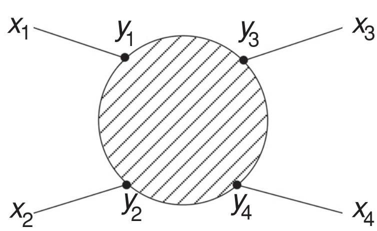
2
Analyticity and unitarity

Firstly, we are going to show that the property of *causality* results in *analyticity* of the scattering amplitude.

2.1 Causality and analyticity

Consider a four-point Green function $A(x_1, x_2; x_3, x_4)$, where the space–time points x_1, x_2 and x_3, x_4 lie in the remote past and future, correspondingly. Let y_1 (y_2) mark the point where the incident particle 1 (2) interacts for the first time, and y_3 (y_4) the point of the last interaction of the particle 3 (4). Then

$$A(x_1, x_2; x_3, x_4) =$$



(2.1)

$$= \int f(y_1, y_2, y_3, y_4) \left\{ \prod_{i=1}^{4} D(y_i - x_i)\, d^4 y_i \right\}.$$

Here $D(y - x)$ describes the propagation of a free particle which we take to be a scalar one since the spin play no rôle in the analysis that follows:

$$D(y_\mu - x_\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{dp_0}{2\pi i} \frac{\exp\{-ip^\mu (y - x)_\mu\}}{m^2 - p^2 - i\epsilon}.$$

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For the time ordering $y_0 > x_0$, one closes the integration contour in energy around the pole at $p_0 = \sqrt{m^2 + \mathbf{p}^2}$ in the lower half-plane, to obtain

$$\begin{aligned} D(y_\mu - x_\mu) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\exp\{-ip^\mu(y-x)_\mu\}}{2p_0} \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, \psi_{\mathbf{p}}(y)\cdot\psi^*_{\mathbf{p}}(x), \qquad y_0 > x_0. \end{aligned}$$

For the final state particles we have $x_{03} > y_{03}$, $x_{04} > y_{04}$, and their propagators take the form

$$D(y_\mu - x_\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, \psi_{\mathbf{p}}(x)\cdot\psi^*_{\mathbf{p}}(y), \qquad x_0 > y_0.$$

Thus, the 'truncated' interaction amplitude f in (2.1) gets multiplied by the product of the wave functions ψ of the incoming particles and of the conjugate wave functions ψ^* of the outgoing ones, evaluated at the 'entry' points y_1, y_2 and the 'exit' points y_3, y_4, correspondingly. The two-particle interaction amplitude in the momentum space, $\mathcal{M}(p_i)$, becomes the Fourier transform:

$$\mathcal{M}(p_i) = \int f(y_1, y_2, y_3, y_4)\, \mathrm{e}^{-i(p_1y_1+p_2y_2)+i(p_3y_3+p_4y_4)} \prod d^4y_i. \qquad (2.2)$$

An integral over the 'centre of gravity' of the four coordinates produces the energy–momentum conservation condition, and we are left with three integrations over the relative positions, $y_i - y_k$. For the sake of simplicity, let us restrict ourselves to to the *forward scattering* case, $p_1 \approx p_3$, $p_2 \approx p_4$. Then (2.2) reduces to

$$\mathcal{M} \Longrightarrow (2\pi)^4\delta(p_1 + p_2 - p_3 - p_4) \int \mathrm{e}^{ip_1(y_3-y_1)} f(y_{13}; p_2) d^4y_{13}, \qquad (2.3)$$

where we have singled out the dependence on one of the momenta, namely p_1. This is sufficient since, because of the Lorentz invariance, the amplitude actually depends on the invariant energy, $\mathcal{M}(p_i) = \mathcal{M}(s)$,

$$s \equiv (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2m_2E_1,$$

which is proportional to the energy E_1 of one of the incident particles in the rest frame of the second one, $E_2 = m_2$.

Causality means that the function f in the integrand of (2.3) must have the form

$$f(y) \;=\; \vartheta(y_0)\vartheta(y_\mu^2)\cdot f_1(y) + f_0(y), \qquad (2.4a)$$

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where f_0 does not contribute to the scattering:

$$\int d^4y \, f_0(y) \, \exp\{ip_1 y\} \; = \; 0. \tag{2.4b}$$

Mark that the condition (2.4b) does not imply $f_0 \equiv 0$, since it is only the Fourier components with a *physical* momentum p_1,

$$p_1 = \left(\sqrt{m^2 + \mathbf{p}^2}, \;\; \mathbf{p}\right),$$

that are required to vanish. How does such a decomposition emerge in the field theory? The easiest way to arrive at (2.4) is to invoke the general operator language.

The interaction amplitude in the coordinate space is related to the time-ordered product of operators describing an absorption of a particle in y_1 and a creation of another one in the space–time point y_3:

$$\begin{aligned} f(y_3, y_1) &\propto \left\langle T\, \psi(y_3)\bar{\psi}(y_1)\right\rangle \\ &\equiv \vartheta(\Delta y_0)\cdot \psi(y_3)\bar{\psi}(y_1) \;\;\pm\;\; \vartheta(-\Delta y_0)\cdot \bar{\psi}(y_1)\psi(y_3) \\ &= \vartheta(\Delta y_0)\left[\psi(y_3)\bar{\psi}(y_1) \mp \bar{\psi}(y_1)\psi(y_3)\right] \pm \bar{\psi}(y_1)\psi(y_3), \end{aligned} \tag{2.5}$$

where Δy stands for the relative coordinate,

$$\Delta y^\mu = y_3^\mu - y_1^\mu \,.$$

Alternative signs $\pm$ in (2.5) and below correspond to bosonic and fermionic operators, i.e. particles with integer and half-integer spin. For the space-like intervals, that is when $(\Delta y)^2 < 0$, by the virtue of *causality* our operators (anti)commute, so that

$$f(y_3, y_1) \propto \vartheta(\Delta y_0)\vartheta((\Delta y)^2)\cdot f_1 \pm \bar{\psi}(y_1)\psi(y_3)\,.$$

The latter piece (f_0) is given by a simple (not T-ordered) product of the two operators, which may be represented as a sum over all possible intermediate states:

$$\langle 0|\, \bar{\psi}(y_1)\psi(y_3)\,|0\rangle = \sum_n \langle 0|\, \bar{\psi}(y_1)|n\rangle \cdot \langle n|\psi(y_3)|0\rangle = \sum_n |C_n|^2 \, \mathrm{e}^{-iP_n(y_1 - y_3)}.$$

Here we have explicitly extracted the coordinate dependence in terms of the total intermediate state momenta, P_n. Substituting into the integral for the scattering amplitude, (2.4a), we immediately get

$$\sum_n \left|C_n^2\right| \int d^4 y_{31} \;\, \mathrm{e}^{ip_1 y_{31}} \cdot \mathrm{e}^{iP_n y_{31}} \propto \delta(p_{0,1} + P_{0,n}) \; = \; 0.$$

Vanishing of this contribution is due to the fact that both the incoming particle and any physical intermediate state n have a *positive energy*. This

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conclusion derives from the stability of the vacuum: any excitation must lie above the vacuum state, $P_{0,n} > 0$.

Finally, we arrive at the integral representation for the amplitude,

$$\mathcal{M}(E_1) = \int d^4y\, f_1(y) \cdot \vartheta(y_0)\vartheta(y_\mu^2)\, \mathrm{e}^{ip_1 y} = \int d^3\mathbf{y} \int_{\sqrt{\mathbf{y}^2}}^{\infty} dt\, \mathrm{e}^{iE_1(t-v_1 z)} f_1(y); \tag{2.6}$$

$$p_1 y \equiv E_1 t - \mathbf{p}_1 \cdot \mathbf{y} = E_1 \cdot (t - v_1 z),$$

where z is the coordinate projection on the direction of the momentum. The theta-functions in (2.6) ensure that the *phase* of the exponent is *positively definite*:

$$t > 0, \quad t > \sqrt{z^2 + \boldsymbol{\rho}_\perp^2} \geq |z| > |v_1 z| \Longrightarrow (t - v_1 z) > 0\,.$$

As a consequence, $\mathcal{M}(E_1) \equiv \mathcal{M}(s)$ is a regular analytic function in the *upper* half-plane of complex energies E_1. Indeed, if the integral (2.6) exists (converges) for real values of the energy, it can be analytically continued onto the upper plane, $\mathrm{Im}\, E_1 > 0$, where it converges even better due to the additional exponentially falling factor, $\exp\{-\mathrm{Im}\, E_1(t - v_1 z)\}$.

2.1.1 Causality and the polynomial boundary for $\mathcal{M}(s)$.

Let us reverse the logic now. The inverse Fourier transform reads

$$f(t) = \int_{-\infty}^{+\infty} dE\, \mathrm{e}^{-iE\,t} \mathcal{M}(E).$$

If $t < 0$, by moving the contour onto the upper half-plane where (as we have just established) $\mathcal{M}(E)$ is regular, we should get zero, to be in accord with causality. This is the case *provided* the amplitude does not increase exponentially along the imaginary axis:

$$|\mathcal{M}(\mathrm{Im}\, E \to +\infty)| \;<\; \exp(\gamma\, \mathrm{Im}\, E) \qquad \text{for arbitrary } \gamma > 0.$$

Otherwise, the exponentially decreasing factor $\exp(t \cdot \mathrm{Im}\, E)$ would not be sufficient to guarantee the vanishing of the response at small but finite negative times

$$-\gamma \leq \; t \; < 0.$$

That is why, to be on the safe side, we will impose an additional restriction on the scattering amplitude by bounding its possible growth with energy

$$|\mathcal{M}(s)| < |s|^N ,$$

with N some finite (though maybe large) power.

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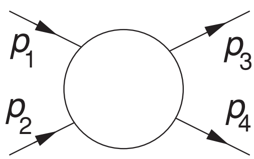
2.2 Cross-channel singularities of Born diagrams from s-channel point of view

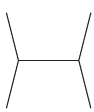
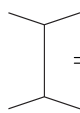
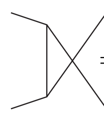
We will employ for simplicity the QFT model scalar particles with a $\lambda\varphi^3$ interaction. This is the simplest example of a renormalizable theory and, though not without a defect (it has no ground state – a stable vacuum), it is well suited for the qualitative analysis of singularities of scattering amplitudes.

Let m and λ be the renormalized mass and interaction constant. We consider a four-particle amplitude characterized by the Mandelstam variables s, t, u:



$$\begin{array}{rcl} s & = & (p_1+p_2)^2; \\ t & = & (p_1-p_3)^2; \quad s+t+u=4m^2, \\ u & = & (p_1-p_4)^2. \end{array}$$

What sort of singularities will we encounter at each order of perturbation theory? In the Born approximation we have three poles, in each of the Mandelstam invariants,

 $= \dfrac{\lambda^2}{m^2-s}$,  $= \dfrac{\lambda^2}{m^2-t}$,  $= \dfrac{\lambda^2}{m^2-u}$. (2.7)

Before moving further we will first discuss the meaning of these poles.

2.2.1 Pole in energy

The first one is the pole in the invariant collision energy s. Recall quantum mechanics. Here the amplitude of elastic scattering of a particle with initial momentum $\mathbf{p}$, $|\mathbf{p}| = |\mathbf{p}'| = \sqrt{2mE}$, has the following representation in terms of the potential V and the incoming state wave function:

$$f(E,\mathbf{q}) = -\frac{2m}{4\pi}\int d^3\mathbf{r}'\, \mathrm{e}^{-i\mathbf{p}'\cdot\mathbf{r}'}V(\mathbf{r}')\psi_{\mathbf{p}}(\mathbf{r}'), \tag{2.8}$$

with $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ the momentum transfer.

In order to extract the energy dependence we are after, we invoke the Green function of the stationary Schrödinger equation:

$$\begin{aligned} &(\widehat{\mathcal{H}} - E)G_E(\mathbf{r}',\mathbf{r}) = \delta(\mathbf{r}'-\mathbf{r}); \\ &\qquad G_E(\mathbf{r}',\mathbf{r}) = \sum_n \frac{\psi_n(\mathbf{r}')\psi_n^*(\mathbf{r})}{E_n - E}\,; \quad \widehat{\mathcal{H}}\,\psi_n = E_n\psi(n), \end{aligned}$$

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where E_n are the exact energy levels of the system. Now we express the exact wave function $\psi_{\mathbf{p}}(\mathbf{r}')$ as

$$\psi_{\mathbf{p}}(\mathbf{r}') = \mathrm{e}^{i\mathbf{p}\cdot\mathbf{r}'} - \int d^3\mathbf{r}\, G_E(\mathbf{r}', \mathbf{r}) V(\mathbf{r})\, \mathrm{e}^{i\mathbf{p}\cdot\mathbf{r}},$$

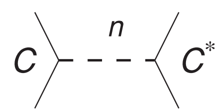
and substitute into (2.8) to derive

$$\begin{aligned} f(E, \mathbf{q}) = -\frac{2m}{4\pi}\Bigg[V(\mathbf{q}) \;-\; \sum_n \frac{1}{E_n - E} \\ \times \left(\int d\mathbf{r}\, \mathrm{e}^{-i\mathbf{p}'\mathbf{r}} V(\mathbf{r}) \psi_n(\mathbf{r}) \right) \left(\int d\mathbf{r}'\, \mathrm{e}^{i\mathbf{p}\,\mathbf{r}'} V(\mathbf{r}') \psi_n^*(\mathbf{r}') \right) \Bigg] \qquad (2.9\text{a}) \\ = f_B \;+\; \sum_n \frac{C_n(\mathbf{p}) C_n^*(\mathbf{p}')}{E_n - E}. \end{aligned}$$

Here the Born scattering amplitude f_B,

$$f_B(\mathbf{q}) \;=\; -\frac{2m}{4\pi} \int d^3\mathbf{r}\, \mathrm{e}^{-i\mathbf{q}\cdot\mathbf{r}} V(\mathbf{r}), \qquad (2.9\text{b})$$

depends only on the momentum transfer $\mathbf{q}$, while the dependence on the energy is contained in the sum over intermediate states n. If the system possesses a bound state – a discrete energy level E_n – its contribution to the amplitude can be depicted as a diagram with $1/(E_n - E)$ as the 'propagator' of the state n, and C as the 'coupling constant'. Thus, in non-relativistic quantum mechanics a pole in energy corresponds to scattering via an intermediate state related to a discrete energy level. In the relativistic theory two particles can transfer into one, and it is this particle which plays the rôle of such an intermediate state.



