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CHOQUET INTEGRALS, HAUSDORFF CONTENT AND FRACTIONAL OPERATORS

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Abstract

We show that the fractional integral operator I_α, $0 < \alpha < n$, and the fractional maximal operator M_α, $0 \le \alpha < n$, are bounded on weak Choquet spaces with respect to Hausdorff content. We also investigate these operators on Choquet–Morrey spaces. The results for the fractional maximal operator M_α are extensions of the work of Tang ['Choquet integrals, weighted Hausdorff content and maximal operators', *Georgian Math. J.* **18**(3) (2011), 587–596] and earlier work of Adams and Orobitg and Verdera. The results for the fractional integral operator I_α are essentially new.

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1. Introduction

The purpose of this paper is to study the boundedness properties of the fractional integral operator I_α, $0 < \alpha < n$, and the fractional maximal operator M_α, $0 \le \alpha < n$, in the framework of Choquet integrals with respect to Hausdorff content.

Let $n \in \mathbb{N}$ and $0 < d \le n$. The symbol $\mathcal{Q}(\mathbb{R}^n)$ denotes the family of all cubes with sides parallel to the coordinate axes in $\mathbb{R}^n$. The d-dimensional Hausdorff content of $E \subset \mathbb{R}^n$ is defined by

$$H^d(E) = \inf\Big\{ \sum_{j=1}^{\infty} \ell(Q_j)^d : E \subset \bigcup_{j=1}^{\infty} Q_j, Q_j \in \mathcal{Q}(\mathbb{R}^n) \Big\},$$

where the infimum is taken over all coverings of the set E by countable families of cubes Q_j and $\ell(Q)$ stands for the side length of the cube Q. It is easily seen that $H^n(E)$

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is just the Lebesgue measure of E, which we will denote by $|E|$. For any cube Q, one has $H^d(Q) = \ell(Q)^d$.

For a nonnegative function f, the integral of f with the respect to H^d is taken in the Choquet sense:

$$\int_{\mathbb{R}^n} f\,dH^d = \int_0^\infty H^d(\{x \in \mathbb{R}^n : f(x) > t\})\,dt.$$

For $0 < p < \infty$, the Choquet space $L^p(H^d)$ and the weak Choquet space $\mathrm{w}L^p(H^d)$ consist of all functions with the properties

$$\|f\|_{L^p(H^d)} := \Big(\int_{\mathbb{R}^n} |f|^p\,dH^d\Big)^{1/p} < \infty$$

and

$$\|f\|_{\mathrm{w}L^p(H^d)} := \sup_{t>0} tH^d(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/p} < \infty,$$

respectively. For $0 < q \le p < \infty$, the Choquet–Morrey space $\mathcal{M}^p_q(H^d)$ consists of all functions with the property

$$\|f\|_{\mathcal{M}^p_q(H^d)} := \sup_{Q\in\mathcal{Q}} \ell(Q)^{d/p-d/q}\Big(\int_Q |f|^q\,dH^d\Big)^{1/q} < \infty.$$

The fractional maximal operator of order α, $0 \le \alpha < n$, is defined by

$$M_\alpha f(x) = \sup_{Q\in\mathcal{Q}(\mathbb{R}^n)} \chi_Q(x)\ell(Q)^{\alpha-n}\int_Q |f(y)|\,dy, \quad x \in \mathbb{R}^n,$$

where χ_E is the characteristic function of the set E. For $\alpha = 0$, the operator M_0 is the usual Hardy–Littlewood maximal operator which is denoted simply by M. The fractional integral operator of order α, $0 < \alpha < n$, is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \quad x \in \mathbb{R}^n.$$

Orobitg and Verdera [6] proved that, for $0 < d < n$ and $p > d/n$,

$$\|Mf\|_{\mathrm{w}L^{d/n}(H^d)} \lesssim \|f\|_{L^{d/n}(H^d)} \quad \text{and} \quad \|Mf\|_{L^p(H^d)} \lesssim \|f\|_{L^p(H^d)}.$$

The tools and ideas that we will use are essentially contained in this classical paper.

We note a disadvantage of the Hausdorff content H^d. It is not true that there exists a constant $C > 0$ such that if $Q_1, \ldots, Q_m$ are nonoverlapping dyadic cubes and $f \ge 0$, then

$$\sum_{j=1}^m \int_{Q_j} f\,dH^d \le C\int_{\bigcup_j Q_j} f\,dH^d.$$

This can be shown by subdividing the interval $[0, 1]$ into 2^m (m large enough) equal intervals and taking $f \equiv 1$.

