's.

PROPOSITION 2.3. *If α is a partial action of G on A then*

(i) *α_e is the identity map ι on A;*
(ii) *$\alpha_{s^{-1}} = \alpha_s^{-1}$ for all $s \in G$.*

PROOF. The statements follow from the following two identities:

$$\iota = \alpha_e\alpha_e^{-1} = \alpha_{ee}\alpha_e^{-1} = \alpha_e\alpha_e\alpha_e^{-1} = \alpha_e$$
$$\alpha_s\alpha_{s^{-1}} = \alpha_{ss^{-1}}|D_s = \alpha_e|D_s = \iota|D_s.$$

LEMMA 2.4. *If α is a partial action of G on A then $\alpha_t(D_{t^{-1}}D_s) = D_tD_{ts}$ for all $s, t \in G$.*

PROOF. By Proposition 2.3 (ii), $\alpha_t(D_{t^{-1}}D_s) = \alpha_{t^{-1}}^{-1}(D_{t^{-1}}D_s) = \alpha_{t^{-1}}^{-1}(D_s)$ which is the domain of $\alpha_{s^{-1}}\alpha_{t^{-1}}$ and hence is contained in the domain $D_{(s^{-1}t^{-1})^{-1}} = D_{ts}$ of $\alpha_{s^{-1}t^{-1}}$. Since the range of α_t is D_t, we have $\alpha_t(D_{t^{-1}}D_s) \subset D_tD_{ts}$ for all $s, t \in G$. Since α_t is an isomorphism, this implies

$$D_{t^{-1}}D_s \subset \alpha_t^{-1}(D_tD_{ts}) = \alpha_{t^{-1}}(D_tD_{ts}) \quad \text{for all} \quad s, t \in G.$$

Replacing t by t^{-1} and s by ts gives $D_tD_{ts} \subset \alpha_t(D_{t^{-1}}D_s)$ for all $s, t \in G$.

LEMMA 2.5. *If α is a partial action of G on A then $\alpha_t(D_{t^{-1}}D_{s_1}\cdots D_{s_n}) = D_tD_{ts_1}\cdots D_{ts_n}$ for all $t, s_1, \ldots, s_n \in G$.*

PROOF. The statement follows from the following calculation using Lemma 2.4:

$$\begin{aligned}\alpha_t(D_{t^{-1}}D_{s_1}\cdots D_{s_n}) &= \alpha_t(D_{t^{-1}}D_{s_1}\cdots D_{t^{-1}}D_{s_n})\\ &= \alpha_t(D_{t^{-1}}D_{s_1}) \cap \cdots \cap \alpha_t(D_{t^{-1}}D_{s_n})\\ &= D_tD_{ts_1} \cap \cdots \cap D_tD_{ts_n}\\ &= D_tD_{ts_1}\cdots D_{ts_n}.\end{aligned}$$

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THEOREM 2.6. *If α is a partial action of G on A then the partial automorphism $\alpha_{s_1} \cdots \alpha_{s_n}$ has domain $D_{s_n^{-1}} D_{s_n^{-1} s_{n-1}^{-1}} \cdots D_{s_n^{-1} \cdots s_1^{-1}}$ and range $D_{s_1} D_{s_1 s_2} \cdots D_{s_1 \cdots s_n}$ for all $s_1, \ldots, s_n \in G$.*

PROOF. The statement about the domain follows by induction from the following calculation using Lemma 2.5:

$$\begin{aligned} \operatorname{domain} \alpha_{s_1} \cdots \alpha_{s_n} &= \alpha_{s_n}^{-1}(\operatorname{domain} \alpha_{s_1} \cdots \alpha_{s_{n-1}}) \\ &= \alpha_{s_n^{-1}}(D_{s_n} D_{s_{n-1}^{-1}} \cdots D_{s_{n-1}^{-1} \cdots s_1^{-1}}) \\ &= D_{s_n^{-1}} D_{s_n^{-1} s_{n-1}^{-1}} \cdots D_{s_n^{-1} \cdots s_1^{-1}}. \end{aligned}$$

The other part now follows since the range of $\alpha_{s_1} \cdots \alpha_{s_n}$ is the domain of $\alpha_{s_n^{-1}} \cdots \alpha_{s_1^{-1}}$.

DEFINITION 2.7. Let α be a partial action of G on A. A *covariant representation* of α is a triple (π, u, H), where $\pi : A \to B(H)$ is a non-degenerate representation of A on the Hilbert space H and for each $g \in G$, u_g is a partial isometry on H with initial space $\pi(D_{g^{-1}})H$ and final space $\pi(D_g)H$, such that

(i) $u_g \pi(a) u_{g^{-1}} = \pi(\alpha_g(a))$ for all $a \in D_{g^{-1}}$;
(ii) $u_{st} h = u_s u_t h$ for all $h \in \pi(D_{t^{-1}} D_{t^{-1}s^{-1}})H$.