2.2.2 Pole in momentum transfer

Having understood the physical meaning of the pole in s, we could repeat the same consideration in the crossing channels where t and u play, correspondingly, the rôle of energy, and thus would make sense of the two other poles in (2.7).

Still, what is the meaning of these poles from the s-channel point of view? The Mandelstam variable t measures the momentum transfer (for elastic scattering with $p_0 = p_0'$, $t = -\mathbf{q}^2$). Are there singularities in the *momentum transfer* in quantum mechanics? Let us examine the Born scattering amplitude (2.9b),

$$f_B(q) \propto \int d^3\mathbf{r}\, \mathrm{e}^{-i\mathbf{q}\cdot\mathbf{r}} V(\mathbf{r}).$$

It develops a singularity at a point where the integral diverges when we continue q onto the complex plane. If the potential has a power tail,

$$V(r) \; \propto \; r^{-n}, \qquad r \to \infty,$$

this happens for an arbitrary small value of $\mathrm{Im}\, q \neq 0$. This means that the singularity emerges at $t = 0$. By dimensional consideration,

$$\int \frac{d^3 r}{r^n} \sim r^{-n+3} \implies q^{n-3},$$

for $n = 1, 2$ it is a pole; for an integer $n \geq 3$ it is a logarithmic singularity. It becomes clear that in order to have a singularity at some finite $t = m^2$, as this is the case of the amplitude (II) in (2.7), the potential has to fall exponentially at large distances. More precisely, it is the Yukawa potential,

$$V(r) = \frac{A}{r}\, \mathrm{e}^{-mr}, \tag{2.10a}$$

whose Fourier image as we saw in the previous lecture gives the pole amplitude, see (1.8),

$$f_B \; \propto \frac{A}{m^2 + \mathbf{q}^2} \; \equiv \; \frac{\lambda^2}{m^2 - t}. \tag{2.10b}$$

We conclude that the relativistic Born amplitude (II) in (2.7) corresponds to a definite potential (Yukawa) with a definite strength ($A = \lambda^2$) and a definite sign (attraction).

Thus, singularities in momentum transfer are related to the *interaction radius*. Let us remark that our 'strong interaction potential' V has a finite radius $r_0 = 1/m$ owing to our basic supposition that all hadrons have non-zero masses.

2.2.3 Exchange potential

Finally, what is the pole in u?

The diagrams (II) and (III) in (2.7) differ by the exchange of final particles (momenta p_3 and p_4). In the non-relativistic quantum mechanics the *exchange potential* is a well known object:

$$V^{(\mathrm{ex})}(\mathbf{r}_{ik}) \; = \; V(\mathbf{r}_{ik}) \cdot P_{ik},$$

with P_{ik} the particle permutation operator. Thus, the diagram (III) determines the exchange potential which in our case *coincides* with the direct

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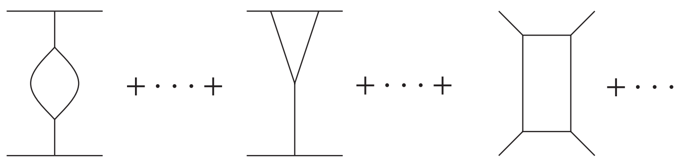
one: the sum of the diagrams (2.7) automatically takes care of the identity of our scalar particles (Bose statistics).

2.3 Higher orders

In higher orders Feynman diagrams become more and more complex. In the second order in λ^2, we will have graphs like



(2.11)

(where the dots stand for the diagrams of the same type with the external lines transmuted). The first diagram seems to have a *double pole* in t. However, this 'self-energy insertion' into the propagator line modifies the particle mass, while we wanted m^2 to represent the true physical mass of the particle in the Green function $(m^2 - t)^{-1}$. Therefore a subtraction has to be made,

$$\frac{1}{m^2-t}\,\Sigma(t)\,\frac{1}{m^2-t} \implies \frac{1}{m^2-t}\left[\Sigma(t)-\Sigma(m^2)\right]\frac{1}{m^2-t},$$

after which the double pole disappears.

Moreover, the *simple pole* in t is effectively absent too. Indeed, by expanding the subtracted self-energy blob one step further,

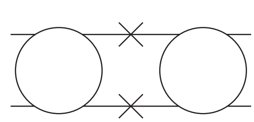
$$\Sigma(t) - \Sigma(m^2) \;=\; (t-m^2)\cdot\Sigma'(m^2) + \Sigma_c(t),$$

we observe that the term proportional to $\Sigma'(m^2)$ must be dropped too, as modifying the value of the on-mass-shell coupling constant λ^2 in the corresponding t-channel Born diagram of (2.7). Since $\Sigma_c(t) \propto (t-m^2)^2$, the remaining contribution is finite. In the same way, the t-channel propagator pole cancels in the second (vertex correction) diagram of (2.11).

The remaining last ('box') graph in (2.11) has no poles either.

The fact that the second-order diagrams do not possess pole singularities does not mean that higher orders do not modify analytic properties of the amplitude. Far from that. This becomes immediately clear if we look at the unitarity condition for the elastic scattering amplitude:

$$2\,\mathrm{Im}\,M_{aa} \;=\; \sum_c M_{ac}M^*_{ac} \;=$$



. (2.12)

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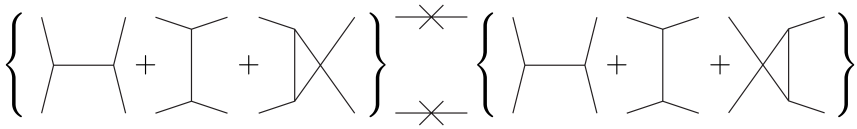
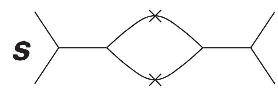


Fig. 2.1 Product of the Born amplitudes in the unitarity condition.

If in the r.h.s. we substitute the Born amplitude $M = \mathcal{O}(\lambda^2)$, see Fig. 2.1, then on the l.h.s. of the equation we will have Im M of the order of λ^4. This shows that starting from the second order in λ^2, the scattering amplitude must be complex above the two-particle threshold, $s > 4m^2$.

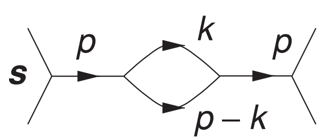
Among the products of the diagrams on the r.h.s. of the unitarity condition (2.12), there is a square of the s-channel Born graph.



Let us examine the corresponding second-order diagram to see where the complexity comes from.

2.3.1 Two-particle thresholds

Consider the second-order diagram



(2.13)

Looking for the origin of the complexity, we can drop the real pole factors and concentrate on the loop integral:

$$\Sigma(s) = \frac{1}{2!} \int \frac{d^4k}{(2\pi)^4 i} \frac{\lambda^2}{(m^2 - k^2 - i\epsilon)(m^2 - (p-k)^2 - i\epsilon)},$$

(with $1/2!$ the symmetry factor characteristic for the loop of two identical particles). In the complex plane of the k_0 variable, singularities of the integrand are positioned at

$$k_0 = \pm\sqrt{m^2 + \mathbf{k}^2} \mp i\epsilon, \tag{2.14a}$$

$$k_0 = p_0 \pm \sqrt{m^2 + (\mathbf{p} - \mathbf{k})^2} \mp i\epsilon. \tag{2.14b}$$

The Feynman $i\epsilon$ prescription displaces the poles from the real axis to tell us on which side of the singularity the contour passes.

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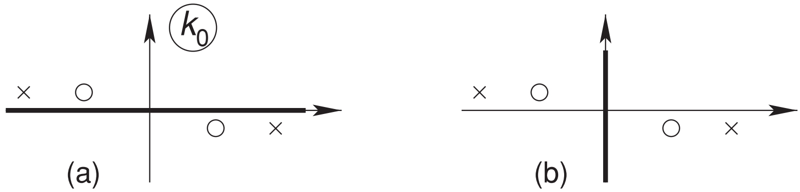


Fig. 2.2 Transformation of the k_0 integration contour; $s < 0$.

Consider the two cases.

$s < 0$. In this case we can always find a reference frame such that $p_0 = 0$, so that the pairs of the poles (2.14) are placed symmetrically around the imaginary axis, as displayed in Fig. 2.2(a). In this situation we can turn the integration path as on Fig. 2.2(b). Introducing a new integration variable, $k_0 = i\kappa$, the self-energy becomes

$$\Sigma(-\mathbf{p}^2) = \frac{1}{2!}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\int \frac{d\kappa}{2\pi}\frac{\lambda^2}{(m^2+\kappa^2+\mathbf{k}^2)(m^2+\kappa^2+(\mathbf{p}-\mathbf{k})^2)},$$

where we have dropped the $i\epsilon$ terms since the denominators vanish nowhere in the integration region. The answer is obviously real valued.

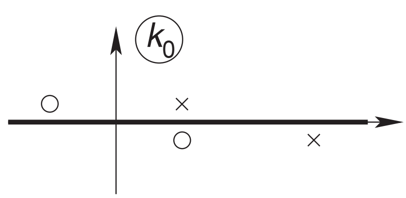
$s > 0$. Now, contrary to the previous case, we can choose the frame $\mathbf{p} = 0$:

$$\begin{aligned} k_0 &= \pm\sqrt{m^2+\mathbf{k}^2} \mp i\epsilon, \\ k_0 &= \pm\sqrt{m^2+\mathbf{k}^2} + p_0 \mp i\epsilon. \end{aligned}$$

With $p_0 > 0$ and increasing, the second pair of poles will move to the right, and at some point the trailing pole of the second pair will collide with the leading pole of the first one:

$$-\sqrt{m^2+\mathbf{k}^2} + p_0 = \sqrt{m^2+\mathbf{k}^2}\,.$$

Two poles *pinch* the contour, and the integral becomes complex.



(2.15)

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Since we are integrating over $\mathbf{k}$, this happens for the first time at

$$s = 4m^2 \;\leq\; \left(2\sqrt{m^2+\mathbf{k}^2}\right)^2. \tag{2.16}$$

This is the value of s corresponding to the two-particle *energy threshold*.

It is straightforward to calculate the *imaginary part* of the self-energy graph in (2.13). To this end we close the contour around the two poles on the upper half-plane in (2.15) and look at the contribution of the pinching one at $k_0 = p_0 - \sqrt{m^2+\mathbf{k}^2}$:

$$\Sigma(p_0^2) = \frac{1}{2!}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\cdot\frac{1}{2(p_0-\sqrt{m^2+\mathbf{k}^2})} \cdot\frac{\lambda^2}{m^2+\mathbf{k}^2-(p_0-\sqrt{m^2+\mathbf{k}^2})^2-i\epsilon} + \text{real},$$

where the first factor is the residue, $2k_0$, and the second – the remaining Green function. Using

$$\mathrm{Im}\frac{1}{a-i\epsilon} = \pi\delta(a),$$

we obtain

$$2\,\mathrm{Im}\,\Sigma(p_0^2) = \frac{\lambda^2}{2!p_0}\int \frac{d^3\mathbf{k}}{2(2\pi)^2\sqrt{m^2+\mathbf{k}^2}}\cdot\delta\left(2\sqrt{m^2+\mathbf{k}^2}-p_0\right). \tag{2.17}$$

As expected, the integral has a non-vanishing support only when $p_0 > 2m$, cf. (2.16). The fact that the amplitude becomes complex means that in the complex plane of the energy variable $s = p_0^2$ it has a *branch cut* which starts at $s = 4m^2$ and runs to infinity. This is the *threshold singularity* corresponding to the production of two particles with equal masses m.