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However, an advantage of the Hausdorff content H^d is that the spaces $L^p(H^d)$, $0 < p \le 1$, have a block decomposition. As a corollary, (see [8, Theorem 2.3]), if T is a subadditive operator and $0 < d, \delta < n$, $0 < p \le 1$ and $q \ge p$, then the following statements are equivalent:

(a) the inequality $\|Tf\|_{L^q(H^\delta)} \le C_1 \|f\|_{L^p(H^d)}$ holds;
(b) the testing inequality $\|T\chi_Q\|_{L^q(H^\delta)} \le C_2 \ell(Q)^{1/p}$ holds for any $Q \in \mathcal{Q}(\mathbb{R}^n)$.

Moreover, the least possible constants C_1 and C_2 are equivalent. Because of this advantage, by using an easy testing inequality, one can verify the following results from [2]. For $d/n < p < d/\alpha$,

$$\|M_\alpha f\|_{L^p(H^{d-\alpha p})} \lesssim \|f\|_{L^p(H^d)}. \tag{1.1}$$

For $0 < d \le n$,

$$\|M_\alpha f\|_{\mathrm{w}L^{d/n}(H^{d-\alpha d/n})} \lesssim \|f\|_{L^{d/n}(H^d)}. \tag{1.2}$$

The reason why the advantage influences the cases $p > 1$ is that, for the fractional maximal operator M_α, the pointwise estimate

$$M_\alpha f(x)^p \le M_{\alpha p}[f^p](x), \quad x \in \mathbb{R}^n,$$

gives a reduction to the case $p = 1$. This no longer works for the fractional integral operator I_α. The difficulty is overcome by using Hedberg's trick (see Lemma 2.7), which is due to the first author (Hatano).

Theorem 1.1. *Let $0 < d \le n$ and $0 \le \alpha < n$. Suppose that $d/n \le r < p < d/\alpha$ and*

$$\frac{d - \alpha r}{q} = \frac{d - \alpha p}{p}.$$

Then:

(i) $\|M_\alpha f\|_{\mathrm{w}L^q(H^{d-\alpha r})} \lesssim \|f\|_{\mathrm{w}L^p(H^d)}$;
(ii) $\|I_\alpha f\|_{\mathrm{w}L^q(H^{d-\alpha r})} \lesssim \|f\|_{\mathrm{w}L^p(H^d)}$ *for $0 < d < n$, $0 < \alpha < n$ and $d/n < r$.*
(iii) $\|I_\alpha f\|_{L^q(H^{d-\alpha r})} \lesssim \|f\|_{L^p(H^d)}$ *for $0 < d < n$, $0 < \alpha < n$ and $d/n < r$.*

Remark 1.2. Taking into account (1.1), one might expect that Theorem 1.1(i) holds for the case $q = r = p$. However, this is not so because, as a special case, it would give the false inequality $\|M_\alpha f\|_{\mathrm{w}L^1(H^{n-\alpha})} \lesssim \|f\|_{\mathrm{w}L^1(H^n)}$.

The key ingredient in the proof of Theorem 1.1(i) is a Kolmogorov-type inequality: for any measurable set $E \subset \mathbb{R}^n$,

$$H^d(E)^{1/p-1/q} \|f\chi_E\|_{L^q(H^d)} \le \left(\frac{q}{p-q}\right)^{1/p} \|f\chi_E\|_{\mathrm{w}L^p(H^d)}, \quad 0 < q < p < \infty.$$

This means that weak L^p integrability implies local L^q integrability provided that $q < p$. In the above inequality, taking the supremum over all measurable sets, we can

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obtain the reverse inequality and, as a consequence, the weak Choquet norm can be estimated as follows (see Proposition 2.5). For $0 < q < p < \infty$,

$$\|f\|_{\mathrm{w}L^p(H^d)} \le \sup_{0<H^d(E)<\infty} H^d(E)^{1/p-1/q}\|f\chi_E\|_{L^q(H^d)} \le \Big(\frac{q}{p-q}\Big)^{1/p}\|f\|_{\mathrm{w}L^p(H^d)}.$$

We notice that the parameter q does not affect the set $\mathrm{w}L^p(H^d)$. However, in the above supremum, if one restricts to the cube $Q \in \mathcal{Q}(\mathbb{R}^n)$ instead of the general set $E \subset \mathbb{R}^n$, then one gets Morrey spaces and one can no longer ignore the influence of the second parameter q. We establish the following results. The first part is from [10, Theorem 2].