Notice that by the Cohen-Hewitt factorization theorem $\pi(D_g)H$ is a closed subspace of H and so the notations for the initial and final spaces make sense. It follows from the conditions of the definition that if (π, u, H) is a covariant representation then $u_e = 1_H$ (the identity map on H) and $u_{s^{-1}} = u_s^*$ for all $s \in G$. Thus condition (3) in McClanahan's definition [5] of a covariant representation is redundant.

Let π_u be the universal representation of a C^*-algebra A. If I is an ideal of A then the double dual I^{**} of I, identified with the strong operator closure of $\pi_u(I)$, is an ideal of the enveloping von Neumann algebra A^{**} of A, which is identified with the strong operator closure of $\pi_u(A)$. As such, I^{**} has the form pA^{**} for some central projection p in A^{**}.

DEFINITION 2.8. Let α be a partial action of G on A. For $s \in G$, p_s denotes the central projection of A^{**} which is the identity of D_s^{**}.

Let (π, u, H) be a covariant representation of (A, G, α). Since π is a non-degenerate representation of A, π can be extended to a normal morphism of A^{**} onto $\pi(A)''$. We will denote this extension also by π. Note that $\pi(D_{s_1} \cdots D_{s_n})H = \pi(p_{s_1} \cdots p_{s_n})H$ for all $s_1, \ldots, s_n \in G$, and $u_s u_s^* = \pi(p_s)$ for all $s \in G$.

THEOREM 2.9. *Let (π, u, H) be a covariant representation. Then $u_{s_1} \cdots u_{s_n}$ is a partial isometry with initial and final spaces*

$$\pi(D_{s_n^{-1}} D_{s_n^{-1} s_{n-1}^{-1}} \cdots D_{s_n^{-1} \cdots s_1^{-1}})H \quad \textit{and} \quad \pi(D_{s_1} D_{s_1 s_2} \cdots D_{s_1 \cdots s_n})H,$$

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for all $s_1, \dots, s_n \in G$.

PROOF. Firstly we show that $u_{g_1} \cdots u_{g_n} u^*_{g_n} \cdots u^*_{g_1} = \pi(p_{g_1} \cdots p_{g_1 \cdots g_n})$. This is clear for $n = 1$. Applying induction and Lemma 2.5 we get

$$\begin{aligned} u_{g_1} \cdots u_{g_n} u^*_{g_n} \cdots u^*_{g_1} &= u_{g_1} u^*_{g_1} u_{g_1} \circ \pi(p_{g_2} \cdots p_{g_2 \cdots g_n}) \circ u^*_{g_1} \\ &= u_{g_1} \circ \pi(p_{g_1^{-1}} p_{g_2} \cdots p_{g_2 \cdots g_n}) \circ u^*_{g_1} \\ &= \pi(\alpha_{g_1}(p_{g_1^{-1}} p_{g_2} \cdots p_{g_2 \cdots g_n})) \\ &= \pi(p_{g_1} \cdots p_{g_1 \cdots g_n}). \end{aligned}$$

The statement about the initial and final spaces now follows from the fact that $\pi(D_{s_1} \cdots D_{s_n})H = \pi(p_{s_1} \cdots p_{s_n})H$ for all $s_1, \dots, s_n \in G$.

The following corollary will be needed in Section 5.

COROLLARY 2.10. *If* (π, u, H) *is a covariant representation then*

$$u_{s_1 \cdots s_n} h = u_{s_1} \cdots u_{s_n} h \quad \text{for all} \quad h \in \pi(D_{s_n^{-1}} D_{s_n^{-1} s_{n-1}^{-1}} \cdots D_{s_n^{-1} \cdots s_1^{-1}})H$$

and

$$\pi(a) u_{s_1 \cdots s_n} = \pi(a) u_{s_1} \cdots u_{s_n} \quad \text{for all} \quad a \in D_{s_1} D_{s_1 s_2} \cdots D_{s_1 \cdots s_n}.$$

PROOF. The first statement follows by induction using Definition 2.7 (ii) and the fact that

$$D_{s_n^{-1}} D_{s_n^{-1} s_{n-1}^{-1} \cdots s_1^{-1}} \supset D_{s_n^{-1}} D_{s_n^{-1} s_{n-1}^{-1}} \cdots D_{s_n^{-1} \cdots s_1^{-1}}.$$

By the first statement we have $u_{s_n^{-1} \cdots s_1^{-1}} \pi(a^*) = u_{s_n^{-1}} \cdots u_{s_1^{-1}} \pi(a^*)$, which implies the second statement after taking conjugates.

COROLLARY 2.11. *If* (π, u, H) *is a covariant representation, then*

$$S = \{u_{s_1} \cdots u_{s_n} : s_1, \dots, s_n \in G\}$$

is a unital inverse semigroup of partial isometries of H.

The situation is more delicate than it may appear at first glance: the following example shows that a set of partial isometries with commuting initial and final projections does not necessarily generate an inverse semigroup of partial isometries.

