THE ISOMORPHISM OF CERTAIN CONTINUOUS RINGS

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1. Statement of theorems to be proved. In this paper we shall prove the following two theorems (the terminology is explained in § 2 below; all rings are assumed to be associative).

THEOREM 1. *Suppose that* $\mathfrak{D}$ *is a division ring of finite order* m *over its centre* Z *and let* $\mu(m)$ *denote the factor sequence* $1, m, m^2, \ldots, m^n, \ldots$. *Then the rings* $\mathfrak{D}_{\mu(m)}$ *and* $Z_{\mu(m)}$ *are isomorphic.*

THEOREM 2. *Suppose that* $\mathfrak{D}$ *is a division ring of infinite order over its centre* Z *such that* $\mathfrak{D}$ *contains an infinite sequence of subdivision rings* $\mathfrak{D}^{(1)}, \mathfrak{D}^{(2)}, \ldots, \mathfrak{D}^{(n)}, \ldots$ *with the following properties:*

(i) *Each* $\mathfrak{D}^{(n)}$ *has centre* Z, *and the order* m_n *of* $\mathfrak{D}^{(n)}$ *over* Z *is finite.*

(ii) *If* $n \neq r$, *then* m_n *and* m_r *are relatively prime and* $uv = vu$ *whenever* $u \in \mathfrak{D}^{(n)}$ *and* $v \in \mathfrak{D}^{(r)}$.

(iii) *For fixed but different* $t_1, \ldots, t_n$, *the set of all finite sums of products*

$$\sum_{i=1}^{N} u_i^{t_1} \ldots u_i^{t_n}$$

with all $u_i^r \in \mathfrak{D}^{(r)}$ *is a subdivision ring of* $\mathfrak{D}$ *with centre* Z *and order* $m_{t_1} \ldots m_{t_n}$ *over* Z. (*If* $t_j = j$ *for* $j = 1, \ldots, n$, *we denote this division ring by* $S^{(n)}$.)

(iv) $\mathfrak{D} = \bigcup_{n=1}^{\infty} S^{(n)}$.

Let μ, ν *denote the factor sequences*

$$\mu = (1, m_1 m_2, \ldots, (m_1 \ldots m_n)^{n-1}, \ldots),$$
$$\nu = (m_1, (m_1 m_2)^2, \ldots, (m_1 \ldots m_n)^n, \ldots).$$

Then $\mathfrak{D}_\mu$ *and* Z_ν *are isomorphic.*

We note that Köthe **(2)**, using tensor products, has constructed such a ring $\mathfrak{D}$ with the field of real rational numbers as centre.

2. Introduction.

2.1. Suppose that $\mathfrak{R}$ is a ring (unit element is not postulated). For each integer $n > 1$ the ring of all $n \times n$ matrices with entries in $\mathfrak{R}$ will be denoted $\mathfrak{R}_n$. If $A \in \mathfrak{R}_p$, we shall write $A \otimes q$ to denote the matrix in $\mathfrak{R}_{pq}$ for which

$$(A \otimes q)_{(s-1)p+i,(s-1)p+j} = \begin{cases} A_{i,j} & \text{for } 1 \leqslant i, j \leqslant p, 1 \leqslant s \leqslant q, \\ 0 & \text{for all other entries,} \end{cases}$$

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in other words, $A \otimes q$ is obtained when A is repeated q times down the diagonal. If $\mathfrak{R}$ has a unit 1, then 1_q will denote the $q \times q$ unit matrix.

Throughout this paper if $A \in \mathfrak{R}_p$ and $B \in \mathfrak{R}_q$, then $A \otimes B \in \mathfrak{R}_{pq}$ will have the following meaning:

$$(A \otimes B)_{(s-1)p+i,(t-1)p+j} = A_{i,j} B_{s,t} \quad \text{for } 1 \leqslant i,j \leqslant p, 1 \leqslant s,t \leqslant q,$$

in other words, $A \otimes B$ is obtained by left multiplying each entry in B by the matrix A. It is easily verified that $\otimes$ is associative (though not commutative). If $\mathfrak{R}$ has a unit, then $A \otimes q = A \otimes 1_q$; if $\mathfrak{R}$ is a division ring, then $A \otimes B = 0$ implies $A = 0$ or $B = 0$. If $p = 1$ or $q = 1$, then $\otimes$ coincides with the usual multiplication.