Theorem 1.3. *Let $0 < d \le n$ and $0 \le \alpha < n$. Suppose that $d/n < r \le p < d/\alpha$ and*

$$\frac{d-\alpha r}{q} = \frac{d-\alpha p}{p}.$$

Then:

(i) $\|M_\alpha f\|_{\mathcal{M}^q_r(H^{d-\alpha r})} \lesssim \|f\|_{\mathcal{M}^p_r(H^d)}$;
(ii) $\|I_\alpha f\|_{\mathcal{M}^q_r(H^{d-\alpha r})} \lesssim \|f\|_{\mathcal{M}^p_r(H^d)}$ *for $0 < d < n$ and $0 < \alpha < n$.*

The paper is organised as follows. In Section 2, we give a proof of the Kolmogorov-type inequality and summarise some elementary properties. In Section 3, we prove the theorems. In Section 4, as an appendix, we gather some further results.

Throughout the paper, we use the following notation. If X and Y are normed spaces with $\|\cdot\|_X \lesssim \|\cdot\|_Y$, then we write $X \hookleftarrow Y$ or $Y \hookrightarrow X$ (sometimes called the embedding). If $X \hookleftarrow Y$ and $X \hookrightarrow Y$, then we write $X = Y$.

For quantities A and B, if $A \le CB$, then we write $A \lesssim B$ or $B \gtrsim A$, and if $A \lesssim B$ and $A \gtrsim B$, then we write $A \sim B$.

2. Preliminaries

We use the following fact about the Hausdorff content, due to Orobitg and Verdera [6, Lemma 3].

Lemma 2.1. *If $0 < d \le n$, $0 < p < \infty$ and $1 \le \theta \le n/d$, then*

$$\|f\|_{L^{\theta p}(H^{\theta d})} \le \theta^{1/\theta p}\|f\|_{L^p(H^d)}. \tag{2.1}$$

Proof. By the substitution $t = s^\theta$,

$$\|f\|^{\theta p}_{L^{\theta p}(H^{\theta d})} = \int_0^\infty H^{\theta d}(\{|f|^{\theta p} > t\})\,dt = \theta\int_0^\infty H^{\theta d}(\{|f|^p > s\})s^{\theta-1}\,ds.$$

Since

$$\Big(\sum_j a_j^q\Big)^{1/q} \le \Big(\sum_j a_j^p\Big)^{1/p} \quad \text{for } a_j \ge 0 \text{ and } 0 < p < q < \infty, \tag{2.2}$$

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we obtain

$$H^{\theta d}(\{|f|^p > s\}) \le (H^d(\{|f|^p > s\}))^\theta.$$

Thus,

$$\begin{aligned}\|f\|_{L^{\theta p}(H^{\theta d})}^{\theta p} &\le \theta \int_0^\infty (H^d(\{|f|^p > s\}))^\theta s^{\theta-1}\, ds \\ &= \theta \int_0^\infty (H^d(\{|f|^p > s\})s)^{\theta-1} H^d(\{|f|^p > s\})\, ds.\end{aligned}$$

Since

$$H^d(\{|f|^p > s\})s = \int_{\{|f|^p>s\}} s\, dH^d \le \int_{\{|f|^p>s\}} |f|^p\, dH^d \le \int_{\mathbb{R}^n} |f|^p\, dH^d,$$

it follows that

$$\|f\|_{L^{\theta p}(H^{\theta d})}^{\theta p} \le \theta \|f\|_{L^p(H^d)}^{\theta p}. \qquad \square$$

We summarise some elementary properties of the weak Choquet spaces.

PROPOSITION 2.2

(W1) (Chebyshev's inequality) *For* $p > 0$, $L^p(H^d) \hookrightarrow \mathrm{w}L^p(H^d)$.
(W2) *If* $0 < d \le \theta d \le n$ *and* $0 < p < \infty$, *then* $\mathrm{w}L^p(H^d) \hookrightarrow \mathrm{w}L^{\theta p}(H^{\theta d})$.
(W3) *If* $d = n$, *then the weak Choquet spaces* $\mathrm{w}L^p(H^d)$ *are the usual weak* L^p *spaces* $\mathrm{w}L^p(\mathbb{R}^n)$.