The integral (2.17) is easy to calculate; the calculation produces nothing but the phase-space volume for the production of two real particles with an aggregate four-momentum p, see (1.25). This is straightforward to see if we recast the answer in the Lorentz covariant form as

$$2\,\mathrm{Im}\,\Sigma(p^2) = \frac{\lambda^2}{2!}\int \frac{d^4k}{(2\pi)^4}2\pi\delta_+((p-k)^2-m^2)\cdot 2\pi\delta_+(k^2-m^2). \tag{2.18}$$

We conclude that in order to calculate $2\,\mathrm{Im}\,\Sigma(p^2)$ one simply has to 'cut through' the diagram (2.13) and replace in the Feynman expression for the amplitude *each cut propagator* by (double) its imaginary part,

$$\frac{1}{m^2-k^2-i\epsilon} \;\Longrightarrow\; (2\pi i)\delta_+\big(k^2-m^2\big)\,.$$

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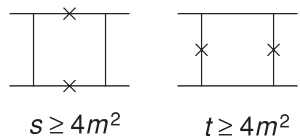
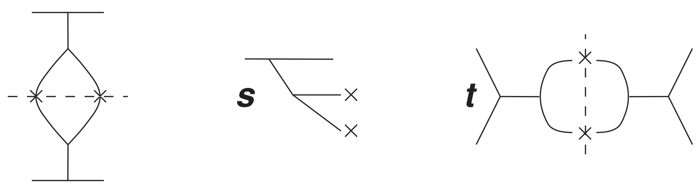


Fig. 2.3 The second-order box graph having branch cuts both in s and t.

This rule is in a perfect accord with the unitarity condition formally applied to the $1 \to 1$ transition via a two-particle intermediate state (take $n = 2$, $\mathcal{M}_{ac} = \mathcal{M}^*_{bc} = \lambda$ in (1.31)), and can be used to calculate imaginary parts of arbitrary diagrams.

When cutting a diagram in order to find its imaginary part, one must make sure that the two graphs that emerge correspond to a real physical process. For example, the diagram with the self-insertion into the t-channel particle exchange can be cut into two as



From the point of view of the s-channel, this looks like squaring an amplitude of the decay of the incoming particle into three, which process is kinematically forbidden. On the other hand, in the t-channel these very subgraphs correspond to a legitimate $2 \to 2$ scattering process, so that this cut describes the t-channel two-particle threshold and makes the amplitude complex for $t > 4m^2$.

Another example. Contrary to the self-energy insertion graph, the 'box' in Fig. 2.3 can be legitimately cut both in the s- and t-channels and therefore possesses two branch cuts. Another second-order graph – the 'crossed box' – shown in Fig. 2.4 has the same t threshold but cannot be cut in the physical region of the s-channel, since such a division would correspond to two-body decays of incoming particles. At the same time, it can be cut in the u-channel describing the $1\bar{4} \to 3\bar{2}$ transition. If we keep t fixed and study analytic properties of the amplitude as a function of s, the u-channel threshold at $u = 4m^2$ will manifest itself as another branch cut in $s = 4m^2 - u - t$ which starts at $s = -t$ and runs *to the left.*

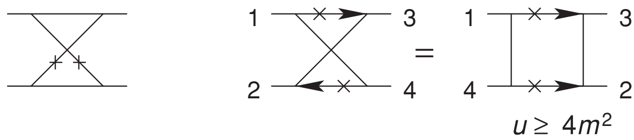


Fig. 2.4 The crossed box graph having branch cuts in t and u. Arrows mark the direction of the positive energy flow.

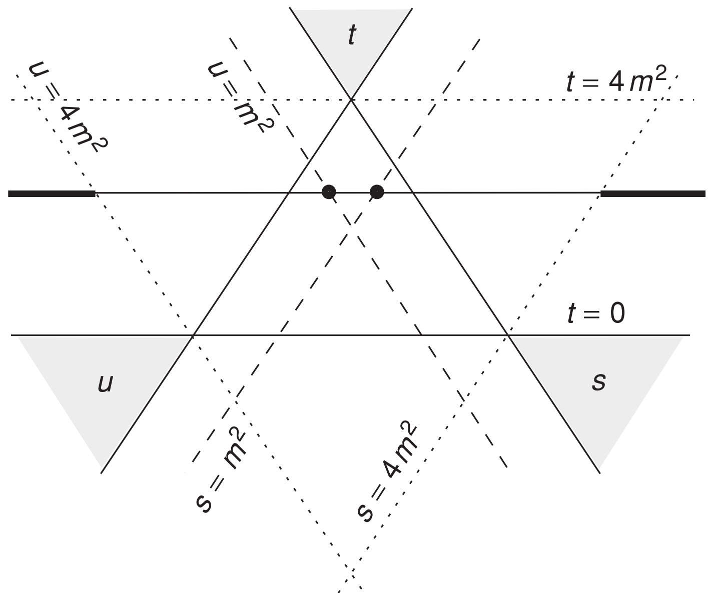


Fig. 2.5 Singularities of the second order amplitude on the Mandelstam plane.

2.3.2 Scattering amplitude as a function of s

Nine cut diagrams contained in Fig. 2.1 describe the imaginary part of the full two particle scattering amplitude in the order λ^4, due to the s-*channel* threshold. The same diagrams with permuted external particles give t- and u-channel complexities.

Thus, the amplitude on the Mandelstam plane is *real* inside the triangle marked by the dotted lines on Fig. 2.5. By fixing the variable t at some value in the interval $0 \le t \le 4m^2$, the regions where the amplitude is *complex* are displayed by two bold lines on Fig. 2.6. The physical s-channel scattering amplitude $A^{(s)}$ is defined for $s \ge 4m^2$ and is given by the value of the invariant amplitude on the *upper side* of the right cut, $\mathrm{Im}\, s = \longrightarrow + i0$. Since for a fixed t we have $s + u = \text{const.}$, the physical u-channel

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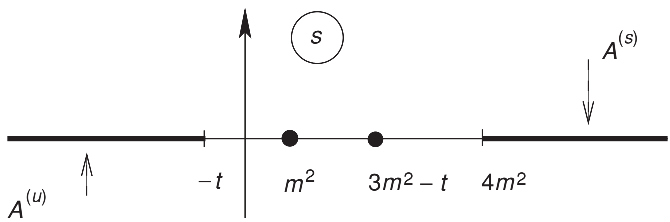
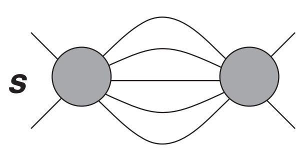


Fig. 2.6 Analytic structure of the amplitude in the complex s-plane and the definition of physical amplitudes of the s- and u-channel reactions.

amplitude $A^{(u)}$ that describes the process $1\bar{4} \to 3\bar{2}$ one obtains by approaching the left cut *from below* $\mathrm{Im}\, s \to -i0$, $\mathrm{Re}\, u > 4m^2$ (see Fig. 2.6).

As we have learnt, due to causality, $A^{(s)}$ is a regular function in the *upper* half-plane of the invariant energy s. Analogously, the *lower* half of the s-plane is free of any singularities as well, thanks to the causality property of the u-channel reaction (which makes the amplitude $A^{(u)}$ regular for $\mathrm{Im}\, u > 0$). Moreover, since the upper and the lower half-planes are analytically connected through a finite interval on which the amplitude is real, we conclude that the invariant amplitude is an analytic function on the entire complex s-plane apart from two poles and two *branch cuts* originating from two-particle thresholds.

In higher orders there appear higher thresholds, and related additional branching points, due to multi-particle thresholds in the intermediate states of the s- and u-channel reactions:



$$s_0 = \left(\sum_{i=1}^{n} m_i \right)^2 \to n^2 m^2. \tag{2.19}$$

2.3.3 Dispersion relation

Once the imaginary part of the analytic function is known, we can restore its real part using the dispersion relation. We write a Cauchy integral around a point s in the complex plane where the amplitude is regular,

$$\int_{\mathcal{C}} \frac{dz}{2\pi i} \frac{A(z)}{z - s} = A(s)\,,$$

and then inflate the contour to embrace the singularities of the amplitude $A(z)$ in the z plane, as shown in Fig. 2.7. If the function falls on the large circle, $|z| \to \infty$, we obtain s and u poles inherited from the Born amplitude, and a sum of integrals of the *discontinuity* across the right

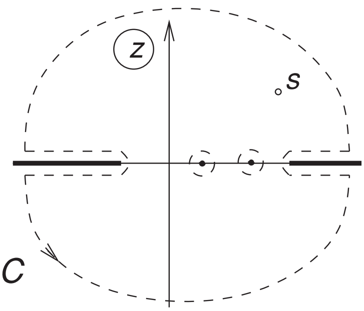


Fig. 2.7 Integration contour in the dispersion relation for the amplitude $A(s)$.

and left branch cuts:

$$\mathrm{Im}_s\, A \;\equiv\; \tfrac{1}{2i}\big[A(s+i0,t) - A(s-i0,t)\big], \quad s > 4m^2, \tag{2.20a}$$

$$\mathrm{Im}_u\, A \;\equiv\; \tfrac{1}{2i}\big[A(u+i0,t) - A(u-i0,t)\big], \quad u > 4m^2. \tag{2.20b}$$

The piece of the Born amplitude responsible for t-channel particle exchange, $\lambda^2/(m^2-t)$, does not fall with s. Therefore in order to eliminate the contribution of the large circle, one has to use the dispersion relation with one subtraction, that is to apply the Cauchy theorem to the function

$$A(s) - A(0) = \int_{\mathcal{C}} \frac{dz}{2\pi i}\left[\frac{A(z)}{z-s} - \frac{A(z)}{z}\right] = \frac{s}{\pi}\int_{\mathcal{C}} \frac{dz}{2i}\,\frac{A(z)}{z(z-s)}\,.$$

The s-independent piece then hides in the subtraction term $A(0,t)$. By combining this result with a complementary information coming from the t channel, we can restore the full second-order amplitude $\mathcal{O}(\lambda^4)$ from the Born one, $\mathcal{O}(\lambda^2)$. In principle, one can move further, order by order, and recursively build up full interaction amplitudes by exploiting unitarity in the cross-channels and analyticity.

It is worthwhile to mention that if the Born amplitude happens to *increase too fast* with energy, the subtraction trick fails to work (as it happens, e.g. in electrodynamics with an anomalous magnetic moment interaction vertex term $\Gamma_\mu \propto \sigma_{\mu\nu}q^\nu$). The number of arbitrary subtraction constants grows with the order of the perturbative expansion, which manifests, in the dispersive theory language, the *non-renormalizability* of the underlying interaction.

The dispersive programme was found indeed to be rather effective in quantum electrodynamics. There were times when it was being enthusiastically explored as a possible way of constructing the theory of hadrons without knowing the internal structure of the interaction. An attractive feature of such a scheme is that it operates only with experimentally accessible physical quantities – on-mass-shell amplitudes.

2.4 Singularities of Feynman graphs: Landau rules

Now we have to address an essentially technical problem, namely how to find singularities of arbitrary Feynman diagrams, and how to determine their position and character.

2.4.1 Position of singularities

Consider a diagram containing n internal lines with four-momenta k_1, $k_2, \ldots, k_n$ which may be expressed as linear combinations of the external momenta p_j and the integration momenta q_m:

$$k_i = \sum_{m=1}^{\ell} b_{im} q_m + \sum_j c_{ij} p_j, \tag{2.21}$$

where b_{im} are either 1 or 0. The number of the independent integration contours (*loops*) ℓ is a function of the topology of the given diagram.

The Feynman integral corresponding to our diagram has the structure

$$A_{n\ell} = \int \frac{d^4 q_1\, d^4 q_2 \cdots d^4 q_\ell}{[(2\pi)^4 i]^\ell} \frac{1}{(m_1^2 - k_1^2)(m_2^2 - k_2^2)\cdots(m_n^2 - k_n^2)}\,. \tag{2.22}$$

We need to find the conditions under which this integral becomes singular in external variables $s_{ik} = (p_i + p_k)^2$ – Lorentz invariants formed by four-momenta of the external particles.

It is worthwhile to stress that for the case of particles with higher spins, matrices and different powers of momenta would appear in the *numerator* of the integrand. This, however, would not affect the singularities of the amplitude which depend exclusively on the structure of the scalar *denominator* in (2.22).

To study the appearance of singularities it is convenient to use the trick invented by R. Feynman in order to get rid of multiple four-vector integrations. Applying to the denominators $a_i = m_i^2 - k_i^2$ in (2.22) the Feynman identity

$$\frac{1}{a_1 a_2 \cdots a_n} = \frac{1}{n!} \int_{\alpha_i \geq 0} \frac{d\alpha_1\, d\alpha_2 \cdots d\alpha_n}{(\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n)^n}\, \delta\big(1 - \sum_{i=1}^{n} \alpha_i\big),$$

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which is not difficult to prove by induction, we get

$$A_{n\ell} = \int \frac{d^4q_1 \cdots d^4q_\ell}{[(2\pi)^4 i]^\ell} \int_0^1 \frac{d\alpha_1\, d\alpha_2 \cdots d\alpha_n}{\square^n}\, \delta\left(1 - \sum_{i=1}^n \alpha_i\right), \tag{2.23}$$

where

$$\square = \square(\{\alpha\}, \{p\}; \{q\}) \equiv \sum_{i=1}^n \alpha_i (m_i^2 - k_i^2). \tag{2.24}$$

The integrand in (2.23) depends analytically on the integration variables; the vanishing of the denominator inside the integration region, $\square = 0$, is the only potential source of singularities.

