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A CAUCHY CRITERION FOR THE SUMMABILITY OF AN INFINITE INTEGRAL

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1.

If $a \in L(1, T)$ for every finite $T>1$, then we say that the infinite integral $\int_1^\infty a(u)du$ is convergent with sum s if $\lim_{T\to\infty}\int_1^T a(u)du = s$. It is well known that a necessary and sufficient condition for $\int_1^\infty a(u)du$ to be convergent (with some finite sum s) is that Cauchy's criterion,

$$\sup_{x>y}\left|\int_y^x a(u)\,du\right| \to 0 \quad \text{as} \quad y\to\infty,$$

holds. The object of this note is to obtain a similar result for summability (C, α) of $\int_1^\infty a(u)\,du$ which reduces to Cauchy's criterion in the case of convergence. The corresponding problem for summable series has been treated by A. F. Andersen in (**1**).

2.

In this section we give the notation and some basic properties.

When $\alpha \geqq 0$, we say that $\int_1^\infty a(u)\,du$ is summable (C, α) with sum s, and write

$$\int_1^\infty a(u)\,du = s(C, \alpha),$$

if

(i) $a \in L(1, T)$ for every finite $T>1$, and

(ii) $t(x) = \int_1^x (1-u/x)^\alpha a(u)\,du \to s$

as $x\to\infty$.

We want to extend the definition to the case $-1<\alpha<0$ and this means altering either (i) or (ii). If we assume that (i) holds, then in the case $-1<\alpha<0$, it does not follow that

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$(x-u)^\alpha a(u)$ is Lebesgue integrable in $(1, x)$; however it is well known that $t(x)$, defined as in (ii), exists and is finite for almost all x. (See, for example, (**4**), p. 146). Thus the limit in (ii) would have to be replaced by an essential limit, so that the condition would be that there exists a function t_1 such that $t(x) = t_1(x) p.p.$ and $t_1(x) \to s$ as $x \to \infty$. This definition is perfectly reasonable and, if it is used, the theorem below holds provided that, in its statement, the supremum is replaced by an essential supremum. However, we prefer a definition which is slightly simpler, though not so general. When $-1 < \alpha < 0$ we say that

$$\int_1^\infty a(u)\,du = s(C, \alpha)$$

if

(i)′ $a \in L^\infty(1, T)$ for every finite $T > 1$,

and

(ii) holds.

It is clear that, under the hypothesis (i)′, $t(x)$ exists everywhere. Moreover t is continuous, as is shown in (**2**, Lemma 5).

We shall need the following Abelian property of summability (C, α): if $-1 < \alpha < \beta$ and

$$\int_1^\infty a(u)\,du = s(C, \alpha)$$

then

$$\int_1^\infty a(u)\,du = s(C, \beta).$$

This is proved in (**6**), p. 27, for the case $0 \leqq \alpha < \beta$ and the proof extends to the case $-1 < \alpha < \beta$. We also need the observation that, if $-1 < \alpha < \beta$, then there is an integral which is summable (C, β) but which is not summable (C, α). (See (**3**), p. 131.)

3.

We are now ready for the main result.

Theorem. *Let $-1 < \alpha < 0$ and $a \in L^\infty(1, T)$ for every finite $T > 1$. Then $\int_1^\infty a(u)\,du = s(C, \alpha)$ if and only if, given $\varepsilon > 0$, there exists $y_0(\varepsilon)$ such that if $y \geqq y_0$,*

$$\sup_{x>y} \left| \int_y^x (1-u/x)^\alpha a(u)\,du \right| < \varepsilon. \tag{1}$$

Proof. Suppose that (1) holds and let $x > y_2 > y_1 > y_0$ so that

$$\left| \int_{y_1}^x (1-u/x)^\alpha a(u)\,du \right| < \varepsilon \quad \text{and} \quad \left| \int_{y_2}^x (q-u/x)^\alpha a(u)\,du \right| < \varepsilon,$$

and hence

$$\left| \int_{y_1}^{y_2} (1-u/x)^\alpha a(u)\,du \right| < 2\varepsilon.$$

If y_2, y_1 are fixed and x tends to infinity, then, by Lebesgue's dominated convergence theorem, we have

$$\left|\int_{y_1}^{y_2} a(u)\,du\right| \leqq 2\varepsilon;$$

i.e. Cauchy's criterion is satisfied and so $\int_1^\infty a(u)\,du$ is convergent. Furthermore, if $1<y<x$, we can write