PROOF. The assertions (W1) and (W3) can be shown immediately from the definition. To prove assertion (W2), we appeal to (2.2) and observe that

$$\begin{aligned}\|f\|_{\mathrm{w}L^{\theta p}(H^{\theta d})} &= \sup_{t>0} t\, H^{\theta d}(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/(\theta p)} \\ &\le \sup_{t>0} t\, H^d(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/p} = \|f\|_{\mathrm{w}L^p(H^d)}. \qquad \square\end{aligned}$$

We next prove the Kolmogorov-type inequality. For $0 < p, r \le \infty$, we define the Lorentz quasinorm

$$\|f\|_{L^{p,r}(H^d)} = \bigg(\int_0^\infty (t^p H^d(\{|f| > t\}))^{r/p} \frac{dt}{t}\bigg)^{1/r}.$$

The Lorentz space $L^{p,r}(H^d)$ is the set of all functions for which this quasinorm is finite.

PROPOSITION 2.3. *Let* $0 < d \le n$, $0 < q < p < \infty$ *and* E *be any measurable set.*

(a) *If* $q \le r \le p$, *then*

$$H^d(E)^{1/p-1/q}\bigg(\int_E |f|^q\, dH^d\bigg)^{1/q} \lesssim q^{1/r} \|f\chi_E\|_{L^{p,q}(H^d)}.$$

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(b) *If $p < r$, then*

$$H^d(E)^{1/p-1/q}\Big(\int_E |f|^q\,dH^d\Big)^{1/q} \lesssim \Big(q\Big(\frac{r-p}{r(p-q)}\Big)^{(r-p)/r}\Big)^{1/p}\|f\chi_E\|_{L^{p,r}(H^d)}. \tag{2.3}$$

PROOF. (a) Let E be an arbitrary measurable set satisfying $0 < H^d(E) < \infty$. For any $A > 0$,

$$\begin{aligned}\int_E |f|^q\,dH^d &= \int_0^\infty H^d(E\cap\{|f|^q > t\})\,dt = q\int_0^\infty t^{q-1}H^d(E\cap\{|f|>t\})\,dt\\ &\le q\int_0^A t^{q-1}H^d(E)\,dt + q\int_A^\infty t^{q-r}t^rH^d(E\cap\{|f|>t\})^{r/p}H^d(E\cap\{|f|>t\})^{1-r/p}\,\frac{dt}{t}\\ &\le H^d(E)^{1-r/p}(A^qH^d(E)^{rp} + qA^{q-r}\|f\chi_E\|^r_{L^{p,r}(H^d)}).\end{aligned}$$

If we take $A^r = q\|f\chi_E\|^r_{L^{p,r}(H^d)}H^d(E)^{-r/p}$, then the two terms on the right-hand side of the above inequality balance. Thus,

$$\begin{aligned}\int_E |f|^q\,dH^d &\le 2H^d(E)^{1-r/p}(q\|f\chi_E\|^r_{L^{p,r}(H^d)})^{q/r}H^d(E)^{r/p-q/p}\\ &\le 2q^{q/r}H^d(E)^{1-q/p}\|f\chi_E\|^q_{L^{p,r}(H^d)},\end{aligned}$$

and this implies

$$H^d(E)^{1/p-1/q}\Big(\int_E |f|^q\,dH^d\Big)^{1/q} \lesssim q^{1/r}\|f\chi_E\|_{L^{p,q}(H^d)}.$$

(b) For any $A > 0$,

$$\begin{aligned}&\int_A^\infty t^{q-1}H^d(E\cap\{|f|>t\})\,dt\\ &\qquad\le \int_A^\infty t^qH^d(\{|f|>t\})\,\frac{dt}{t} = \int_A^\infty t^{q-p}\cdot t^pH^d(\{|f|>t\})\,\frac{dt}{t}\\ &\qquad\le \Big(\int_A^\infty t^{(q-p)r/(r-p)}\,\frac{dt}{t}\Big)^{(r-p)/r}\Big(\int_A^\infty (t^pH^d(\{|f|>t\}))^{r/p}\,\frac{dt}{t}\Big)^{p/r}\\ &\qquad\le \Big(\frac{r-p}{r(p-q)}\Big)^{(r-p)/r}A^{q-p}\|f\chi_E\|^p_{L^{p,r}(H^d)}.\end{aligned}$$