Substituting the decomposition (2.21), the characteristic function $\square$ of (2.24) becomes an inhomogeneous quadratic form. This form can be diagonalized by an orthogonal transformation:

$$\square(\alpha, p; q) = \Delta(\alpha, p) - \sum_{m=1}^\ell \delta_m \cdot \tilde{q}_m^2, \quad \delta_m = \delta_m(\alpha) > 0, \tag{2.25}$$

with $\tilde{q} = \{\tilde{q}_m\}$ the set of new integration momenta.

In (2.25) $\Delta = \Delta(\alpha, s_{ik})$ is a function of invariant energies s_{ik}. Let us start from the kinematical domain where all these invariants are *negative*, $s_{ik} < 0$. (This is easy to achieve by taking all the energy components $p_j = 0$ in some reference frame.) Then the energy integrations can be transformed as we did above when we studied the self-energy graph,

$$\tilde{q}_{0m} = i\kappa_m, \quad \frac{d^4\tilde{q}_m}{(2\pi)^4 i} = \frac{d\kappa_m\, d^3\tilde{\mathbf{q}}_m}{(2\pi)^4} \equiv \frac{d^4 Q_m}{(2\pi)^4}.$$

The expression for the denominator becomes positively definite,

$$\square = \sum_i \alpha_i \left(m_i^2 + \left[\sum \kappa_s\right]^2 + \mathbf{k}_i^2\right) = \Delta(\alpha, s_{ik}) + \sum_{m=1}^\ell \delta_m \left[\kappa_m^2 + \tilde{\mathbf{q}}_m^2\right],$$

and consequently the integral (2.23) yields a regular, real amplitude $A_{n\ell}$. Rescaling the Q variables, the integral can be evaluated as

$$\begin{aligned} A_{n\ell} &= \int_0^1 \frac{\delta(1-\Sigma\alpha) \prod_i d\alpha_i}{(2\pi)^{4\ell}} \int \frac{d^4Q_1\, d^4Q_2 \cdots d^4Q_\ell}{[\Delta + \sum_{m=1}^\ell \delta_m Q_m^2]^n} \\ &= \int_0^1 \frac{\delta(1-\Sigma\alpha) \prod_i d\alpha_i}{\prod_{m=1}^\ell \delta_m^2(\alpha)} \cdot \frac{1}{\left[\Delta(\alpha, s_{ik})\right]^{n-2\ell}} \times [\text{number}]. \end{aligned} \tag{2.26}$$

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As a function of the integration momenta $\tilde{q}$, $\square$ has a *minimum* at $\tilde{q}_m = 0$. At this point $\square = \Delta$, which permits us to determine Δ from the equation

$$\Delta(\alpha, s_{ik}) = \square\,(\alpha, p, q^{(0)}), \tag{2.27a}$$

where $\{q^{(0)}\}$ is the position of the *extremum* of $\square$, simultaneously in 4ℓ integration variables:

$$\left.\frac{\partial \square\,(\{q\})}{\partial q_k}\right|_{\{q\}=\{q^{(0)}\}} = 0, \quad k = 1, 2, \ldots, \ell. \tag{2.27b}$$

Now we start changing s_{ik} to see when (2.26) becomes singular. We are left with $n-1$ integrations over Feynman parameters $0 \le \alpha_i \le 1$ restricted by the condition $1 - \sum_{i=1}^{n-1} \alpha_i = \alpha_n \ge 0$. An equation $\Delta(\alpha, s_{ik}) = 0$ determines a surface in the n-dimensional space of αs. A singularity appears when this surface *touches* for the first time the integration domain.

For each variable α_i this can happen in two ways: either a zero of Δ collides with the *endpoint* of the integration interval, $\alpha_i = 0$, or *two zeroes* simultaneously arrive from the complex plane and assume a common real value *inside* the interval, $\alpha_i > 0$:

$$\alpha_i = 0, \qquad \text{or} \quad \frac{d\,\Delta}{d\alpha_i} = 0, \qquad i = 1, 2, \ldots, n-1. \tag{2.28}$$

By virtue of (2.27b),

$$\frac{d\,\Delta(\alpha; q^{(0)})}{d\alpha_i} = \frac{\partial \square}{\partial \alpha_i} + \sum_k \frac{\partial \square}{\partial q_k}\,\frac{dq_k^{(0)}}{d\alpha_i} = \frac{\partial \square}{\partial \alpha_i}. \tag{2.29}$$

Moreover, since $\square$, by its definition (2.24), is a homogeneous linear function of αs,

$$\square = \sum_{i=1}^{n} \alpha_i \frac{\partial \square}{\partial \alpha_i}. \tag{2.30}$$

By successively applying equations (2.27a), (2.30), (2.29) and (2.28) to the point $\Delta = 0$, we derive

$$0 = \Delta = \square = \sum_{i=1}^{n} \alpha_i \frac{\partial \square}{\partial \alpha_i} = \sum_{i=1}^{n-1} \alpha_i \frac{d\Delta}{d\alpha_i} + \alpha_n \frac{\partial \square}{\partial \alpha_n} = \alpha_n \frac{\partial \square}{\partial \alpha_n}.$$

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Finally, the condition for the appearance of singularity reads

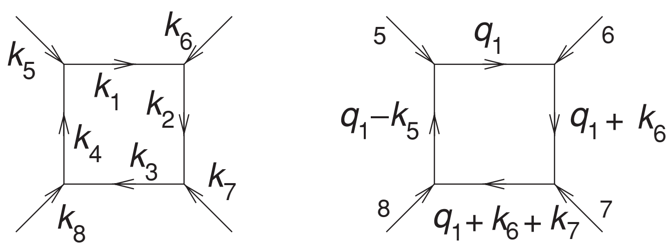
$$\frac{\partial \square}{\partial q_k} = 0, \qquad k = 1, 2, \ldots, \ell; \tag{2.31a}$$

$$\frac{\partial \square}{\partial \alpha_i} = 0 \quad (\text{or} \quad \alpha_i = 0), \quad i = 1, 2, \ldots, n. \tag{2.31b}$$

Equations (2.31b), combined with (2.30), guarantee that in this point $\square = 0$, so that we need not watch the relation $\square = \Delta = 0$ anymore.

The relations (2.31) together with the restriction $\sum \alpha_i = 1$ impose $4\ell + n + 1$ conditions on $4\ell + n$ variables q_k and α_i. This means that a solution may exist only for specific values of external momenta. The corresponding equation $f(s_{ik}) = 0$ determines the '*Landau surface*' for the position of a singularity of the amplitude in the space of invariants s_{ik}. This equation may be resolved, e.g. to determine the position of a singularity in the invariant energy s for fixed momentum transfer variables: $s = s_0(t, u, \ldots)$, or vice versa.

Consider some closed contour inside the diagram:



Introducing the loop momentum $q_1 = k_1$ and applying the first extremum condition (2.31a),

$$\frac{\partial}{\partial q_1}\left(\alpha_1 q_1^2 + \alpha_2(q_1 + k_6)^2 + \alpha_3(q_1 + k_6 + k_7)^2 + \alpha_4(q_1 - k_5)^2\right) = 0,$$

we obtain a system of linear equations stating that

$$\sum_i \alpha_i k_i^\mu = 0 \qquad \text{along each loop.} \tag{2.32a}$$

(It resembles Kirchhoff current law equations for electric circuits, with momentum k_i playing the rôle of the current, and α_i that of resistance.) In addition, the second condition (2.31b) tells us that each line either has to have an on-mass-shell momentum or should be dropped from

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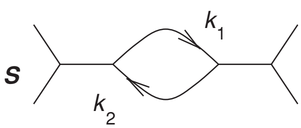
consideration (short-circuited):

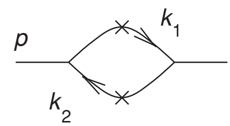
$$k_i^2 = m_i^2 \quad \text{or} \;\; \alpha_i = 0\,. \tag{2.32b}$$

Let us consider some examples.

2.4.2 Threshold singularities

We first take the diagram which has a two-particle threshold singularity, in order to see how the *Landau equations* (2.32) reproduce the result that we have already learnt.





The Landau condition

$$\alpha_1 k_1 + \alpha_2 k_2 = 0 \tag{2.33}$$

shows that the energy components of the momenta k_1 and k_2 have *opposite signs* ($\alpha_i > 0$), so that the singularity we are looking for corresponds physically to the decay of the external particle p into two.

Our elementary loop is too simple to sustain *reduction* (putting one of αs to zero), so the pinch singularity remains the only option:

$$k_1^2 = m_1^2\,, \;\; k_2^2 = m_2^2\,; \quad \alpha_i > 0.$$

Projecting (2.33) onto k_i we obtain a system of linear equations for α_1 and α_2,

$$\begin{aligned} \alpha_1 k_1^2 + \alpha_2 (k_1 k_2) &= 0 \\ \alpha_1 (k_1 k_2) + \alpha_2 k_2^2 &= 0, \end{aligned} \tag{2.34}$$

whose determinant must be zero for a solution to exist. Substituting

$$2(k_1 k_2) = k_1^2 + k_2^2 - (k_1 - k_2)^2 = m_1^2 + m_2^2 - p^2\,,$$

we obtain the *solvability* condition for the system (2.34) in the form

$$\text{Det} = m_1^2 m_2^2 - \frac{1}{4}\big(m_1^2 + m_2^2 - p^2\big)^2 = 0\,,$$

that is,

$$(p^2 - (m_1 + m_2)^2) \cdot (p^2 - (m_1 - m_2)^2) = 0\,.$$

Only one of the two possible solutions,

$$p^2 = (m_1 + m_2)^2,$$

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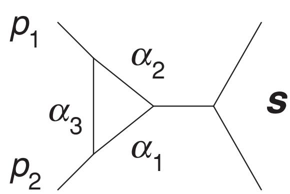
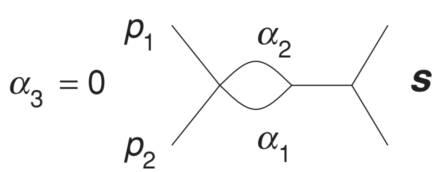
satisfies the condition $\alpha_i > 0$ (one has to have $(k_1 k_2) < 0$ in (2.34)). We conclude that the singularity emerges when the mass of the initial object (invariant energy s of the system) exceeds the threshold value $m_1 + m_2$ sufficient for a real decay into two particles m_1 and m_2 to take place.

2.4.3 Anomalous singularity and deuteron form factor

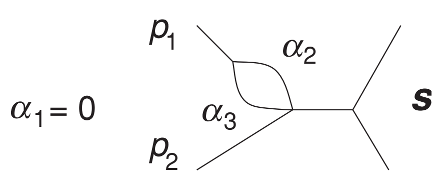
Now take a diagram with three lines in the loop:



This graph can be reduced to a simpler one by setting one of αs to zero. For example, such a reduction



gives us back the threshold singularity at $s = (m_1 + m_2)^2$. If we choose to nullify another Feynman parameter, e.g.



the emerging singularity at $p_1^2 = (m_2 + m_3)^2$ has nothing to do with the energy s but corresponds to the situation when the external particle p_1 is *unstable* (acquires a 'complex mass').

Let us look for a genuine singularity of the triangle diagram that corresponds to three on-mass-shell lines:

$$\begin{aligned} \alpha_1 k_1^2 + \alpha_2 (k_1 k_2) + \alpha_3 (k_1 k_3) &= 0\,, \\ \alpha_1 (k_1 k_2) + \alpha_2 k_2^2 + \alpha_3 (k_2 k_3) &= 0\,, \\ \alpha_1 (k_1 k_3) + \alpha_2 (k_2 k_3) + \alpha_3 k_3^2 &= 0\,. \end{aligned} \tag{2.35}$$

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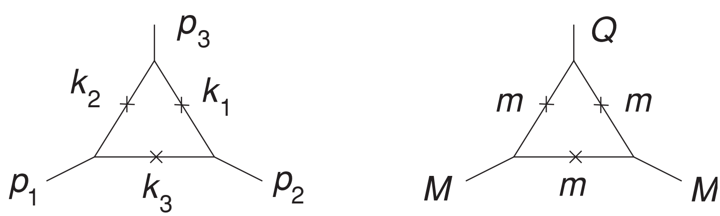
We take masses of the internal particles to be the same,

$$k_1^2 = k_2^2 = k_3^2 = m^2,$$

and, at the same time, allow the virtual momenta of external particles to be different. For the sake of simplicity we take two of them equal,

$$p_1^2 = p_2^2 = M^2\,, \quad p_3^2 = Q^2\,,$$

and look for a singularity in the virtuality Q^2:



(2.36)

Kinematical relations between internal and external momenta result in

$$\begin{aligned}(k_1k_2) &= \tfrac{1}{2}\left[\,k_1^2 + k_2^2 - (k_1-k_2)^2\,\right] = m^2 - \tfrac{1}{2}Q^2\,,\\ (k_1k_3) &= (k_2k_3) \;=\; m^2 - \tfrac{1}{2}M^2.\end{aligned} \tag{2.37}$$