(We shall not make use of the fact that $\{A \otimes B \mid A \in \mathfrak{R}_p \text{ and } B \in \mathfrak{R}_q\}$ generates a subring of $\mathfrak{R}_{pq}$ which is isomorphic to the tensor product of $\mathfrak{R}_p$ and $\mathfrak{R}_q$ over the centre of $\mathfrak{R}$.)

We define an injective ring homomorphism $\phi_{kn,n} : \mathfrak{R}_n \to \mathfrak{R}_{kn}$ by the rule $A \mapsto A \otimes k$ (the symbol $X \to Y$ denotes a mapping of the *set* X into the *set* Y whereas the symbol $x \mapsto y$ means that the *element* y is assigned as value to the *element* x).

2.2. By definition a *factor sequence* μ is an infinite sequence of integers n_i, all $\geqslant 1$, such that $n_i \to \infty$ when $i \to \infty$ and for each $i : n_{i+1} = k_i n_i$ for some integer $k_i \geqslant 1$.

By $\mathfrak{R}_\infty$ and $\mathfrak{R}_\mu$, we denote the inductive limits

$$\mathfrak{R}_\infty = \varinjlim (\mathfrak{R}_n, \phi_{kn,n}), \qquad \mathfrak{R}_\mu = \varinjlim (\mathfrak{R}_{n_i}, \phi_{k_i n_i, n_i}).$$

If $A \in \mathfrak{R}_n$, we write $A(\infty)$ to denote the element in $\mathfrak{R}_\infty$ (equivalence class) determined by A; if $n \in \mu$, we write $A(\mu)$ to denote the element in $\mathfrak{R}_\mu$ determined by A. Clearly the rule $A(\mu) \mapsto A(\infty)$ determines an injective ring homomorphism $\mathfrak{R}_\mu \to \mathfrak{R}_\infty$ which is, in general not bijective; however, it is bijective if, for example, the nth term of μ is the nth power of the product of the first n primes.

2.3. Suppose now that $\mathfrak{D}$ is a division ring with centre Z. Then $Z_\infty \subset \mathfrak{D}_\infty$ and $Z_\mu \subset \mathfrak{D}_\mu$ for each factor sequence μ. Each of the rings $\mathfrak{D}_n$, $\mathfrak{D}_\mu$, $\mathfrak{D}_\infty$ possesses a unit 1, is regular (in the sense of von Neumann), is irreducible, and possesses a unique normalized rank function R (normalized by the condition $R(1) = 1$). The rank R is determined by the rule

$$R(A) = \frac{\text{usual right column rank of } A}{n};$$

see **(3; 4; 1)**.

On every regular rank ring $\mathfrak{R}$ the function $d(A,B) = R(A - B)$ is a metric and the metric completion $\mathfrak{R}\hat{}$ is again a regular rank ring **(1)**. Thus $\mathfrak{D}_\mu\hat{}$ and $\mathfrak{D}_\infty\hat{}$ are regular rank rings.

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Von Neumann stated the following theorem.

THEOREM N. (i) *For every division ring* $\mathfrak{D}$ *and every factor sequence* μ, *the rings* $\mathfrak{D}_\infty\hat{}$ *and* $\mathfrak{D}_\mu\hat{}$ *are isomorphic.*

(ii) *If the order of* $\mathfrak{D}$ *over its centre* Z *is finite, then* $Z_\infty\hat{}$ *and* $\mathfrak{D}_\infty\hat{}$ *are isomorphic.*

A detailed proof of (i) was given by von Neumann in an unpublished manuscript, but for (ii) he left only some indication of his proof (**4**). Our Theorem 1 (which we shall prove without assuming (i) or (ii) of Theorem N) states that if the order of $\mathfrak{D}$ over Z is m (finite), then $\mathfrak{D}_{\mu(m)}$ and $Z_{\mu(m)}$ are isomorphic (hence $\mathfrak{D}_{\mu(m)}\hat{}$ and $Z_{\mu(m)}\hat{}$ are isomorphic). Theorem N (i) combined with our Theorem 1 supplies a proof of N (ii).

A detailed exposition of von Neumann's proof of N (i) will be given in a subsequent issue of this Journal.