This implies

$$\int_E |f|^q\,dH^d \lesssim A^qH^d(E) + q\Big(\frac{r-p}{r(p-q)}\Big)^{(r-p)/r}A^{q-p}\|f\chi_E\|^p_{L^{p,r}(H^d)}$$

and, proceeding as before,

$$\int_E |f|^q\,dH^d \lesssim \Big(q\Big(\frac{r-p}{r(p-q)}\Big)^{(r-p)/r}\Big)^{q/p}H^d(E)^{1-q/p}\|f\chi_E\|^q_{L^{p,r}(H^d)},$$

which yields (2.3). □

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REMARK 2.4. Taking $E = Q \in \mathcal{Q}(\mathbb{R}^n)$ and $r = \infty$ in (2.3) yields

$$\ell(Q)^{d/p-d/q}\|f\chi_Q\|_{L^q(H^d)} \lesssim \|f\chi_Q\|_{L^{p,\infty}(H^d)}.$$

For an arbitrary r with $p \le r < \infty$, multiplying by $\ell(Q)^{d/r-d/p}$ on both sides gives

$$\ell(Q)^{d/r-d/q}\|f\chi_Q\|_{L^q(H^d)} \lesssim \ell(Q)^{dr-d/p}\|f\chi_Q\|_{L^{p,\infty}(H^d)}.$$

This means that, for $0 < q < p \le r < \infty$,

$$\|f\|_{\mathcal{M}^r_q(H^d)} \lesssim \|f\|_{\mathcal{M}^r_{p,\infty}(H^d)} := \sup_{Q\in\mathcal{Q}(\mathbb{R}^n)} \ell(Q)^{d/r-d/p}\|f\chi_Q\|_{L^{p,\infty}(H^d)}. \tag{2.4}$$

The estimate (2.4) can be found in [4] and the book [9]. It is interesting because one always has

$$\|f\|_{\mathcal{M}^r_q(H^d)} \le \|f\|_{\mathcal{M}^r_p(H^d)} \quad \text{and} \quad \|f\|_{\mathcal{M}^r_{p,\infty}(H^d)} \le \|f\|_{\mathcal{M}^r_p(H^d)}.$$

In the case $r = \infty$ in Proposition 2.3, we can show the reverse inequality.

PROPOSITION 2.5. *Let* $0 < d \le n$ *and* $0 < q < p < \infty$. *Then*

$$\|f\|_{\mathrm{w}L^p(H^d)} \le \sup_{0<H^d(E)<\infty} H^d(E)^{1/p-1/q}\Big(\int_E |f|^q\, dH^d\Big)^{1/q} \le \Big(\frac{q}{p-q}\Big)^{1/p}\|f\|_{\mathrm{w}L^p(H^d)}.$$

PROOF. The second inequality follows by taking $r = \infty$ in (2.3). We prove the reverse inequality. For all $t > 0$,

$$\begin{aligned} &tH^d(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/p} \\ &\quad= H^d(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/p-1/q} \cdot t\,H^d(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/q} \\ &\quad\le H^d(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/p-1/q}\Big(\int_{\{|f|>t\}} |f|^q\, dH^d\Big)^{1/q}. \end{aligned}$$

Hence,

$$tH^d(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/p} \le \sup_{0<H^d(E)<\infty} H^d(E)^{1/p-1/q}\Big(\int_E |f|^q\, dH^d\Big)^{1/q},$$

as desired. □

We summarise some elementary properties of the Choquet–Morrey spaces.

PROPOSITION 2.6

(M1) *If* $0 < p = q < \infty$, *then* $\mathcal{M}^p_p(H^d) = L^p(H^d)$.
(M2) *If* $0 < q < p < \infty$, *then* $\mathrm{w}L^p(H^d) \hookrightarrow \mathcal{M}^p_q(H^d)$.
(M3) *If* $d = n$, *then the Choquet–Morrey spaces* $\mathcal{M}^p_q(H^d)$ *are the usual Morrey spaces* $\mathcal{M}^p_q(\mathbb{R}^n)$.
(M4) *If* $0 < q \le r \le p < \infty$, *then* $\mathcal{M}^p_r(H^d) \hookrightarrow \mathcal{M}^p_q(H^d)$.
(M5) *If* $0 < d \le \theta d \le n$ *and* $0 < q \le p < \infty$, *then* $\mathcal{M}^p_q(H^d) \hookrightarrow \mathcal{M}^{\theta p}_{\theta q}(H^{\theta d})$.