EXAMPLE 2.12. Let $\alpha \in (0, \pi/2)$ and

$$U = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

be partial isometries on $\mathbb{C}^3$. A short calculation shows that U^2, V^2, UV, VU are all partial isometries and so all the initial and final projections of U and V commute, but $(UV)^2$ is not a partial isometry. This example is a modification of an idea of Marcelo Laca.

3. Action of an inverse semigroup

In this section we define an action of a unital inverse semigroup and a covariant representation of such an action. The assumption of the identity of the semigroup is for technical reasons. In the absence of an identity we can easily add one.

DEFINITION 3.1. Let A be a C^*-algebra and S be a unital inverse semigroup with identity e. An *action* of S on A is a semigroup homomorphism $s \mapsto (\beta_s, E_{s^*}, E_s)$: $S \to \mathrm{PAut}(A)$, with $E_e = A$.

Notice that $\beta_{s^*} = \beta_s^{-1}$ for all $s \in S$ so the notation E_{s^*} and E_s makes sense. It can be shown as in Proposition 2.3 that β_e is the identity map ι on A. Also if $f \in S$ is an idempotent then so is β_f, which means β_f is the identity map on $E_{f^*} = E_f$.

LEMMA 3.2. *If β is an action of the unital inverse semigroup S on A then $\beta_t(E_{t^*}E_s)$ $= E_{ts}$ for all $s, t \in S$.*

PROOF. The proof follows from the following calculation:

$$\beta_t(E_{t^*}E_s) = \mathrm{image}(\beta_t\beta_s) = \mathrm{image}(\beta_{ts}) = E_{ts}.$$

There is an important inverse semigroup action associated with a partial action, which we are going to use in Section 5.

PROPOSITION 3.3. *Let α be a partial action of a group G on the C^*-algebra A, and let (π, u, H) be a covariant representation of α. Let $S = \{(\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) :$ $g_1, \ldots, g_n \in G\}$. Then S is a unital inverse semigroup with coordinatewise multiplication. For $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) \in S$ let*

$$E_s = D_{g_1} D_{g_1 g_2} \cdots D_{g_1 \cdots g_n}, \qquad \beta_s = \alpha_{g_1} \cdots \alpha_{g_n} : E_{s^*} \to E_s.$$

Then β is an action of S on A.

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PROOF. By Corollary 2.11, S is a unital inverse semigroup with identity (α_e, u_e), where e is the identity of G. It is clear that β is a semigroup homomorphism with $E_{(\alpha_e, u_e)} = D_e = A$.

DEFINITION 3.4. Let β be an action of the unital inverse semigroup S on A. A *covariant representation* of β is a triple (π, v, H) where $\pi : A \to B(H)$ is a non-degenerate representation of A on the Hilbert space H and $v : S \mapsto B(H)$ is a multiplicative map such that

(i) $v_s \pi(a) v_{s^*} = \pi(\beta_s(a))$ for all $a \in E_{s^*}$;
(ii) v_s is a partial isometry with initial space $\pi(E_{s^*})H$ and final space $\pi(E_s)H$.

It is easy to show that $v_e = 1_H$ and $v_{s^*} = v_s^*$.

PROPOSITION 3.5. *Keeping the notation of Proposition 3.3, define* $v : S \to B(H)$ *by* $v_s = u_{g_1} \cdots u_{g_n}$, *where* $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n})$. *Then* (π, v, H) *is a covariant representation of* (A, S, β). *Conversely if* (ρ, z, K) *is a covariant representation of* (A, S, β) *then the function*

$$w : G \to B(K) \quad \text{defined by} \quad w_g = z(\alpha_g, u_g)$$

gives a covariant representation (ρ, w, K) *of* (A, G, α).

PROOF. It is clear that v is a semigroup homomorphism from S into an inverse semigroup of partial isometries on H. To check Definition 3.4 (i) let $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) \in S$ and $a \in E_{s^*} = D_{g_n^{-1}} D_{g_n^{-1} g_{n-1}^{-1}} \cdots D_{g_n^{-1} \cdots g_1^{-1}}$. Using Definition 2.7 (i) and Lemma 2.5 we have

$$\begin{aligned} v_s \pi(a) v_{s^*} &= \operatorname{Ad} u_{g_1} \cdots u_{g_n} \circ \pi(a) = \operatorname{Ad} u_{g_1} \cdots u_{g_{n-1}} \circ \pi \circ \alpha_{g_n}(a) \\ &= \cdots = \pi \circ \alpha_{g_1} \cdots \alpha_{g_n}(a) = \pi(\beta_s(a)). \end{aligned}$$