Bearing in mind that $\alpha_i > 0$, from the last line of the system (2.35),

$$(\alpha_1+\alpha_2)\cdot\left(m^2 - \tfrac{1}{2}M^2\right) + \alpha_3 m^2 = 0\,,$$

we derive the necessary condition for the existence of singularity:

$$M^2 \;>\; 2m^2\,. \tag{2.38}$$

Substituting (2.37) into (2.35), the characteristic equation follows:

$$\begin{aligned}0 = \mathrm{Det}&\begin{bmatrix} m^2 & m^2-\frac{1}{2}Q^2 & m^2-\frac{1}{2}M^2\\ m^2-\frac{1}{2}Q^2 & m^2 & m^2-\frac{1}{2}M^2\\ m^2-\frac{1}{2}M^2 & m^2-\frac{1}{2}M^2 & m^2\end{bmatrix}\\ &= \tfrac{1}{2}Q^2\cdot\left[\,m^2\left(2m^2-\tfrac{1}{2}Q^2\right) - 2(m^2-\tfrac{1}{2}M^2)^2\,\right].\end{aligned} \tag{2.39}$$

We obtain the so-called ‘anomalous singularity’ positioned at

$$Q_0^2 = 4M^2\left[\,1-\left(\frac{M}{2m}\right)^2\,\right]. \tag{2.40}$$

The graph (2.36) describes the scattering of an object of mass M with momentum transfer $p_3^2 = Q^2$. If this object is *stable* ($M < 2m$), the singularity (2.40) lies at $Q_0^2 > 0$, that is *outside* the physical region of the scattering reaction. However, if the ‘mass defect’ is small, $2m - M \ll M$,

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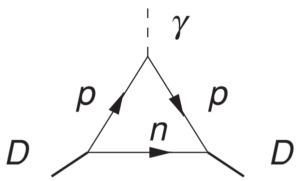
the singularity may occur very close to the physical region, $t = Q^2 \le 0$, in which case it would strongly affect the behaviour of the amplitude. A physical example is provided by a *deuteron* D – the lightest nucleus, a loosely bound state of a proton and a neutron, with the graph (2.36) describing, e.g. the electromagnetic electron–deuteron scattering:



$$M_D - (m_p + m_n) = \epsilon \ll m \quad (m = m_p \simeq m_n).$$

In non-relativistic quantum mechanics, the electron scattering amplitude

$$f(q) = \frac{e^2}{q^2} F(q)$$

is proportional to the *form factor* $F(q)$, given by the Fourier component of the distribution of electric charge inside the nucleus:

$$F(q) = \int d^3\mathbf{r}\, \psi^2(\mathbf{r})\, \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}}, \quad F(0) = 1. \tag{2.41}$$

Here ψ is the proton wave function inside the deuteron:

$$\psi(\mathbf{r}) \propto \mathrm{e}^{-r\sqrt{\epsilon m}},$$

with ϵ the binding energy. Expanding (2.41) at small momentum transfer,

$$F(q) = 1 - \mathbf{q}^2 \int d^3\mathbf{r} \frac{\mathbf{r}^2}{2} \psi^2(\mathbf{r}) + \mathcal{O}(\mathbf{q}^4) \simeq 1 - \frac{\mathbf{q}^2 \langle \mathbf{r}^2 \rangle}{2}, \quad \langle \mathbf{r}^2 \rangle \sim (\epsilon\, m)^{-1},$$

we see that the amplitude starts falling at characteristic momentum transfers of the order of the inverse deuteron radius.

On the other hand, in the relativistic-theory framework we have discussed that the interaction radius is determined by the t-channel singularities. If the triangle amplitude as a function of $t = Q^2$ had the *normal threshold* at $Q^2 = 4m^2 \gg \langle \mathbf{r}^2 \rangle^{-1}$ as the closest singularity to the physical region, this would contradict the non-relativistic expectation for a loosely bound large-size system. It is the *anomalous singularity* (2.40),

$$Q_0^2 \simeq 16\, m \cdot \epsilon \ll m^2,$$

that is responsible for a fast decrease of the elastic form factor with momentum transfer and reconciles the two approaches.

If, having started from the vicinity of $2m$, we decrease the mass M of the external particle down to $M = \sqrt{2}m$, the position of the anomalous singularity $Q_0^2(M)$ reaches its maximum, where it hits the two-particle threshold branch point, $Q_0^2 = 4m^2$, and disappears from the physical sheet of the amplitude, diving under the branch cut.

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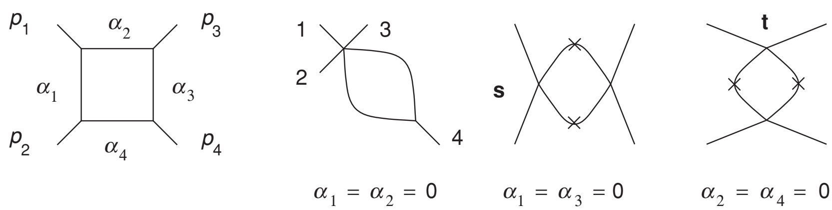
Thus, the anomalous singularity is present only inside a specific interval of masses, $2m^2 < M^2 < 4m^2$. In particular, it is absent when masses of all (external and internal) particles are the same (as in the $\lambda\varphi^3$ theory).

2.4.4 When imaginary part acquires imaginary part

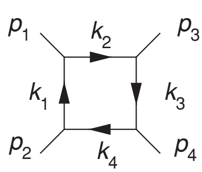
For the last example consider the diagram with four internal lines:



By setting to zero two neighbouring α parameters, we get a singularity in the virtual mass of one of the external particles; setting $\alpha_1 = \alpha_3 = 0$ reproduces the s-channel two-particle threshold, and $\alpha_2 = \alpha_4 = 0$ – the corresponding t-channel singularity.

The reduction of one line leads us back to the triangle graphs which, as we already know, may possess anomalous singularities.

Let us examine a new genuine singularity of the box graph corresponding to $\alpha_i \neq 0$. Taking for simplicity all the masses to be equal, $p_i^2 = k_i^2 = m^2$, for the products of internal momenta we have



$$\begin{aligned} 2k_1k_2 = 2k_2k_3 = 2k_3k_4 = 2k_4k_1 \;&=\; m^2\,; \\ 2k_1k_3 = 2m^2 - t\,, \quad 2k_2k_4 = 2m^2 - s\,. \end{aligned} \tag{2.42}$$

The system of Landau equations can be simplified using the symmetry of the solution following from the structure of the graph itself: $\alpha_3 = \alpha_1$, $\alpha_4 = \alpha_2$. Then the 4×4 system reduces to

$$\begin{aligned} \alpha_1 \cdot (m^2 + k_1k_3) + \alpha_2 \cdot 2k_1k_2 \;&=\; 0, \\ \alpha_1 \cdot 2k_1k_2 + \alpha_2 \cdot (m^2 + k_2k_4) \;&=\; 0. \end{aligned}$$

Evaluating the determinant,

$$\mathrm{Det} = (m^2 + k_1k_3)(m^2 + k_2k_4) - 4(k_1k_2)^2 = 0,$$

and using the kinematical relations (2.42) we obtain the equation for the *Landau surface* in the form

$$(s - 4m^2)(t - 4m^2) = 4m^4. \tag{2.43}$$

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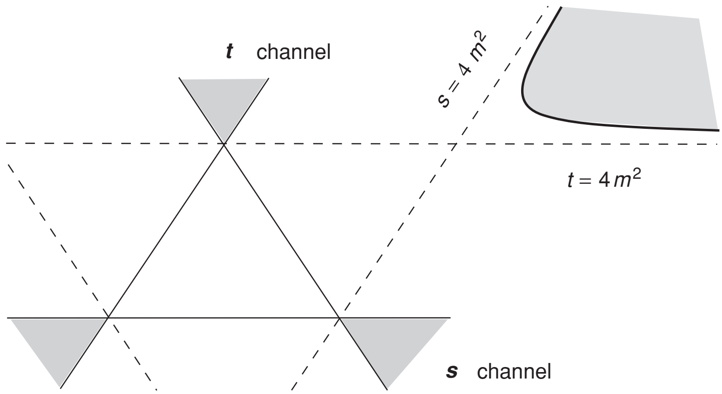
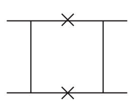


Fig. 2.8 Landau curve of the 's–t' box graph on the Mandelstam plane.

On the Mandelstam plane, it is a *hyperbola* (known as a *Karplus curve*), limited by the asymptotes $s = 4m^2$ $(t = \infty)$ and $t = 4m^2$ $(s = \infty)$. This curve is lying in the unphysical domain, 'between' the physical regions of s- and t-channel reactions, see Fig. 2.8. But for $s > 4m^2$, as well as for $t > 4m^2$, the amplitude is already complex due to the threshold singularities. Where then does the additional singularity come from and what is its meaning?

In the discussion of the analytic features of the second-order scattering amplitude and their relation to the unitarity in the $\lambda\varphi^3$ theory, we saw that for $s > 4\mu^2$ the two 'horizontal' lines may both turn on-shell, and the amplitude develops an imaginary part:

$s > 4\mu^2$:  $= 2i\,\mathrm{Im}_s\, A(s,t) \equiv A(s+i0,t) - A(s-i0,t).$ (2.44)

In the physical region of s-channel scattering, $t < 0$, the 'vertical' propagators stay always off-shell. However, the imaginary part (discontinuity) (2.44) is itself an analytic function of t. If we start to increase t, somewhere above $t = 4\mu^2$ the vertical lines will be able to go on-shell too. This happens precisely on the Landau curve (2.43) where the 'imaginary part' $\mathrm{Im}_s\, A$ becomes complex, that is develops its own 'imaginary part': $\mathrm{Im}_t\, \mathrm{Im}_s\, A(s,t) \neq 0$.

2.4.5 Character of singularities

Let us find out the way the amplitude behaves near the singularity, that is the *character* of the latter. To do that we return to the original integral (2.23). As we know, when s approaches the Landau surface,

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$s \to s_0(t, u, \ldots)$, the characteristic function $\square$ defined in (2.24) develops a simultaneous extremum in integration momenta, $q = q^{(0)}$, and in $n-1$ independent Feynman parameters, $\alpha = \alpha^{(0)}$. It may be represented therefore as follows:

$$\square = \sum_{ij}^{4\ell}(q_i - q_i^{(0)})(q_j - q_j^{(0)})a_{ij} + \sum_{ij}^{n-1}(\alpha_i - \alpha_i^{(0)})(\alpha_j - \alpha_j^{(0)})b_{ij} + \gamma(s_0 - s).$$

Let us rescale the integration variables as

$$q_i - q_i^{(0)} = \sqrt{s_0 - s} \cdot y_i\,, \quad \alpha_i - \alpha_i^{(0)} = \sqrt{s_0 - s} \cdot \beta_i.$$

Then the s-dependence of the integral factors out,

$$A_{n\ell} \sim \frac{(s_0 - s)^{\frac{4\ell}{2}}\,(s_0 - s)^{\frac{n-1}{2}}}{(s_0 - s)^n} \int \frac{d^4y_1 \cdots d^4y_\ell\, d\beta_1 \cdots d\beta_{n-1}}{\square^{\,n}(y, \beta)},$$

$$\square\,(y, \beta) = \sum_{ij}^{4\ell} a_{ij} y_i y_j + \sum_{ij}^{n-1} b_{ij}\beta_i\beta_j + \gamma,$$

giving

$$A_{n\ell} = \left(\sqrt{s_0 - s}\right)^{4\ell - n - 1} \cdot \mathcal{N}. \tag{2.45}$$

Due to finite limits of the integrals over αs, the factor $\mathcal{N}$ here may have some residual s-dependence. With $s \to s_0$, however, the integration range in βs expands,

$$|\Delta\beta_i| \;\propto\; 1/\sqrt{s_0 - s} \;\to\; \infty.$$

Hence the integrals over β_i may be replaced by the s-independent integration over $-\infty < \beta_i < +\infty$, *provided* the multiple y–β integral so obtained does not diverge. A simple power counting shows that the integral for $\mathcal{N}$ converges when

$$(4\ell + n - 1) - 2n = 4\ell - n - 1 \;< 0.$$

Looking at (2.45) we conclude that the latter condition is equivalent to the amplitude $A_{n\ell}$ *increasing* towards the singularity. If this is the case, one has two possibilities for the character of the singularity, depending on the value of the characteristic exponent

$$\mathcal{E} \equiv 4\ell - n - 1, \tag{2.46}$$

namely:

(1) the branching point when $\mathcal{E}$ is *odd*, $\mathcal{E} = -(2k+1)$,

$$A_{n\ell} \propto (s_0 - s)^{-k-\frac{1}{2}} \quad (k \geq 0); \tag{2.47a}$$

(2) the pole in $(s_0 - s)$ of degree k for the case of *even* $\mathcal{E} = -2k$:

$$A_{n\ell} \propto (s^0 - s)^{-k} \quad (k \geq 1). \tag{2.47b}$$

What kind of singularity appears when $\mathcal{E} \geq 0$ and the integral diverges? To answer the question we should treat not the amplitude $A_{n\ell}$ itself, but its s-derivative:

$$\frac{d^m A_{n\ell}}{ds^m} \sim \int \frac{d^4 q_1 \cdots d^4 q_\ell \, d\alpha_1 \, d \cdots d\alpha_{n-1}}{\square^{n+m}} \, (-\gamma)^m \frac{(n+m)!}{n!} \, .$$