$$t(x)-\int_1^\infty a(u)\,du = I_1+I_2+I_3$$

where

$$I_1 = t(x)-\int_1^y (1-u/x)^\alpha a(u)\,du,$$

$$I_2 = \int_1^y (1-u/x)^\alpha a(u)\,du - \int_1^y a(u)\,du,$$

and

$$I_3 = \int_1^y a(u)\,du - \int_1^\infty a(u)\,du.$$

Given $\varepsilon>0$, by (1), there exists $y_0(\varepsilon)$ such that, if $x>y\geqq y_0$, $|I_1|<\varepsilon$. Also, since $\int_1^\infty a(u)\,du$ is convergent, there exists $y_1(\varepsilon)$ such that if $y\geqq y_1$, $|I_3|<\varepsilon$. To deal with I_2, fix $y\geqq \max(y_0, y_1)$; then, by Lebesgue's dominated convergence theorem, there exists $x_0(\varepsilon)$ such that, if $x\geqq x_0$, $|I_2|<\varepsilon$. Putting these together we see that $\int_1^\infty a(u)\,du$ is summable (C, α).

To prove that (1) is necessary we first show that we can assume $t(x)\to 0$ as $x\to\infty$. If $t(x)\to s$ as $x\to\infty$, then define $\hat{a}(u)=a(u)-s$ for $1\leqq u\leqq 2$ *and* $\hat{a}(u)=a(u)$ if $u>2$. For $x>2$,

$$\int_1^x (1-u/x)^\alpha \hat{a}(u)\,du = \int_1^x (1-u/x)^\alpha a(u)\,du - s\int_1^2 (1-u/x)^\alpha\,du$$

$$\to s-s \quad \text{as} \quad x\to\infty$$

and we only have to observe that (1) remains unchanged if we replace a by $\hat{a}$ (for $y>2$). Thus we can assume that $t(x)\to 0$ as $x\to\infty$. Since, for $1<y<x$,

$$\int_y^x (1-u/x)^\alpha a(u)\,du = t(x)-\int_1^y (1-u/x)^\alpha a(u)\,du,$$

we see that a necessary and sufficient condition for (1) to hold is that, given $\varepsilon>0$, there exists $y_0(\varepsilon)$ such that, if $y\geqq y_0$, then

$$\sup_{x>y}\left|\int_1^y (1-u/x)^\alpha a(u)\,du\right| < \varepsilon.$$

We now use the following identity due to M. Riesz: if $-1<\alpha<0$ and $1<y<x$, then

$$\Gamma(\alpha+1)\Gamma(-\alpha)\int_1^y (1-u/x)^\alpha a(u)\,du = (1-y/x)^{\alpha+1}\int_1^y (1-u/x)^{-1}(y-u)^{-\alpha-1}u^\alpha t(u)\,du.$$

A proof of this is given in (**2**). (See equations (1.1), (3.1) and (5.18.) It is worthwhile to point out that there is an elementary proof of equation (5.18) of (**2**) using the following identity†; if $y<v<x$, $-1<\alpha<0$, then

$$\int_v^x (x-u)^{\alpha}(u-y)^{-\alpha-2}\,du=\int_v^x\left(\frac{x-y}{u-y}-1\right)^{\alpha}(u-y)^{-2}\,du$$

$$=\frac{1}{\alpha+1}(x-y)^{-1}(x-v)^{\alpha+1}(v-y)^{-\alpha-1}.$$

Thus we wish to show that, if $t(x)\to 0$ as $x\to\infty$, then

$$\sup_{x>y}\left|(1-y/x)^{\alpha+1}\int_1^y (1-u/x)^{-1}(y-u)^{-\alpha-1}u^{\alpha}t(u)\,du\right|\to 0 \tag{2}$$

as $y\to\infty$. We first prove that

$$\sup_{x>3y/2}\left|(1-y/x)^{\alpha+1}\int_1^y (1-u/x)^{-1}(y-u)^{-\alpha-1}u^{\alpha}t(u)\,du\right|\to 0 \tag{3}$$

as $y\to\infty$.

For $x>3y/2$, $(x-u)^{-1}\leqq(x-y)^{-1}$, and so

$$\left|(1-y/x)^{\alpha+1}\int_1^y (1-u/x)^{-1}(y-u)^{-\alpha-1}u^{\alpha}t(u)\,du\right|\leqq(1-y/x)^{\alpha}\int_1^y (y-u)^{-\alpha-1}u^{\alpha}\,|t(u)|\,du.$$

Now $\sup_{x>3y/2}(1-y/x)^{\alpha}=3^{-\alpha}$ and, by using Theorem 6 of (**5**) or giving an easy direct proof,

$$\int_1^y (y-u)^{-\alpha-1}u^{\alpha}\,|t(u)|\,du\to 0$$

as $y\to\infty$, and this proves (3).