By Theorem 2.9, v_s has the desired initial and final spaces. For the second part of the theorem let $a \in D_{g^{-1}} = E_{s^*}$, where $s = (\alpha_g, u_g)$. Then

$$w_g \rho(a) w_{g^{-1}} = z_s \rho(a) z_{s^*} = \rho(\beta_s(a)) = \rho(\alpha_g(a)),$$

and so w satisfies Definition 2.7 (i). To check Definition 2.7 (ii), let $g_1, g_2 \in G$, $h \in \rho(D_{g_2^{-1}} D_{g_2^{-1} g_1^{-1}})K$ and let $s = (\alpha_{g_1 g_2}, u_{g_1 g_2})$, $s_1 = (\alpha_{g_1}, u_{g_1})$, $s_2 = (\alpha_{g_2}, u_{g_2}) \in S$. By Definition 2.2 (ii), $\alpha_{g_1 g_2}(\alpha_{g_1}\alpha_{g_2})^* = \alpha_{g_1}\alpha_{g_2}(\alpha_{g_1}\alpha_{g_2})^*$. By Definition 2.7 (ii) and Theorem 2.9, $u_{g_1 g_2}(u_{g_1} u_{g_2})^* = u_{g_1} u_{g_2}(u_{g_1} u_{g_2})^*$. Hence $s(s_1 s_2)^* = s_1 s_2 (s_1 s_2)^*$ and so $z_s z_{(s_1 s_2)^*} = z_{s_1 s_2} z_{(s_1 s_2)^*}$. Since the final space of $z_{(s_1 s_2)^*}$ is $\rho(D_{g_2^{-1}} D_{g_2^{-1} g_1^{-1}})K$, it follows that $z_s h = z_{s_1 s_2} h$. Thus

$$w_{g_1 g_2} h = z_s h = z_{s_1 s_2} h = z_{s_1} z_{s_2} h = w_{g_1} w_{g_2} h$$

as desired. It is clear that w_g has the required initial and final spaces.

Notice that if in the previous theorem we let $z = v$, then the construction gives $w = u$.

Not every unital inverse semigroup action arises from a partial action via the construction of Proposition 3.3, as we can see in the next example.

EXAMPLE 3.6. Let $S = \{e, f\}$ be the unital inverse semigroup that contains the identity e and an idempotent $f \neq e$. Let $A = \mathbb{C}$ and β_s be the identity map ι of A for all $s \in S$. Suppose there is a partial action (A, G, α) and a covariant representation (π, u, H) of α so that S can be identified with the inverse semigroup $\{(\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) : g_1, \ldots, g_n \in G\}$ and $\beta_s = \alpha_{g_1} \cdots \alpha_{g_n}$ for all $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) \in S$. Clearly e is identified with $(\iota, 1_H)$, where 1_H is the identity of $B(H)$. Suppose f is identified with $(\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n})$. By the definition of β_s, for all $g_1, \ldots, g_n \in G$ we have $\alpha_{g_1} \cdots \alpha_{g_n} = \iota$. Since f is an idempotent $u_{g_1} \cdots u_{g_n}$ is an idempotent too. Hence for all $h \in H$ we have

$$h = \pi(1)(h) = \pi(\beta_f(1))(h) = (u_{g_1} \cdots u_{g_n} \pi(1) u_{g_1} \cdots u_{g_n})(h) = u_{g_1} \cdots u_{g_n}(h).$$

This means that $u_{g_1} \cdots u_{g_n}$ must be the identity of $B(H)$. But this is a contradiction since e and f are different elements of S.

4. The crossed product

McClanahan [5] defines the *partial crossed product* $A \times_\alpha G$ of the C^*-algebra A and the group G by the partial action α as the enveloping C^*-algebra of $L = \{x \in l^1(G, A) : x(g) \in D_g\}$ with multiplication and involution

$$(x * y)(g) = \sum_{h \in G} \alpha_h[\alpha_{h^{-1}}(x(h)) y(h^{-1}g)], \qquad x^*(g) = \alpha_g(x(g^{-1})^*).$$

He shows that there is bijective correspondence $(\pi, u, H) \leftrightarrow (\pi \times u, H)$ between covariant representations of (A, G, α) and non-degenerate representations of $A \times_\alpha G$, where $\pi \times u$ is the extension of the representation of L defined by $x \mapsto \sum_{g \in G} \pi(x(g)) u_g$. We are going to follow his footsteps constructing the *crossed product* of a C^*-algebra and a unital inverse semigroup by an action β.

Let β be an action of the unital inverse semigroup S on the C^*-algebra A. Define multiplication and involution on the closed subspace

$$L = \{x \in l^1(S, A) : x(s) \in E_s\}$$

of $l^1(S, A)$ by

$$(x * y)(s) = \sum_{rt=s} \beta_r[\beta_{r^*}(x(r))y(t)], \qquad x^*(s) = \beta_s(x(s^*)^*).$$

Notice that by Lemma 3.2, $(x * y)(s) \in E_s$. A routine calculation shows that $\|x * y\| \leq \|x\|\|y\|$, $x * y \in L$, $\|x * y\| \leq \|x\|\|y\|$ and $\|x^*\| = \|x\|$, and so $x^* \in L$. We are going to denote by $a\delta_s$ the function in L taking the value a at s and zero at every other element of S. Notice that $a_s\delta_s * a_t\delta_t = \beta_s(\beta_{s^*}(a_s)a_t)\delta_{st}$ and $(a\delta_s)^* = \beta_{s^*}(a^*)\delta_{s^*}$.