2.4. To show that $\mathfrak{D}_{\mu(m)}$ and $Z_{\mu(m)}$ are isomorphic it is sufficient to exhibit injective ring homomorphisms for all $n \geqslant 1$:

$$\alpha_n : \mathfrak{D}_{m^{n-1}} \to Z_{m^n}, \qquad \beta_n : Z_{m^n} \to \mathfrak{D}_{m^n}$$

such that $\beta_n \alpha_n A = A \otimes m$ and $\alpha_{n+1} \beta_n A = A \otimes m$. It will follow immediately that the rule

$$A \mapsto \alpha_n(\mathrm{A}) \quad \text{whenever} \quad A \in \mathfrak{D}_{m^{n-1}}, n \geqslant 1$$

determines a bijective ring isomorphism $\mathfrak{D}_{\mu(m)} \to Z_{\mu(m)}$ as required to prove Theorem 1 (a similar technique suffices to prove Theorem 2).

3. Construction of α_n and β_n.

3.1. Suppose that $\mathfrak{D}$ is a division ring of finite order m over its centre Z. We consider $\mathfrak{D}$ as an m-dimensional vector space over the commutative scalars Z and we choose a fixed basis $e_1, \ldots, e_m$ for this vector space.

For each v in $\mathfrak{D}$ two Z-linear mappings $\mathfrak{D} \to \mathfrak{D}$ are determined by the rules $x \mapsto vx$ and $x \mapsto xv$ respectively. With respect to the fixed basis for $\mathfrak{D}$, each of these linear mappings is represented by an $m \times m$ matrix with entries in Z, $\Psi(v)$ amd $\Psi'(v)$ respectively.

The following known facts are easily verified:

$$\Psi(u)\Psi'(v) = \Psi'(v)\Psi(u) \qquad \textit{for all } u, v \text{ in } \mathfrak{D};$$

$$\Psi(1) = \Psi'(1) = 1_m;$$

the rule $v \mapsto \Psi(v)$ determines an injective ring homomorphism $\mathfrak{D} \to Z_m$;

the rule $v \mapsto \Psi'(v)$ determines an injective ring anti-homomorphism $\mathfrak{D} \to Z_m$ (anti-homomorphism because $\Psi'(uv) = \Psi'(v)\Psi'(u)$).

For the convenience of the reader we recall the proof of a known lemma.

LEMMA 1. *Suppose that all $u_i, v_i \in \mathfrak{D}$ and that*

$$\sum_{i=1}^{N} u_i x v_i = 0$$

for all $x \in \mathfrak{D}$ (this condition is equivalent to the condition

$$\sum_{i=1}^{N} \Psi(u_i)\Psi'(v_i) = 0$$

and is implied by the condition

$$\sum_{i=1}^{N} u_i \Psi'(v_i) = 0).$$

Then if $u_1, \ldots, u_N$ are Z-independent, it follows that $v_i = 0$ for all i; if $v_1, \ldots, v_N$ are Z-independent, it follows that $u_i = 0$ for all i.

Proof. If $N = 1$, the hypothesis implies that $u_1 v_1 = 0$. Since a Z-independent element must be non-zero and hence must possess an inverse, the lemma follows for this case.

Now suppose that the lemma has been established for some $n \geqslant 1$ and that $u_1, \ldots, u_{n+1}$ are Z-independent and

$$\sum_{i=1}^{n+1} u_i x v_i = 0 \qquad \text{for all } x \text{ in } \mathfrak{D}.$$

To show that all v_i are 0, we may assume that $v_{n+1} \neq 0$ (because of the inductive hypothesis) and we need only derive a contradiction.

For all x, y in $\mathfrak{D}$ we have:

$$\left(\sum_{i=1}^{n+1} u_i(x v_{n+1}^{-1})v_i\right)y = 0 = \sum_{i=1}^{n+1} u_i(xy\, v_{n+1}^{-1})v_i.$$

By subtraction we obtain

$$\sum_{i=1}^{n} u_i x(v_{n+1}^{-1} v_i y - y v_{n+1}^{-1} v_i) = 0 \qquad \text{for all } x \text{ in } \mathfrak{D}.$$

Because of the inductive hypothesis (since $u_1, \ldots, u_n$ are Z-independent along with $u_1, \ldots, u_{n+1}$) we have, for each $i \leqslant n$:

$$v_{n+1}^{-1} v_i y = y v_{n+1}^{-1} v_i \qquad \text{for all } y \text{ in } \mathfrak{D},$$

in other words

$$v_i = z_i v_{n+1} \qquad \text{for some } z_i \in Z.$$

Then

$$\left(u_{n+1} + \sum_{i=1}^{n} z_i u_i\right) x v_{n+1} = 0 \qquad \text{for all } x \text{ in } \mathfrak{D};$$

hence

$$u_{n+1} + \sum_{i=1}^{n} z_i u_i = 0,$$

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contradicting the assumed Z-independence of $u_1, \ldots, u_{n+1}$. By induction on n, this proves part of Lemma 1; the remaining part is proved in the same way.