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PROOF. The assertions (M1)–(M4) are obvious. For assertion (M5), we use (2.1) and set $\tilde{d} = \theta d$, $\tilde{p} = \theta p$ and $\tilde{q} = \theta q$, to give

$$\ell(Q)^{\tilde{d}/\tilde{p}-\tilde{d}/\tilde{q}}\|f\chi_Q\|_{L^{\tilde{q}}(H^{\tilde{d}})} \le \theta\ell(Q)^{d/p-d/q}\|f\chi_Q\|_{L^q(H^d)}. \qquad \square$$

It is well known, from Hedberg [5], that for $1 \le p < \infty$, the pointwise estimate

$$|I_\alpha f(x)| \lesssim \|f\|_{L^p(\mathbb{R}^n)}^{\alpha p/n} Mf(x)^{1-\alpha p/n}$$

holds. To prove the boundedness of the fractional integral operators on weak Choquet and Choquet–Morrey spaces with Hausdorff content, we extend Hedberg's inequality.

LEMMA 2.7. *Let $0 \le \beta < \alpha < n$ and $d/n \le p < q < \infty$ with*

$$\frac{d-\beta p}{q} = \frac{d-\alpha p}{p}.$$

Then

$$|I_\alpha f(x)| \lesssim \left(\frac{d}{p}-\alpha\right)^{(p/q)-1}(\alpha-\beta)^{-p/q}\|f\|_{\mathcal{M}^p_{d/n}(H^d)}^{1-p/q} M_\beta f(x)^{p/q} \quad \text{for } x \in \mathbb{R}^n.$$

PROOF. Fix $r > 0$. We decompose

$$\begin{aligned}
|I_\alpha f(x)| &\le \sum_{j=-\infty}^{\infty} \int_{2^j r \le |x-y| < 2^{j+1} r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy \\
&\lesssim \sum_{j=-\infty}^{\infty} \frac{1}{(2^j r)^{n-\alpha}} \int_{|x-y|<2^{j+1}r} |f(y)|\,dy \\
&= \sum_{j=-\infty}^{-1} \frac{1}{(2^j r)^{n-\alpha}} \int_{|x-y|<2^{j+1}r} |f(y)|\,dy + \sum_{j=0}^{\infty} \frac{1}{(2^j r)^{n-\alpha}} \int_{|x-y|<2^{j+1}r} |f(y)|\,dy \\
&=: J_1 + J_2.
\end{aligned}$$

We estimate

$$J_1 \lesssim \sum_{j=-\infty}^{-1} (2^j r)^{\alpha-\beta} M_\beta f(x) \lesssim \frac{r^{\alpha-\beta}}{\alpha-\beta} M_\beta f(x),$$

where we have used $2^{\alpha-\beta} - 1 \gtrsim \alpha - \beta$. By (2.1),

$$J_2 \lesssim \sum_{j=1}^{\infty} (2^j r)^{\alpha-d/p} \|f\|_{\mathcal{M}^p_{d/n}(H^d)} \lesssim \frac{r^{\alpha-d/p}}{(d/p)-\alpha} \|f\|_{\mathcal{M}^p_{d/n}(H^d)},$$

where we have used $2^{(d/p)-\alpha} - 1 \gtrsim (d/p) - \alpha$. Then, taking the optimal quantity $r > 0$ in the previous inequalities and noticing that

$$\frac{p}{q} = \frac{d-\alpha p}{d-\beta p},$$

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we obtain

$$|I_\alpha f(x)| \lesssim \Big(\frac{d}{p} - \alpha\Big)^{p/q-1} (\alpha - \beta)^{-p/q} \|f\|^{1-p/q}_{\mathcal{M}^p_{d/n}(H^d)} M_\beta f(x)^{p/q},$$

as desired. □

3. Proof of the theorems

PROOF OF THEOREM 1.1(i). Set $\delta = d - \alpha r$. For $t > 0$, we see that

$$\begin{aligned}\Omega(M_\alpha f; t) &:= \{x \in \mathbb{R}^n : M_\alpha f(x) > t\} \\ &= \{x \in \mathbb{R}^n : M_\alpha[f\chi_{\Omega(M_\alpha f;t)}](x) > t\}.\end{aligned}$$