PROPOSITION 4.1. *L is a Banach *-algebra.*

PROOF. Let $x, y, z \in L$ and $a \in \mathbb{C}$. Routine calculations show that $(x + y)^* = x^* + y^*$, $(ax)^* = \bar{a}x^*$, $x^{**} = x$ and $(x * y)^* = y^* * x^*$. We show that $(x * y) * z = x * (y * z)$. It suffices to show this for $x = a_r\delta_r$, $y = a_s\delta_s$ and $z = a_t\delta_t$. If $\{u_\lambda\}$ is an approximate identity for E_{s^*}, then we have

$$\begin{aligned}
(a_r\delta_r * a_s\delta_s) * a_t\delta_t &= \beta_r(\beta_{r^*}(a_r)a_s)\delta_{rs} * a_t\delta_t \\
&= \beta_{rs}(\beta_{s^*r^*}(\beta_r(\beta_{r^*}(a_r)a_s))a_t)\delta_{rst} \\
&= \lim_\lambda \beta_{rs}(\beta_{s^*}(\beta_{r^*}(a_r)a_s)u_\lambda a_t)\delta_{rst} \\
&= \lim_\lambda \beta_r(\beta_{r^*}(a_r)a_s\beta_s(u_\lambda a_t))\delta_{rst} \\
&= \lim_\lambda \beta_r(\beta_{r^*}(a_r)\beta_s(\beta_{s^*}(a_s)u_\lambda a_t))\delta_{rst} \\
&= \beta_r(\beta_{r^*}(a_r)\beta_s(\beta_{s^*}(a_s)a_t))\delta_{rst} \\
&= a_r\delta_r * \beta_s(\beta_{s^*}(a_s)a_t)\delta_{st} \\
&= a_r\delta_r * (a_s\delta_s * a_t\delta_t).
\end{aligned}$$

DEFINITION 4.2. If (π, v, H) is a covariant representation of (A, S, β) then define $\pi \times v : L \to B(H)$ by $(\pi \times v)(x) = \sum_{s\in S} \pi(x(s))v_s$.

PROPOSITION 4.3. *$(\pi \times v)$ is a non-degenerate representation of L.*

PROOF. Routine calculations show that $(\pi \times v)$ is a *-homomorphism. If $\{u_\lambda\}$ is a bounded approximate identity for A, then $\{u_\lambda\delta_e\}$ is a bounded approximate identity for L, since for $a \in E_s$ we have $\lim_\lambda u_\lambda\delta_e * a\delta_s = \lim_\lambda u_\lambda a\delta_s = a\delta_s$, and $\lim_\lambda a\delta_s * u_\lambda\delta_e = \lim_\lambda \beta_s(\beta_s^{-1}(a)u_\lambda)\delta_s = \beta_s(\beta_s^{-1}(a))\delta_s = a\delta_s$. Since π is a non-degenerate representation, $\big(\pi \times v\big)(u_\lambda\delta_e) = \pi(u_\lambda)$ converges strongly to $1_{B(H)}$ and so $(\pi \times v)$ is non-degenerate.

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DEFINITION 4.4. Let A be a C^*-algebra and β be an action of the unital inverse semigroup S on A. Define a seminorm $\|.\|_c$ on L by

$$\|x\|_c = \sup\{\|(\pi \times v)(x)\| : (\pi, v) \text{ is a covariant representation of } (A, S, \beta)\}.$$

Let $I = \{x \in L : \|x\|_c = 0\}$. The *crossed product* $A \times_\beta S$ is the C^*-algebra obtained by completing the quotient L/I with respect to $\|.\|_c$. We denote the quotient map of L onto L/I by Φ.

For any covariant representation (π, v) of (A, S, β), the associated representation $\pi \times v$ of L factors through a non-degenerate representation of $A \times_\beta S$, which we also denote by $\pi \times v$.

The following lemma shows that the ideal I may be non-trivial:

LEMMA 4.5. *If $s \leq t$ in S, that is, $s = ft$ for some idempotent $f \in S$, then $\Phi(a\delta_s) = \Phi(a\delta_t)$ for all $a \in E_s$. In particular $\Phi(a\delta_s) = \Phi(a\delta_e)$ if s is an idempotent.*

PROOF. It is clear that $a \in E_t$. If (π, v) is a covariant representation of (A, S, β) then

$$(\pi \times v)(a\delta_{ft} - a\delta_t) = \pi(a)v_f v_t - \pi(a)v_t = 0,$$

which shows $\Phi(a\delta_s - a\delta_t) = 0$. The second statement follows from the fact that $s = se$.

In spite of the above lemma, we identify $a\delta_s$ with its image in $A \times_\beta S$.