By choosing a proper value of m, we make the integral convergent and can repeat the above analysis to obtain

$$\frac{d^m A_{n\ell}}{ds^m} = \text{const.} \cdot (s_0 - s)^{\frac{4\ell-n-1}{2} - m}.$$

Now we integrate m times over s to restore the amplitude and face two cases as before:

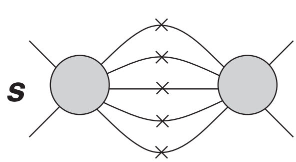
(1) the branching point singularity when $\mathcal{E} = 2k+1 \quad [m = k]$,

$$A_{n\ell} \propto (s_0 - s)^{k+\frac{1}{2}} \qquad (k \geq 0)\,; \tag{2.48a}$$

(2) the logarithmic singularity for $\mathcal{E} = 2k \quad [m = k+1]$,

$$A_{n\ell} \propto (s_0 - s)^k \, \ln(s_0 - s) \qquad (k \geq 0). \tag{2.48b}$$

Example 1. Return to multi-particle threshold singularities (2.19). Here we have n internal lines and $\ell = n-1$ independent momentum integrations. The exponent (2.46) equals $\mathcal{E} = 3n - 5$, and when the number of particles in the intermediate state is *even*, the amplitude near the threshold singularity behaves as



$$A \sim (s_0 - s)^{\frac{3}{2}(n-2)+\frac{1}{2}} \qquad (n \ \text{even}). \tag{2.49a}$$

For $n = 2$ we recover the known square-root singularity characterizing the two-particle threshold. If the number of particle is *odd*, for $n = 1$ our formula gives a simple particle pole, $A \sim (m^2 - s)^{-1}$; otherwise the exponent

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is a positive integer, and the threshold singularity becomes logarithmic:

$$A \sim (s_0 - s)^{\frac{3}{2}(n-1)-1} \cdot \ln(s_0 - s) \qquad (n \text{ odd}). \tag{2.49b}$$

This result is easy to get directly from the unitarity relation:

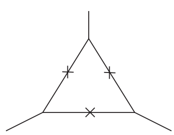
$$2\,\mathrm{Im}\,A \propto \int \frac{d^4k_1 \cdots d^4k_n}{(2\pi)^{3n}} (2\pi)^4 \delta^4\big(p - \Sigma\, k_i\big) \prod_{i=1}^{n} \delta(k_i^2 - m_i^2).$$

Near the threshold, three-momenta of all particles are small, and the integration produces the non-relativistic phase space volume:

$$\sim \prod_{i=1}^{n-1} d^3\mathbf{k}_i \cdot \delta\left(\sqrt{s} - \sqrt{s_0} - \sum_{i=1}^{n} \frac{\mathbf{k}_i^2}{2m_i}\right) \sim |\mathbf{k}|^{3\cdot(n-1)-2}; \quad \sqrt{s_0} = \sum_{i=1}^{n} m_i\,.$$

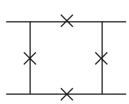
Each additional particle brings in the suppression factor $|\mathbf{k}|^3 \sim (s_0 - s)^{3/2}$, and the singularity weakens.

Example 2. Anomalous 'vertex' singularity:



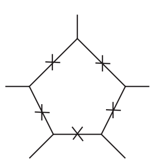
$$n = 3,\ \ \ell = 1, \quad \mathcal{E} = 4 \cdot 1 - 3 - 1 = 0; \quad A \sim \ln(s_0 - s).$$

Example 3. Box diagram:



$$n = 4,\ \ \ell = 1, \quad \mathcal{E} = 4 \cdot 1 - 4 - 1 = -1; \quad A \sim \frac{1}{\sqrt{s_0 - s}}.$$

Example 4. A five-leg amplitude possesses a genuine *pole* singularity:



$$n = 5,\ \ \ell = 1, \quad \mathcal{E} = 4 \cdot 1 - 5 - 1 = -1; \quad A \sim \frac{1}{s_0 - s}.$$

We observe that the strength of the singularity *grows* when we increase the number of lines in the loop. According to the Landau rules, multi-leg one loop amplitudes would seem to develop stronger and stronger singularities: $A_{61} \sim (s - s_0)^{-3/2}$, $A_{71} \sim (s - s_0)^{-2}$, etc.

Would they? Rather not. It is most likely that the strongest singularity a Feynman diagram may have is a *pole* and the reason is the following.

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While studying the equation for the Landau surface that determines the position of singularities,

$$f(\{s_{ik}\}) = 0\,, \quad 1 \le i \ < \ k \le N-1 \tag{2.50}$$

(with N the number of external momenta), we were treating the Lorentz invariants s_{ik} that characterize the amplitude as *independent* variables. The number of *linearly independent* pair products of the external momenta $p_i p_k$ (not counting the masses, $i \neq k$) in (2.50) is

$$\frac{(N-1)(N-2)}{2} - 1, \tag{2.51a}$$

where the subtracted unity stands for the additional on-mass-shell relation, $m_N^2 = s_{NN}$ for the last particle with momentum $p_N = -\sum_{i=1}^{N-1} p_i$. At the same time, we have calculated in (1.33) the total number of Lorentz invariant variables, the N-point amplitude depends on:

$$3N - 10. \tag{2.51b}$$

The two numbers (2.51) coincide for $N = 5$, which is just the case of the pole singularity in the Example 4 above.

If we take $N > 5$, the number of the pair products (2.51a) takes over, which means that certain *kinematical relations* between s_{ik} appear, undermining our analysis of the character of singularities.

What remains to be done about the character of Landau singularities is to verify the case when some of the αs are zero.

Consider a singularity that emerges when $\alpha_1 = 0$. Then there is no extremum with respect to α_1, and the function $\square$ has the expansion

$$\square \simeq c\alpha_1 + \sum_{i,k}^{n-1} (\alpha_i - \alpha_i^{(0)})(\alpha_k - \alpha_k^{(0)}) b_{ik} + \cdots + \gamma(s_0 - s). \tag{2.52}$$

Near the singularity, the characteristic magnitude of α_1 under the integral, $\alpha_1 \sim (s_0 - s)$, is *much smaller* than all other deviations, $|\alpha_i - \alpha_i^{(0)}| \sim \sqrt{s_0 - s}$. Therefore α_1 can be neglected in the quadratic form in (2.52), as well as in the sum,

$$\delta\left(1 - \sum_{i=1}^{n} \alpha_i\right) \simeq \delta\left(1 - \sum_{i=2}^{n} \alpha_i\right).$$

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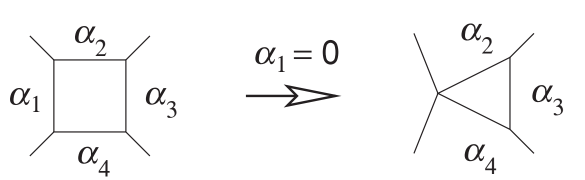
Hence, we obtain an expression identical to that for the amplitude without the line k_1, α_1:

$$\int_0^1 d\alpha_2 \cdots d\alpha_n \int \frac{d\alpha_1\, \delta\big(1 - \sum_{i=1}^n \alpha_i\big)}{\big(c\alpha_1 + \square_{\,\alpha_1=0}\big)^n}$$

$$\Longrightarrow \int_0^1 \frac{d\alpha_2 \cdots d\alpha_n}{\square^{\,n-1}}\, \delta\left(1 - \sum_{i=2}^n \alpha_i\right).$$

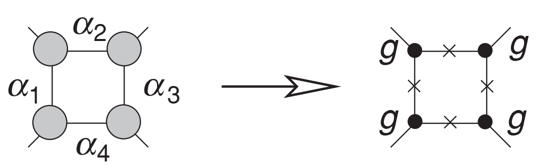
Thus, not only the position of the singularity but also its character can be derived from a *reduced graph* with the line α_1 contracted:



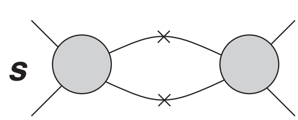
2.4.6 Amplitude near singularity

Now that we learnt how to determine the position and the character of the singularities, let us address the question of the *magnitude* of the amplitude near a singularity.

Let us take some complicated amplitude and set to zero all αs but four, to form a square graph. At the corresponding Landau singularity, all internal particles are on the mass shell, $k_i^2 = m_i^2$, therefore the full sub-amplitudes that determine their interaction vertices may each be equated with the renormalized on-mass-shell interaction constant g:



If we consider instead a simpler threshold branch-cut singularity, its magnitude will be determined by the square of the physical scattering amplitude near the threshold:



$$\propto A^2\,(s = s_0). \tag{2.53}$$

In general, if we want to calculate the magnitude of an arbitrary Feynman diagram near the singularity, it will be always determined by the exact on-mass-shell interaction amplitudes.

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2.5 Beyond perturbation theory: relation to unitarity

By using the language of Feynman diagrams we have arrived at the pattern of singularities of the interaction amplitudes.

Namely, for each singularity:

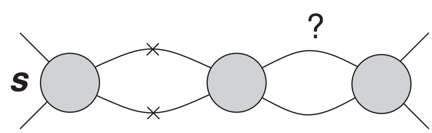
(1) its position is determined by masses of real hadrons;

(2) its character derives from the topology of the interaction process;

(3) the coefficient in front of a singularity is expressed in terms of the physical on-mass-shell amplitudes.

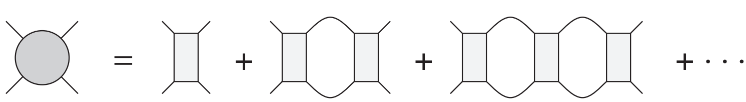
This conclusion goes beyond the perturbation theory which we have employed to derive it. The reason for that lies in the *unitarity* property: the series of Feynman diagrams (though having little sense in a theory of strongly interacting objects) formally solve the unitarity conditions.

The blocks in (2.53) are supposed to correspond to *exact* two-particle scattering amplitudes which become complex themselves when we move above the threshold. The particles in the intermediate state are allowed to interact many times: as a result, the threshold singularities overlay.



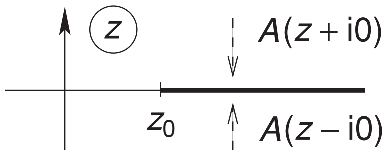
How to treat such an eventuality?

To take into account successive two-particle scatterings in the full amplitude, let us single out the block that has no two-particle intermediate state and, therefore, no threshold at $s = 4m^2$:


(2.54)

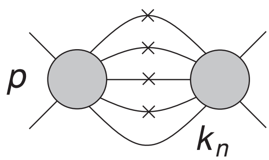
According to Landau rules, to find the s-channel threshold singularity, we must pick one of the two-particle states and put the two lines on the mass shell. Then the chains of two-particle irreducible blocks sum up into the full amplitudes, on the left and on the right from the 'cut', resulting in (2.53). A branch cut singularity of the function is characterized by the *discontinuity* across the cut:

$$\Delta A(z) = \frac{1}{2i}\big[A(z+i0) - A(z-i0)\big].$$



This is what we use to call the '*imaginary part*', cf. (2.20). The name is justified when the function below $z = z_0$ is *real*; hence it assumes complex conjugate values on the sides of the cut: $A(z - i0) = [A(z + i0)]^*$.

Integrating over energy components of $n - 1$ loop momenta, we may close the contours around the positive energy poles of all but one intermediate state particles:



$$\propto \frac{1}{m_n^2 - k_{0n}^2 + \mathbf{k}_n^2}. \tag{2.55}$$

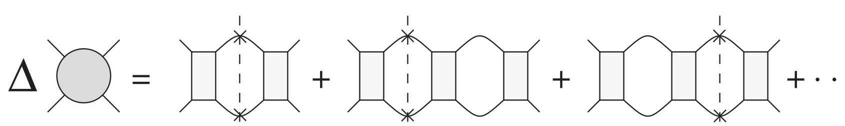
The last line we must also put on the mass shell by replacing the remaining propagator (2.55) by $2\pi\delta(m_n^2 - k_n^2)$. Using the energy conservation, $k_{0n} = p_0 - \sum_{i=1}^{n-1} k_{0i} > 0$, this procedure is equivalent to evaluating *discontinuity* of the amplitude with respect to the incoming energy p_0:

$$\frac{1}{2i}\left[\frac{1}{m_n^2 - (p_0 + i\epsilon - \sum k_{0i})^2 + \mathbf{k}_n^2} - \frac{1}{m_n^2 - (p_0 - i\epsilon - \sum k_{0i})^2 + \mathbf{k}_n^2}\right].$$

The amplitude (2.54) has symbolically the structure of the product:

$$A(s) = \sum_{n=1}^{\infty} F_1 \cdot F_2 \cdot \cdots \cdot F_n\,.$$

To find the discontinuity of the iterated amplitude,



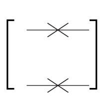
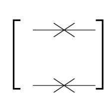
we must calculate

$$A_n(s + i0) - A_n(s - i0) = (F^+)^n - (F^-)^n, \quad \text{with } F^{\pm} = F(s \pm i0).$$

An evaluation of the discontinuity of the product of functions is algebraically similar to taking a derivative:

$$\Delta(AB) = A \cdot (\Delta B) + (\Delta A) \cdot B^*;$$

iterating this rule we obtain

$$\Delta A = \sum_{i,k=1} F_1^+ \cdots F_i^+$$

$$F_1^- \cdots F_k^- = A(s + i0)$$

$$A(s - i0).$$

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The r.h.s. of this expression is real, correcting (2.53).