Next we show that

$$\sup_{y<x\leqq 3y/2}\left|(1-y/x)^{\alpha+1}\int_1^y (1-u/x)^{-1}(y-u)^{-\alpha-1}u^{\alpha}t(u)\,du\right|\to 0 \tag{4}$$

as $y\to\infty$. To do this we split the integral in (4) into three parts as follows:

$$\int_1^y=\int_1^{y/2}+\int_{y/2}^{2y-x}+\int_{2y-x}^{y}.$$

We consider each integral separately, and during the course of the argument we shall employ the letter M to denote constants which need not be the same at different occurrences.

In the first integral, $1\leqq u\leqq y/2$; and since also $y<x\leqq 3y/2$, we have

$$(x-u)^{-1}(y-u)^{-\alpha-1}\leqq My^{-1}y^{-\alpha-1}.$$

Then

$$\sup_{y<x\leqq 3y/2}\left|(1-y/x)^{\alpha+1}\int_1^{y/2}(1-u/x)^{-1}(y-u)^{-\alpha-1}u^{\alpha}t(u)\,du\right|\leqq My^{-\alpha-1}\int_1^{y/2}u^{\alpha}\,|t(u)|\,du\to 0$$

as $y\to\infty$.

† This was pointed out to me by Professor D. Borwein.

When $y/2 \leqq u \leqq 2y-x$ and $y < x \leqq 3y/2$, we have

$$(x-u)^{-1}u^{\alpha} \leqq M(y-u)^{-1}y^{\alpha},$$

and so

$$\sup_{y<x\leqq 3y/2}\left|(1-y/x)^{\alpha+1}\int_{y/2}^{2y-x}(1-u/x)^{-1}(y-u)^{-\alpha-1}u^{\alpha}t(u)\,du\right|$$

$$\leqq M \sup_{y<x\leqq 3y/2}(x-y)^{\alpha+1}\int_{y/2}^{2y-x}(y-u)^{-\alpha-2}\,|t(u)|\,du$$

$$\leqq M \sup_{y<x\leqq 3y/2}(x-y)^{\alpha+1}\left\{\sup_{y/2\leqq u\leqq 2y-x}|t(u)|\int_{y/2}^{2y-x}(y-u)^{-\alpha-2}\,du\right\}$$

$$\leqq M \sup_{y<x\leqq 3y/2}(x-y)^{\alpha+1}\left\{\sup_{y/2\leqq u\leqq 2y-x}|t(u)|.(x-y)^{-\alpha-1}\right\}$$

$$\leqq M \sup_{y/2<u\leqq y}|t(u)|\to 0$$

as $y\to\infty$.

Finally, when $2y-x \leqq u \leqq y$ and $y < x \leqq 3y/2$, we have

$$(x-u)^{-1}u^{\alpha} \leqq M(x-y)^{-1}y^{\alpha}.$$

Hence

$$\sup_{y<x\leqq 3y/2}\left|(1-y/x)^{\alpha+1}\int_{2y-x}^{y}(1-u/x)^{-1}(y-u)^{-\alpha-1}u^{\alpha}t(u)\,du\right|$$

$$\leqq M \sup_{y<x\leqq 3y/2}(x-y)^{\alpha}\int_{2y-x}^{y}(y-u)^{-\alpha-1}\,|t(u)|\,du$$

$$\leqq M \sup_{y<x\leqq 3y/2}(x-y)^{\alpha}\left\{\sup_{2y-x\leqq u\leqq y}|t(u)|\int_{2y-x}^{y}(y-u)^{-\alpha-1}\,du\right\}$$

$$\leqq M \sup_{y<x\leqq 3y/2}(x-y)^{\alpha}\left\{\sup_{2y-x\leqq u\leqq y}|t(u)|.(x-y)^{-\alpha}\right\}$$

$$\leqq M \sup_{y<x\leqq 3y/2}|t(u)|\to 0$$

as $y\to\infty$. This proves (4), and so (2) holds.

4.

In the case $\alpha > 0$, (1) is a sufficient condition for $\int_1^{\infty} a(u)\,du$ to be summable (C, α) since it implies the stronger result that $\int_1^{\infty} a(u)\,du$ is convergent (by the first part of the proof of the theorem which only required $\alpha > -1$). However, if $\alpha > 0$, (1) is not a necessary condition for summability (C, α). This follows from the observation made at the end of Section 2, that there exist integrals summable (C, α) but not convergent.

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