COROLLARY 4.6. *If β is an action of a semilattice S on a C^*-algebra A, then $A \times_\beta S$ is isomorphic to A.*

PROPOSITION 4.7. *Let (Π, H) be a non-degenerate representation of $A \times_\beta S$. Define a representation π of A on H and a map $v : S \to B(H)$ by*

$$\pi(a) = \Pi(a\delta_e), \qquad v_s = \text{s-lim}_\lambda\, \Pi(u_\lambda \delta_s)$$

where $\{u_\lambda\}$ is an approximate identity for E_s and s-lim$_\lambda$ *denotes the strong operator limit. Then (π, v, H) is a covariant representation of (A, S, β).*

PROOF. π is a non-degenerate representation, since $\{u_\lambda \delta_e\}$ is an approximate identity for $A \times_\beta S$ whenever $\{u_\lambda\}$ is an approximate identity for A.

We show that v_s is well-defined. If $h \in \pi(E_{s^*})H$ then $h = \Pi(a\delta_e)k$ for some $a \in E_{s^*}$ and $k \in H$ and a short calculation shows that

$$\lim_\lambda \Pi(u_\lambda \delta_s)h = \Pi(\beta_s(a)\delta_s)k,$$

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since $\beta_{s^*}(u_\lambda)$ is an approximate identity for E_{s^*}. Note that the limit is independent of the choice of $\{u_\lambda\}$, since the expression $h = \Pi(a\delta_e)k$ was. On the other hand, if $h \perp \pi(E_{s^*})H$ then

$$\begin{aligned}\lim_\lambda \Pi(u_\lambda\delta_s)h &= \lim_\lambda \Pi(\beta_s(\beta_{s^*}(\sqrt{u_\lambda})\beta_{s^*}(\sqrt{u_\lambda}))\delta_s)h \\ &= \lim_\lambda \Pi(\sqrt{u_\lambda}\delta_s * \beta_{s^*}(\sqrt{u_\lambda})\delta_e)h \\ &= \lim_\lambda \Pi(\sqrt{u_\lambda}\delta_s)\pi(\beta_{s^*}(\sqrt{u_\lambda}))h.\end{aligned}$$

But $\pi(E_{s^*})h = 0$, so $v_s h = 0$. Hence v_s is well-defined. Clearly v_s is a bounded linear transformation, and if f is an idempotent then v_f is the orthogonal projection onto $\pi(E_f)H$. The following calculation shows that $v_s^* = v_{s^*}$:

$$v_s^* = \text{s-lim}_\lambda \Pi(u_\lambda\delta_s)^* = \text{s-lim}_\lambda \Pi(\beta_{s^*}(u_\lambda)\delta_{s^*}) = v_{s^*}.$$

For $s, t \in S$ let $\{u_\lambda^s\}$ and $\{u_\mu^t\}$ be bounded approximate identities for E_s and E_t respectively. Then

$$\begin{aligned}v_s v_t &= \lim_{\lambda,\mu} \Pi(u_\lambda^s\delta_s * u_\mu^t\delta_t) \\ &= \lim \Pi(\beta_s(\beta_{s^*}(u_\lambda^s)u_\mu^t)\delta_{st}) \\ &= v_{st},\end{aligned}$$

since the net $\{\beta_s(\beta_{s^*}(u_\lambda^s)u_\mu^t)\}$ with the product direction is an approximate identity for $\beta_s(E_{s^*}E_t) = E_{st}$ (using boundedness of $\{u_\lambda^s\}$ and $\{u_\mu^t\}$). Thus v is multiplicative. We have $v_s^*v_s = v_{s^*}v_s = v_{s^*s}$, which is the projection onto $\pi(E_{s^*s})H = \pi(E_{s^*})H$. Hence v_s is a partial isometry with initial space $\pi(E_{s^*})H$, hence final space $\pi(E_s)H$ (since $v_s^* = v_{s^*}$).

The covariance condition is satisfied, since if $a \in E_{s^*}$ then

$$\begin{aligned}v_s\pi(a)v_{s^*} &= \text{s-lim}_{\lambda,\mu} \Pi(u_\mu\delta_s)\Pi(a\delta_e)\Pi(\beta_{s^*}(u_\lambda)\delta_{s^*}) \\ &= \text{s-lim}_{\lambda,\mu} \Pi(u_\mu\beta_s(a)u_\lambda\delta_{ss^*}) = \Pi(\beta_s(a)\delta_e).\end{aligned}$$

PROPOSITION 4.8. *The correspondence* $(\pi, v, H) \leftrightarrow (\pi \times v, H)$ *is a bijection between covariant representations of* (A, S, β) *and non-degenerate representations of* $A \times_\beta S$.

PROOF. This follows from computations similar to the end of the proof of [5, Proposition 2.8].

The following example shows that, unlike in the partial action case, the crossed product $A \times_\beta S$ is not the enveloping C^*-algebra of L in general.

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EXAMPLE 4.9. Let $S = \{e, f\}$ be the unital inverse semigroup that contains the identity e and an idempotent f. Let $A = \mathbb{C}$ and β_s be the identity map of A for all $s \in S$ as in Example 3.6. It is clear that $L = l^1(S)$. Wordingham [10] shows that the left regular representation of $l^1(S)$ on $l^2(S)$ is faithful and so the enveloping C^*-algebra cannot be the same as $A \times_\beta S$, which is isomorphic to $\mathbb{C}$ by Corollary 4.6.