Summing together the discontinuities of the amplitude across n-particle threshold branchings, we finally derive

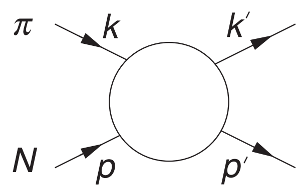
$$\Delta A_{2\to 2}(s) \;=\; \sum_n \tau_n(s) A_{2\to n}(s) A^*_{2\to n}(s)\,, \tag{2.56}$$

which is nothing but the unitarity relation, with τ_n the n-particle phase space volume.

2.6 Checking analytic properties of physical amplitudes

We will conclude the discussion of analyticity in this lecture by considering two practically important examples.

2.6.1 Dispersion relation for forward πN scattering



Consider pion–nucleon scattering. Due to the isotopic symmetry of strong interactions, πN interaction amplitude depends on the total isospin of the system, rather than on the individual isospin state (electric charge) of each of the participating particles. Pions π^+, π^0, π^- form an isotopic triplet ($I = 1$), and nucleons p, n – the doublet ($I = \frac{1}{2}$). Therefore, the full amplitude contains two independent functions describing the interaction: $1 \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$, or, from the t-channel point of view, $\pi\pi = 1 \otimes 1 = 0 \oplus 1$ (the $\bar{N} + N$ pair cannot have isospin 2).

We will study the *forward* pion–nucleon scattering, $t = 0$:

$$s = (p+k)^2 = M^2 + \mu^2 + 2M\nu,$$
$$u = (p-k')^2 = M^2 + \mu^2 - 2M\nu = 2(M^2+\mu^2) - s,$$

with ν the energy of the pion, $p_0 = p_0'$, in the rest frame of the target nucleon.

Let $\mathbf{U}$ denote the dublet of nucleons, and $\boldsymbol{\phi}_\alpha$ the isovector pion field, $\alpha = 1, 2, 3$. The general form of the scattering amplitude is

$$A = \overline{\mathbf{U}}(p')\boldsymbol{\phi}_\alpha(k')\big(f_+(\nu)\delta_{\alpha\beta}\cdot \mathbf{I} + f_-(\nu)\varepsilon_{\alpha\beta\gamma}\cdot \boldsymbol{\tau}_\gamma\big)\boldsymbol{\phi}_\beta(k)\mathbf{U}(p), \tag{2.57}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the anti-symmetric unit tensor, and $\boldsymbol{\tau}_\alpha$ is the triplet of Pauli matrices in the 2×2 space of nucleon isospinors. The diagonal term proportional to the unit matrix $\mathbf{I}$ takes care of elastic scattering, while

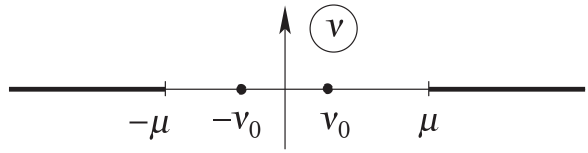


Fig. 2.9 Complex pion energy plane for forward πN scattering amplitude.

the $\boldsymbol{\tau}$ term anti-symmetric in isospin indices is responsible for reactions with electric charge transfer, like $\pi^+ n \to \pi^0 n$.

The variable

$$\nu = \frac{s-u}{2M} = \frac{s-(M^2+\mu^2)}{M}$$

changes sign under the permutation of the initial and final pions, $s \leftrightarrow u$. Therefore, since pions are Bose particles, the amplitudes $f_\pm$ in (2.57) are, correspondingly, even and odd with respect to the crossing:

$$f_+(-\nu) = f_+(\nu), \quad f_-(-\nu) = -f_-(\nu).$$

We will study the symmetric part, f_+. The amplitude has a nucleon pole,

$$s = M^2 \implies \nu_0 = -\frac{\mu^2}{M},$$

and the branch cut that starts from the πN threshold:

$$s = (M+\mu)^2 \implies \nu_{\text{thresh}} = \mu.$$

The u-channel singularities mirror the s-channel ones on the complex plane of the variable ν, see Fig. 2.9. Let us try to write down the dispersion relation for $f = f_+$ as a function of the complex variable ν:

$$\begin{aligned} f(\nu) = \frac{r}{\nu_0 - \nu} + \frac{1}{\pi}\int_\mu^\infty \frac{d\nu' \,\mathrm{Im}\, f(\nu')}{\nu' - \nu} \\ + \frac{r}{\nu_0 + \nu} + \frac{1}{\pi}\int_{-\mu}^{-\infty} \frac{d\nu' \,\mathrm{Im}\, f(\nu')}{\nu' - \nu}. \end{aligned} \tag{2.58a}$$

Combining the contributions of two poles and s- and u-channel cuts, with account of $f(-\nu) = f(\nu)$, we get a more compact expression

$$f(\nu) = \frac{2r\,\nu_0}{\nu_0^2 - \nu^2} + \frac{1}{\pi}\int_\mu^\infty \frac{d\nu'^2 \,\mathrm{Im}\, f(\nu')}{\nu'^2 - \nu^2}. \tag{2.58b}$$

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This relation makes sense only if the integral converges at $\nu' \to \infty$. The optical theorem (1.32) tells us that

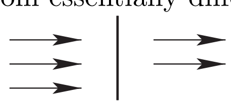
$$\operatorname{Im} A(\nu) = \tfrac{1}{2} J \sigma_{\text{tot}}(\nu) = 2M|\mathbf{k}| \cdot \sigma_{\text{tot}}(\nu), \quad |\mathbf{k}| = \sqrt{\nu^2 - \mu^2},$$

where we have calculated the invariant flux (1.26) in the nucleon rest frame, $J = 4M|\mathbf{k}_\pi|$. We conclude that the amplitude actually *grows* as $A \propto \nu$, since the total cross section is approximately constant at large collision energies. Therefore we must modify the dispersion relation by performing the subtraction, $f \to f(\nu) - f(0)$:

$$f(\nu) = f(0) \;+\; \frac{2r}{\nu_0} \frac{\nu^2}{\nu_0^2 - \nu^2} \;+\; \frac{\nu^2}{\pi} \int_\mu^\infty \frac{d\nu'^2}{\nu'^2} \frac{\operatorname{Im} f(\nu')}{(\nu'^2 - \nu^2)}\,. \tag{2.59}$$

Since the imaginary part of the amplitude is directly related to the total cross section, $\operatorname{Im} f = 2Mk\,\sigma_{\text{tot}}$, once we have measured the total cross section, we know the integrand and may restore the amplitude which is a measurable quantity itself.

Importantly, information about the amplitude and cross section comes from essentially different sources. The total cross section $\sigma_{\text{tot}}(\nu)$ can be



accessed by observing the *loss* of particles by an incident beam. On the other hand, the amplitude is extracted from a completely different experiment. One measures differential angular distribution of the elastic scattering and reconstructs the amplitude from the Legendre expansion:

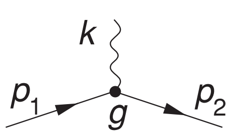
$$f(\nu, \Theta) = \sum_{\ell=1}^{\infty} f_\ell(\nu)\, P_\ell(\cos\Theta).$$

The integral of the cross section converges, so that the main contribution comes from not too large energies.

Confronting the experimental information on the forward amplitude, $f(\nu, 0)$, and $\sigma_{\text{tot}}(\nu)$ the relation (2.59) was found to hold. This is a verification of the analyticity.

The dispersion relation allows one to determine experimentally the value of the residue r. But the residue in the pole of the amplitude is the *renormalized coupling constant*, in the field-theory framework.

Let us take a neutral meson π^0 and try to model the pion–nucleon interaction vertex in the QFT language:



$$= \; g \cdot \overline{\mathbf{U}}(p_2)\, i\gamma_5 \mathbf{U}(p_1) \cdot \boldsymbol{\phi}(k).$$

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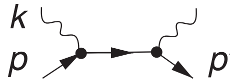
(We have included into the vertex the factor $i\gamma_5$ since π is pseudoscalar.) The pole diagram constructed on the base of this vertex reads

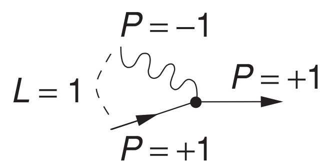
$A_{\text{pole}} =$ 

$$= -g^2\,\overline{\mathbf{U}}(p')\gamma_5 \frac{1}{M-(\hat{p}+\hat{k})}\gamma_5 \mathbf{U}(p)$$
$$= -g^2\overline{\mathbf{U}}\frac{\gamma_5(M+\hat{p}+\hat{k})\gamma_5}{M^2-s}\mathbf{U} = -g^2\overline{\mathbf{U}}\frac{M-\hat{p}-\hat{k}}{M^2-s}\mathbf{U}. \tag{2.60}$$

Using the Dirac equation, $(M-\hat{p})\mathbf{U}(p)=0$,

$$A_{\text{pole}} \;=\; \overline{\mathbf{U}}(p')\frac{\hat{k}}{M^2-s}\mathbf{U}(p)\,. \tag{2.61}$$

The pole amplitude has an interesting feature: why there is only a pion momentum in the numerator of the amplitude? The pion has a negative internal parity; to be allowed to fuse into a nucleon, the incident pion has to have an *odd* orbital momentum, $L=1$, in order to match the spatial parity of the πN pair, $P_\pi P_N(-1)^L$, to that of the nucleon, P_N. The p-wave amplitude must be proportional to $\mathbf{k}$, explaining the structure of the pole amplitude (2.61).



In the forward scattering limit, $p \simeq p'$, we have

$$\begin{aligned} \overline{\mathbf{U}}(p)\gamma_\mu\mathbf{U}(p) &= 2p_\mu, \\ \overline{\mathbf{U}}(p)\hat{k}\mathbf{U}(p) \equiv k^\mu\,\overline{\mathbf{U}}(p)\gamma_\mu\mathbf{U}(p) &= 2kp = s - M^2 - \mu^2, \end{aligned} \tag{2.62}$$

and in the numerator of the Born amplitude there appears the combination $s-M^2$ which cancels the pole. The remaining *true pole* contribution becomes

$$A_{\text{pole}} \Longrightarrow -\frac{g^2\,\mu^2}{M^2-s} = -\frac{g^2\,\mu^2}{M}\,\frac{1}{\nu_0-\nu}, \quad \nu_0 = -\frac{\mu^2}{M}.$$

Comparing with the pole term in the dispersion relation (2.58a), we relate the residue r with the coupling constant g as

$$r \;=\; -g^2\mu^2/M\,. \tag{2.63}$$

We may roughly estimate the magnitude of the residue. Since the interaction is strong, its cross section is determined by the interaction radius, the latter being inverse proportional to the mass of the lightest hadron – the π meson: $\sigma_{\text{tot}} \sim \mu^{-2}$. Taking moderate pion energies of the order of

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its mass, $\nu \sim |\mathbf{k}| \sim \mu$, the estimate follows:

$$f \sim \operatorname{Im} f = 2M|\mathbf{k}|\sigma_{\text{tot}} \sim 2M\mu \cdot \frac{1}{\mu^2} = \frac{2M}{\mu}.$$

Suppose that the contributions of the pole and of the cut to the dispersion relation are of the same order:

$$\frac{r}{\nu_0 - \nu} \sim f\,; \quad -\frac{r}{\mu} \sim \frac{2M}{\mu} \quad \Longrightarrow \quad r \simeq -2M.$$

From (2.63) we then have

$$g^2 \simeq 2M^2/\mu^2 \simeq 100.$$

Real experimental measurement of the residue r yields

$$g^2/4\pi \simeq 14.$$

The closeness of the two numbers shows that indeed the pole term and the dispersion integral over the cut contribute equally.

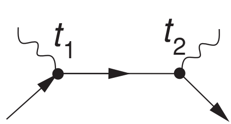
Thus, although there is certainly no perturbation theory, we can obtain the pole term in the dispersion relation from the first graph (2.60).

It should be stressed that the pole term differs essentially from the Feynman diagram. Indeed, if we *did not drop* in the numerator of the Born amplitude (2.62) the piece $(s - M^2)$ which cancels the pole,

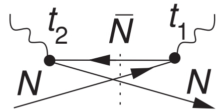
$$\frac{s - M^2 - \mu^2}{M^2 - s} = -\frac{\mu^2}{M(\nu_0 - \nu)} - 1,$$

the new estimate based on the *full amplitude* would have been $M/\mu \simeq 7$ times larger!

Is there a reason why the pure pole term gives a reasonable size contribution while the estimate based on the Feynman diagram fails?