Thus,

$$\begin{aligned}tH^\delta(\Omega(M_\alpha f; t))^{1/q} &= H^\delta(\Omega(M_\alpha f; t))^{1/q-1/r} \cdot t\, H^\delta(\Omega(M_\alpha f; t))^{1/r} \\ &\lesssim H^\delta(\Omega(M_\alpha f; t))^{1/q-1/r} \cdot \|f\chi_{\Omega(M_\alpha f;t)}\|_{L^r(H^d)},\end{aligned}$$

where we have used (1.1), or (1.2) when $r = d/n$. By the Kolmogorov-type inequality (2.3), we need only verify that, for a compact set $E \subset \mathbb{R}^n$, $H^\delta(E)^{1/r-1/q} \geq H^d(E)^{1/r-1/p}$. By (2.2) and since $\delta < d$,

$$H^\delta(E)^{1/r-1/q} \geq H^d(E)^{(1/r-1/q)\delta/d},$$

and

$$\frac{\delta}{d}\Big(\frac{1}{r} - \frac{1}{q}\Big) = \frac{1}{r} - \frac{\alpha}{d} - \frac{d - \alpha p}{dp} = \frac{1}{r} - \frac{1}{p}.$$

This completes the proof. □

PROOF OF THEOREM 1.1(ii). Thanks to the continuity, one can choose θ and β so that $1 < \theta < n/d$, $r\alpha/p < \beta < \alpha$ and

$$\theta(d - \beta p) = d - \alpha r = d - \frac{r}{p}\alpha p.$$

Setting $\delta = \theta d$ and $u = \theta p$ gives $\delta - \beta u = d - \alpha r$. This equation and

$$\frac{d - \alpha r}{q} = \frac{d - \alpha p}{p}$$

immediately imply

$$\frac{\delta - \beta u}{q} = \frac{d - \alpha r}{q} = \frac{d - \alpha p}{p} = \frac{d}{p} - \alpha = \frac{\delta}{u} - \alpha = \frac{\delta - \alpha u}{u}.$$

Since $\beta < \alpha$, we see that $p < u < q$. The equation

$$\frac{\delta - \beta u}{q} = \frac{\delta - \alpha u}{u}$$

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and Lemma 2.7 yield

$$|I_\alpha f(x)| \lesssim \|f\|_{\mathcal{M}^u_{\delta/n}(H^\delta)}^{1-u/q} M_\beta f(x)^{u/q} \quad \text{for } x \in \mathbb{R}^n.$$

The equations

$$\frac{d-\alpha r}{u} = \frac{\delta - \beta u}{u} = \frac{d-\alpha p}{p}$$

and Theorem 1.1(i) yield

$$\|(M_\beta f)^{u/q}\|_{\mathrm{w}L^q(H^{d-\alpha r})} = \|M_\beta f\|_{\mathrm{w}L^u(H^{d-\alpha r})}^{u/q} \lesssim \|f\|_{\mathrm{w}L^p(H^d)}^{u/q}.$$

Since we always have $\|f\|_{\mathcal{M}^u_{\delta/n}(H^\delta)} \lesssim \|f\|_{\mathcal{M}^p_{d/n}(H^d)} \lesssim \|f\|_{\mathrm{w}L^p(H^d)}$, this completes the proof. □

PROOF OF THEOREM 1.1(iii). Setting $\beta = r\alpha/p$, we have

$$\frac{d-\beta p}{q} = \frac{d-\alpha r}{q} = \frac{d-\alpha p}{p}.$$

From Lemma 2.7,

$$\|I_\alpha f\|_{L^q(H^{d-\beta p})} \lesssim \|f\|_{\mathcal{M}^p_{d/n}(H^d)}^{1-p/q} \|(M_\beta f)^{p/q}\|_{L^q(H^{d-\beta p})}.$$

By (1.1), $\|M_\beta f\|_{L^p(H^{d-\beta p})} \lesssim \|f\|_{L^p(H^d)}$ and, by the inclusion property, $\|f\|_{\mathcal{M}^p_{d/n}(H^d)} \lesssim \|f\|_{L^p(H^d)}$. These estimates complete the proof. □

PROOF OF THEOREM 1.3(ii). This theorem can be proved in the same manner as Theorem 1.1(ii). □

4. Appendix

In this section, we give some further consequences of our theorems and the embedding lemma (Lemma 2.1).