The next two results describe two quite different crossed products associated with an inverse semigroup itself.

PROPOSITION 4.10. *Let S be a unital inverse semigroup, and let β_s be the identity map of $\mathbb{C}$ for all $s \in S$. Then β_s is an action of S on $\mathbb{C}$, and $\mathbb{C} \times_\beta S$ is isomorphic to the C^*-algebra of the maximal group homomorphic image of S.*

PROOF. For $s \in S$ let $[s] = \{t \in S : fs = ft$ for some idempotent $f\}$ be the associated element of the maximal group homomorphic image G. The formula $\Psi([s]) = \Phi(\delta_s)$ determines a homomorphism of $C^*(G)$ onto $\mathbb{C} \times_\beta S$. The formulas $\pi(a) = ae$ and $v_s = [s]$ define a covariant representation of $(\mathbb{C}, S, \beta)$ in $C^*(G)$ such that $\pi \times v$ is an inverse of Ψ.

PROPOSITION 4.11. *Let S be a unital inverse semigroup with idempotent semilattice E. Define a semigroup action β of S on $C^*(E)$ so that $\beta_s : E_{s^*} \to E_s$ is determined by $\beta_s(\delta_f) = \delta_{sfs^*}$, where E_s is the closed span of the set $\{\delta_f : f \leq ss^*\}$ in $C^*(E)$. Then $C^*(S)$ is isomorphic to $C^*(E) \times_\beta S$.*

PROOF. The formula $\Psi(s) = ss^*\delta_s$ for $s \in S$ determines a homomorphism of $C^*(S)$ to $C^*(E) \times_\beta S$. The canonical injections of E and S in $C^*(S)$ determine a covariant representation (π, v) of $(C^*(E), S, \beta)$ in $C^*(S)$ such that $\pi \times v$ is an inverse of Ψ.

5. Connection between the crossed products

In this section we show that every crossed product by a partial action is isomorphic to a crossed product by a suitably chosen inverse semigroup action.

LEMMA 5.1. *Let (A, G, α) and (A, S, β) be as in Proposition 3.3. Let (ρ, z, K) be a covariant representation of (A, S, β), and define a covariant representation (ρ, w, K) of (A, G, α) by $w_g = z(\alpha_g, u_g)$ as in Proposition 3.5. Then $(\rho \times z)(A \times_\beta S) = (\rho \times w)(A \times_\alpha G)$.*

PROOF. One can easily check that

$$\sum_{s \in S} \rho(E_s) z_s \supset \sum_{g \in G} \rho(D_g) w_g.$$

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If $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n})$ and $a \in E_s = D_{g_1} D_{g_1 g_2} \cdots D_{g_1 \cdots g_n}$, then by Corollary 2.10 we have

$$\rho(a) z_s = \rho(a) z(\alpha_{g_1}, u_{g_1}) \cdots z(\alpha_{g_1}, u_{g_n}) = \rho(a) w_{g_1} \cdots w_{g_n} = \rho(a) w_{g_1 \cdots g_n},$$

showing the other inclusion.

THEOREM 5.2. *Let α be a partial action of a group G on a C^*-algebra A, and let (π, u, H) be a covariant representation of (A, G, α) such that the representation $\pi \times u$ of $A \times_\alpha G$ is faithful. Define an inverse semigroup by*

$$S = \{(\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n}) : g_1, \dots, g_n \in G\}$$

and an action β of S by $\beta_s = \alpha_{g_1} \cdots \alpha_{g_n}$ for $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n})$, as in Proposition 3.3. Then the crossed products $A \times_\alpha G$ and $A \times_\beta S$ are isomorphic.

PROOF. Let $v_s = u_{g_1} \cdots u_{g_n}$ for $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n})$. We know from Proposition 3.5 that (π, v, H) is a covariant representation of (A, S, β). Note that $(\pi \times u)(A \times_\alpha G) = (\pi \times v)(A \times_\beta S)$ by Lemma 5.1. It suffices to show that $\pi \times v$ is a faithful representation of $A \times_\beta S$ because then $(\pi \times v)^{-1} \circ (\pi \times u) : A \times_\alpha G \to A \times_\beta S$ is an isomorphism. Consider another representation of $A \times_\beta S$, which by Proposition 4.8 must be in the form $\rho \times z$ for some covariant representation (ρ, z) of (A, S, β). By Proposition 3.5 the definition $w_g = z(\alpha_g, u_g)$ gives a covariant representation (ρ, w, K) of (A, G, α) and we have $(\rho \times w)(A \times_\alpha G) = (\rho \times z)(A \times_\beta S)$, again by Lemma 5.1. Since $\pi \times u$ is a faithful representation, there is a homomorphism Θ such that $\Theta \circ (\pi \times u) = \rho \times w$. We are going to show that $\Theta \circ (\pi \times v) = \rho \times z$, which is going to prove that $\pi \times v$ is faithful. It suffices to check this on a generator $a\delta_s$, where $s = (\alpha_{g_1} \cdots \alpha_{g_n}, u_{g_1} \cdots u_{g_n})$ and $a \in E_s = D_{g_1} D_{g_1 g_2} \cdots D_{g_1 \cdots g_n}$:

$$\begin{aligned}
\Theta((\pi \times v)(a\delta_s)) &= \Theta(\pi(a) v_s) = (\rho \times w) \circ (\pi \times u)^{-1} (\pi(a) u_{g_1} \cdots u_{g_n}) \\
&= (\rho \times w) \circ (\pi \times u)^{-1} (\pi(a) u_{g_1 \cdots g_n}) = (\rho \times w)(a \delta_{g_1 \cdots g_n}) \\
&= \rho(a) w_{g_1 \cdots g_n} = \rho(a) w_{g_1} \cdots w_{g_n} \\
&= \rho(a) z(\alpha_{g_1}, u_{g_1}) \cdots z(\alpha_{g_n}, u_{g_n}) = \rho(a) z_s \\
&= (\rho \times z)(a\delta_s),
\end{aligned}$$

where we have appealed to Corollary 2.10 twice more.

EXAMPLE 5.3. Let $A = \mathbb{C}^2$, $G = \mathbb{Z}$, $D_0 = A$, $D_{-1} = \{(a, 0) : a \in A\}$, $D_1 = \{(0, a) : a \in A\}$ and $D_n = \{(0, 0)\}$ for $n \in G \setminus \{-1, 0, 1\}$. Let α_0 be the identity map, α_1 be the forward shift $\alpha_1(a, 0) = (0, a)$, and define $\alpha_n = \alpha_1^n$ for $n \neq 0$. Then $A \times_\alpha G$ is isomorphic to the matrix algebra M_2 ([3], [5, Example 2.5]).

We construct a faithful representation $\pi \times u$ of the partial crossed product $A \times_\alpha G$. Let π be the representation of A on the Hilbert space $H = \mathbb{C}^2$ by multiplication operators; that is, $\pi(a_1, a_2)(h_1, h_2) = (a_1 h_1, a_2 h_2)$ for $(a_1, a_2) \in A$ and $(h_1, h_2) \in H$. Let u_1 be the forward shift, u_{-1} the backward shift on H, and let u_n be the constant zero map for all $n \in G \setminus \{-1, 0, 1\}$.

The unital inverse semigroup S generated by $\{(\alpha_n, u_n) : n \in G\}$ contains six elements $S = \{e, f, s, s^*, s^*s, ss^*\}$ where $e = 1_H$ is the identity of S, the zero element f of S is the constant zero map and $s = (\alpha_1, u_1)$. Let $E_e = A$, $E_f = \{(0, 0)\}$, $E_{s^*} = E_{s^*s} = D_{-1}$ and $E_s = E_{ss^*} = D_1$. Define the semigroup action β of S as in Proposition 3.3. Then β_s is the forward shift, β_{s^*} is the backward shift and β_t is the identity map for all other $t \in S$. As we have seen in Theorem 5.2 the crossed product $A \times_\beta S$ is isomorphic to the matrix algebra M_2.

Notice that in the last example the semigroup S is isomorphic to the inverse semigroup generated by $\{\alpha_n : n \in N\}$ as well as to the inverse semigroup generated by $\{u_n : n \in N\}$. Based upon experience with group actions, it might seem natural to expect that S is isomorphic to the inverse semigroup generated by the range of u whenever $\pi \times u$ is a faithful representation of $A \times_\alpha G$. Perhaps surprisingly this is not the case. All three semigroups can be non-isomorphic as the following example shows.

EXAMPLE 5.4. Let $A = C[0, 1]$, $G = \mathbb{Z}_2$, and $D_0 = A$, and let α_1 be the identity map on $D_1 = \{x \in A : x(1) = 0\}$. We construct a faithful representation $\pi \times u$ of the partial crossed product $A \times_\alpha G$. Let π be the representation of A on the Hilbert space $L^2[0, 1] \times L^2[0, 1]$ defined by $\pi(f) = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$, and let $u_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $u_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By [5, Propositions 3.4 and 4.2], $\pi \times u$ is faithful since Z_2 is amenable. The inverse semigroup generated by $\{u_0, u_1\}$ is isomorphic to Z_2. It is clear that $\{\alpha_0, \alpha_1\}$ is a semilattice, hence is definitely not isomorphic to the inverse semigroup $\{u_0, u_1\}$. The inverse semigroup generated by $\{(\alpha_0, u_0), (\alpha_1, u_1)\}$ contains three elements $\{(\alpha_0, u_0), (\alpha_1, u_1), (\alpha_1, u_0)\}$.

Although every partial crossed product is isomorphic to a crossed product by an action of a unital inverse semigroup, this semigroup action may not be unique up to isomorphism. For all we know different faithful representations $\Pi = \pi \times u$ of the crossed product $A \times_\alpha G$ could generate essentially different semigroup actions. If we want to talk about a canonical semigroup action associated with $A \times_\alpha G$ then we can choose Π to be the universal representation of $A \times_\alpha G$.

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