Recall the meaning of a Feynman graph, in the space–time language. The pion–nucleon interaction graph actually incorporates *two* space–time configurations, including the one with the *inverse time ordering*, $t_2 < t_1$, which corresponds to three coexisting nucleons. It is clear, however, that for moderate energies, such a state has nothing to do with the process we are considering. The proton mass is very large (compared to that of the pion) and long before the $NN\bar{N}$ state there will be many additional pions present in the intermediate state.



When the coupling of the field theory is small, the expansion of the amplitude is organized in powers of this coupling. In particular, in QED

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it is easy to verify that the Compton scattering amplitude $\gamma e \to \gamma e$ is dominated just by the 'time inverted' Born diagram.

However, if the interaction is strong, so that the coupling is large and the perturbative expansion makes no sense, it is not the first-order graph but the *nearest singularities* that determine the answer.

This observation constitutes the core idea of the dispersive approach.

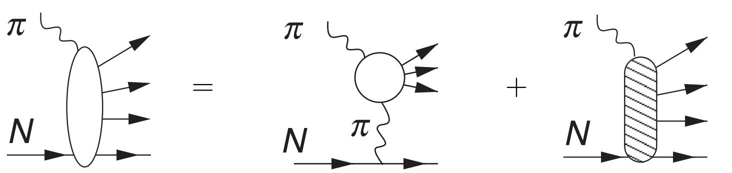
2.6.2 Chew–Low method

What can we measure directly? We actually have only one stable target – the proton. Plus a relatively stable neutron. All other hadrons are unstable. Nevertheless, we are able to prepare a beam of some unstable particles and scatter them off a proton, for example.

But how to measure an amplitude when the projectile and the target are both unstable, for example that of $\pi\pi$ scattering? One has to undertake a flanking manoeuvre in order to extract it from available data (Chew and Low, 1959).

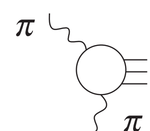
Suppose a pion scatters off a nucleon, producing some hadron state. Could we single out some part of the process into which $\pi\pi$ interaction amplitude, of even cross section, would enter?

Such a diagram is easy to invent:



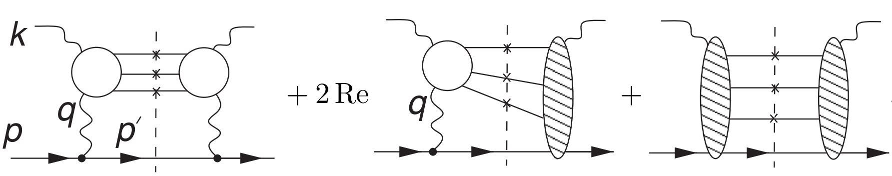
(2.64)

The first diagram on the r.h.s. of (2.64) contains a pion exchanged in the t-channel, and the upper block of this diagram can be looked upon as pion–pion scattering amplitude.



The problem is, how to see and extract this particular term from the background of all other possible contributions. How to do that when the perturbative approach is not applicable? We must look for specific features of the pion exchange amplitude that might help us in our task.

Let us square the amplitude and look at the contributions to the cross section:



What is there remarkable about the first graph? If we do not integrate over the recoil nucleon momentum p' but measure the differential cross section in p', then this graph will contain the square of the pion propagator at given momentum transfer

$$\frac{d\sigma}{dt} \propto \left(\frac{1}{\mu^2 - t}\right)^2, \qquad t = (p' - p)^2.$$

(The interference term contains the pion propagator in the first power, $1/(\mu^2 - t)$, and the third one has none.) We may adopt the specific sharp t-dependence of the cross section as the means for extracting the pion exchange.

Let us focus on the t dependence in the sense of analytic properties.

Estimate of momentum transfer. For the momentum transfer we have

$$-q^2 = -(p - p')^2 = 2(p_0 p_0' - \mathbf{p} \cdot \mathbf{p}' - m^2), \tag{2.65}$$

where we made use of the on-mass-shell conditions $p^2 = p'^2 = m^2$. Here m is the proton mass, and the capital M we reserve for the invariant mass of the produced hadron system (minus the recoil nucleon): $M^2 = (k + q)^2$.

It is clear that the physical region corresponds to *negative* $t = q^2$. Indeed, since q^2 is Lorentz invariant, we can calculate it in an arbitrary reference frame. In the 'laboratory frame' where the nucleon target is at rest, $\mathbf{p} = 0$, (2.65) reduces to

$$-q^2 = 2m(p_0' - m) > 0,$$

which expression is obviously positive (the energy p_0' of the recoiling nucleon cannot be smaller than its mass m).

What is a typical momentum transfer? In particular, how small the virtuality of the exchanged pion may be? One can derive $\left|q^2\right|_{\min}$ from the kinematical relations at our disposal:

$$\begin{aligned} s &= (k + p)^2 = \mu^2 + m^2 + 2(kp); \\ M^2 &= (k + q)^2 = \mu^2 + 2(kq) + q^2, \\ m^2 &= (p - q)^2 = m^2 - 2(pq) + q^2. \end{aligned}$$

The resulting expression is rather cumbersome. It is important, however, to remark that $\left|q^2\right|_{\min}$ becomes *extremely small* at high collision energies.

To see this we introduce the *rapidity* variable,

$$p_0 = m\cosh\eta, \;\; |\mathbf{p}| = m\sinh\eta; \quad \eta = \ln\frac{p_0 + |\mathbf{p}|}{m},$$

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and represent (2.65) as

$$\begin{aligned} \left|q^2\right| &= 2m^2(\cosh\eta\cosh\eta' - \sinh\eta\sinh\eta'\cdot\cos\Theta - 1) \\ &= 4m^2\left[\cosh^2\tfrac{1}{2}(\eta-\eta') \;+\; \sinh\eta\sinh\eta'\cdot\sin^2\tfrac{1}{2}\Theta\right], \end{aligned} \tag{2.66}$$

where Θ is the nucleon scattering angle in the chosen frame of reference. In the πN centre of mass system the variation of Θ does not affect the modulus of the nucleon's three-momentum $|\mathbf{p}'| = p'_c$ and, therefore, the rapidity η'. This makes it obvious that the minimum of $\left|q^2\right|$ in (2.66) corresponds to $\Theta_c = 0$. Now we recall the expression (1.27) for the cms momentum as a function of masses:

$$\begin{aligned} p_c\cdot 2\sqrt{s} &= \sqrt{s^2 - 2(m^2+\mu^2)s + (m^2-\mu^2)^2}\,, \\ p'_c\cdot 2\sqrt{s} &= \sqrt{s^2 - 2(m^2+M^2)s + (m^2-M^2)^2}\,. \end{aligned}$$

Calculating the cms energies of the initial- and final-state nucleons,

$$E_N\cdot 2\sqrt{s} = s + m^2 - \mu^2, \quad E'_N\cdot 2\sqrt{s} = s + m^2 - M^2,$$

we then construct the difference of rapidities,

$$\eta_c - \eta'_c = \ln\frac{E_N+p_c}{E'_N+p'_c} = \ln\frac{(s+m^2-\mu^2+\cdots)\;+\;(s-m^2-\mu^2+\cdots)}{(s+m^2-M^2+\cdots)\;+\;(s-m^2-M^2+\cdots)}.$$

In the large s limit $(s \gg M^2, m^2)$ this gives

$$\frac{\eta_c-\eta'_c}{2} \;=\; \frac{M^2-\mu^2}{s} \;+\; \mathcal{O}\big(s^{-2}\big)\,,$$

so that

$$\left|q^2\right| \geq \left|q^2\right|_{\min} \simeq 4m^2\left(\frac{M^2-\mu^2}{s}\right)^2. \tag{2.67}$$

We see that the virtuality q^2 can actually be very small. Still, it is *negative* while in order to extract the $\pi\pi$ interaction amplitude we need to find the residue of the second-order pole at the *positive* virtuality $q^2 = \mu^2$. Could this be done? It is clear that, mathematically speaking, this is not a well defined problem. We have to find specific conditions under which the pole diagram gives a *significant* contribution to the physical cross section, because if it is small in the physical region, no extrapolation would help us to extract it!

Pion exchange contribution. We know how to calculate the cross section. We average over the initial (and sum over final) nucleon spin, sum over

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all final states and write

$$\begin{aligned}\sigma_{\pi N} \quad = \quad & \frac{1}{J}\sum_n \int \frac{d^4p'}{(2\pi)^3}\delta_+(p'^2-m^2)\prod_{i\le n}\frac{d^4k_i}{(2\pi)^3}\delta_+(k_i^2-m_i^2) \\ & \times (2\pi)^4\delta\left(p+k-p'-\sum_i k_i\right)\cdot\left(\frac{1}{\mu^2-q^2}\right)^2\cdot g_{\pi N}^2 \qquad (2.68) \\ & \times \frac{1}{2}\,\mathrm{Tr}\big[\,(m+\hat{p})\,i\gamma_5(m+\hat{p}')\,i\gamma_5\,\big]\cdot A_{kq}(\{k_i\})A_{kq}^*(\{k_i\}).\end{aligned}$$

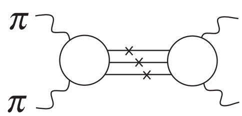
Here A_{kq} marks the amplitude of the $\pi\pi$ interaction with the production of n particles with momenta $\{k_i\}$.

The nucleon trace gives

$$\tfrac{1}{2}\,\mathrm{Tr}\big[\,(m+\hat{p})\,i\gamma_5(m+\hat{p}')\,i\gamma_5\,\big] = -q^2.$$

Collecting all the ingredients that depend on the momentum q, we observe that by virtue of the unitarity relation they combine into the imaginary part of the forward scattering amplitude of a real pion k and the *virtual* pion q:

$$2\,\mathrm{Im}\,A_{\pi\pi} =$$



$$\begin{aligned} = & \sum_n \int \prod_{i\le n}\frac{d^4k_i}{(2\pi)^3}\delta_+(k_i^2-m_i^2) \\ & \times (2\pi)^4\delta^4\left(q+k-\sum_i k_i\right)\cdot A_{kq}(\{k_i\})A_{kq}^*(\{k_i\}).\end{aligned} \qquad (2.69)$$

We may write

$$\frac{d\sigma_{\pi N}}{d^4q} = \frac{g_{\pi N}^2}{(2\pi)^3 J}\delta(q^2-2pq)\frac{-q^2}{(\mu^2-q^2)^2}\times 2\,\mathrm{Im}\,A_{\pi\pi}((k+q)^2;q^2), \quad (2.70)$$

where $(k+q)^2$ is the invariant energy of the $\pi\pi$ collision. Strictly speaking, we must keep q^2 as the argument of the $\pi\pi$ amplitude which may depend on the pion virtuality. However, in the vicinity of the pole we may substitute $q^2=\mu^2$ everywhere in the numerator of (2.70). Then enters the true physical pion scattering amplitude, $A_{\pi\pi}((k+q)^2;q^2)\to A_{\pi\pi}((k+q)^2;\mu^2)$, and we have

$$\frac{d\sigma_{\pi N}^{\text{pole}}}{d^4q} = \frac{g_{\pi N}^2}{(2\pi)^3 J}\delta(q^2-2pq)\frac{-\mu^2}{(\mu^2-q^2)^2}\times 2\,\mathrm{Im}\,A_{\pi\pi}((k+q)^2). \quad (2.71)$$

The first sad fact: in the physical region we had $-q^2>0$, and the cross section (2.68) was positive, while now we have $\sigma\to-\infty$ in the pole! So the residue in the form of (2.71) makes little sense.

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Actually, the contribution of the pion exchange *vanishes* at $q^2 = 0$, as the original expression (2.70) for the cross section shows. This property is essential. If in the experiment it were observed that with the decrease of $|q^2|$ the cross section remained large, this would have meant that our pole graph was insignificant!

Another important property follows from the fact that the exchanged pion is spinless. Therefore it can transfer no information about the direction of $\mathbf{q}$ into the upper block. This means that the distribution of secondary particles must be *isotropic* in the cms of the $\pi\pi$ pair (the distribution in the so-called Treiman–Yang angle).

I have described two checks of whether the one-pion exchange term σ^{pole} contributes significantly in the physical region. If both these criteria are met, one can write

$$\frac{d\sigma}{dq^2\, d\Omega} \cdot (\mu^2 - q^2)^2 \;\simeq\; F(s_{12}, q^2), \qquad q^2 \leq -\left|q^2\right|_{\min},$$

makes fit to the data for the differential distribution and then extrapolate into the unphysical point $q^2 = \mu^2$ where

$$F(s_{12}, \mu^2) \;=\; \text{const.} \cdot \operatorname{Im} A_{\pi\pi}(s_{12}).$$

Strangely enough, this way one obtains reasonably consistent results from different experiments.

In spite of an error margin of the order of 100%, the knowledge of the $\pi\pi$ interaction amplitude, so obtained, is nevertheless extremely important. From an abstract position, the $\pi\pi$ interaction amplitude could differ significantly from directly measurable nucleon interaction amplitudes because, in principle, it could be determined by physical quantities that are totally different from those that govern the nucleon–nucleon interaction.

The Chew–Low method of studying the $\pi\pi$ interaction constitutes another example of how the knowledge of the analytic properties (pole at $q^2 = \mu^2$; distant other singularities) allows us to extract valuable information and to verify this way the basic concepts put in the foundation of the theory.