PROPOSITION 4.1. *Let* $0 < d, \delta < n$ *and* $0 \le \alpha < n$. *If* $q > p$, $d/n < p < d/\alpha$ *and*

$$\frac{\delta}{q} = \frac{d}{p} - \alpha,$$

then

$$\|M_\alpha f\|_{\mathrm{w}L^q(H^\delta)} \lesssim \|f\|_{\mathrm{w}L^p(H^d)}.$$

PROOF. Because $\delta > d - \alpha p$, we can choose u, r and $\theta > 1$ so that $d/n < r < p$ and

$$q = \theta u, \quad \delta = \theta(d - \alpha r).$$

Since

$$\frac{d-\alpha r}{u} = \frac{d-\alpha p}{p},$$

Theorem 1.1(i) yields $\|M_\alpha f\|_{\mathrm{w}L^u(H^{d-\alpha r})} \lesssim \|f\|_{\mathrm{w}L^p(H^d)}$, which gives, by the embedding in Proposition 2.2, $\|M_\alpha f\|_{\mathrm{w}L^q(H^\delta)} \lesssim \|f\|_{\mathrm{w}L^p(H^d)}$. This completes the proof. □

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PROPOSITION 4.2. *Let $0 < d, \delta \le n$ and $0 \le \alpha < n$. Suppose that $d/n < r \le p < d/\alpha$, $q \ge p$, $\delta \ge d - \alpha r$, $s \ge r$ and*

$$\frac{\delta}{q} = \frac{d}{p} - \alpha, \qquad \frac{\delta}{s} = \frac{d}{r} - \alpha.$$

Then

$$\|M_\alpha f\|_{\mathcal{M}^q_s(H^\delta)} \lesssim \|f\|_{\mathcal{M}^p_r(H^d)}.$$

PROOF. We can choose u and $\theta \ge 1$ so that $q = \theta u$ and $\delta = \theta(d - \alpha r)$. Since

$$\frac{d - \alpha r}{u} = \frac{d - \alpha p}{p},$$

it follows from Theorem 1.3(i) that $\|M_\alpha f\|_{\mathcal{M}^u_r(H^{d-\alpha r})} \lesssim \|f\|_{\mathcal{M}^p_r(H^d)}$. Since

$$\frac{d}{r} - \alpha = \frac{d - \alpha r}{r} = \frac{\delta}{\theta r},$$

we must have $s = \theta r$. Thus, by the inclusion property (Proposition 2.6),

$$\|M_\alpha f\|_{\mathcal{M}^q_s(H^\delta)} \lesssim \|M_\alpha f\|_{\mathcal{M}^u_r(H^{d-\alpha r})} \lesssim \|f\|_{\mathcal{M}^p_r(H^d)},$$

as desired. □

The case $d = \delta = n$ of Proposition 4.2 was first studied in unpublished work of Spanne (for the fractional integral operator) and Peetre published in [7, Theorem 5.4]. After that, the condition used by Spanne was generalised to

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, \qquad \frac{s}{q} = \frac{r}{p}$$

by Adams [1], and Chiarenza and Frasca [3].

PROPOSITION 4.3. *Let $0 < d \le n$, $0 < \alpha < n$, $d/n < r \le p < \infty$ and $0 < s \le q < \infty$. If*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}, \qquad \frac{s}{q} = \frac{r}{p},$$

then

$$\|I_\alpha f\|_{\mathcal{M}^q_s(H^d)} \lesssim \|f\|_{\mathcal{M}^p_r(H^d)}.$$

PROOF. Taking $\beta = 0$ in Lemma 2.7 gives $|I_\alpha f(x)| \lesssim \|f\|^{1-p/q}_{\mathcal{M}^p_{d/n}(H^d)} Mf(x)^{p/q}$. By the boundedness of M on the Morrey spaces,

$$\begin{aligned}\|I_\alpha f\|_{\mathcal{M}^q_s(H^d)} &\lesssim \|f\|^{1-p/q}_{\mathcal{M}^p_{d/n}(H^d)} \|(Mf)^{p/q}\|_{\mathcal{M}^q_s(H^d)} \\ &= \|f\|^{1-p/q}_{\mathcal{M}^p_{d/n}(H^d)} \|Mf\|^{p/q}_{\mathcal{M}^p_r(H^d)} \\ &\lesssim \|f\|^{1-p/q}_{\mathcal{M}^p_{d/n}(H^d)} \|f\|^{p/q}_{\mathcal{M}^p_r(H^d)} \lesssim \|f\|_{\mathcal{M}^p_r(H^d)},\end{aligned}$$

where, in the last inequality, we have used Proposition 2.6. □

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