LEMMA 2. *Suppose that* $n \geqslant 1$ *and all* $v_{i,j} \in \mathfrak{D}$. *Then the following assertions are equivalent.*

(i) $\sum_{i=1}^{N} \Psi(v_{i,1})\Psi'(v_{i,2}) \otimes \ldots \otimes \Psi(v_{i,2n-3})\Psi'(v_{i,n-2}) \otimes \Psi(v_{i,2n-1}) = 0,$

(ii) $\sum_{i=1}^{N} v_{i,1} \otimes \Psi'(v_{i,2})\Psi(v_{i,3}) \otimes \ldots \otimes \Psi'(v_{i,2n-2})\Psi(v_{i,2n-1}) = 0.$

Proof. When $n = 1$, (i) becomes

$$\sum_{i=1}^{N} \Psi(v_{i,1}) = 0$$

and (ii) becomes

$$\sum_{i=1}^{N} v_{i,1} = 0;$$

so Lemma 2 holds for $n = 1$ since Ψ is additive and injective.

We now prove Lemma 2 for the case $n > 1$. We first express $v_{1,1}, \ldots, v_{N,1}$ in terms of independent elements. Then we apply Lemma 1 to (i); this shows that it is sufficient to prove the equivalence of

(i)′ $\sum_{i=1}^{N} v_{i,2} \otimes \Psi(v_{i,3})\Psi(v_{i,4}) \otimes \ldots \otimes \Psi(v_{i,2n-3})\Psi'(v_{i,2n-2}) \otimes \Psi(v_{i,2n-1}) = 0,$

(ii)′ $\sum_{i=1}^{N} \Psi'(v_{i,2})\Psi(v_{i,3}) \otimes \ldots \otimes \Psi'(v_{i,2n-2})\Psi(v_{i,2n-1}) = 0.$

Now we express $v_{1,2}, v_{2,2}, \ldots, v_{N,2}$ in terms of independent elements and apply Lemma 1 to (ii)′; after that we see that it is sufficient to establish the equivalence of

(i)″ $\sum_{i=1}^{N} \Psi(v_{i,3})\Psi'(v_{i,4}) \otimes \ldots \otimes \Psi(v_{i,2n-1}) = 0,$

(ii)″ $\sum_{i=1}^{N} v_{i,3} \otimes \ldots \otimes \Psi'(v_{i,2n-2})\Psi(v_{i,2n-1}) = 0.$

By repeating such reductions we arrive at an equivalent assertion which has the form of Lemma 2 but with $n = 1$.

This completes the proof of Lemma 2.

COROLLARY. *For each* $n \geqslant 1$,

(iii) $\sum_{i=1}^{N} v_{i,1} \otimes \Psi'(v_{i,2})\Psi(v_{i,3}) \otimes \ldots \otimes \Psi'(v_{i,2n-2})\Psi(v_{i,2n-1}) \otimes \Psi'(v_{i,2n}) = 0$

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if and only if

(iv) $$\sum_{i=1}^{N} \Psi(v_{i,1})\Psi'(v_{i,2}) \otimes \ldots \otimes \Psi(v_{i,2n-1})\Psi'(v_{i,2n}) = 0.$$

Proof. This can be derived from Lemma 2 or proved directly in the same way, using Lemma 1 repeatedly.

LEMMA 3. *For each* $n \geqslant 1$:

(i) *every element in* Z_{m^n} *can be represented in the form*

$$\sum_{i=1}^{N} \Psi(v_{i,1})\Psi'(v_{i,2}) \otimes \ldots \otimes \Psi(v_{i,2n-1})\Psi'(v_{i,2n});$$

(ii) *every element in* $\mathfrak{D}_{m^{n-1}}$ *can be represented in the form*

$$\sum_{i=1}^{N} v_{i,1} \otimes \Psi'(v_{i,2})\Psi(v_{i,3}) \otimes \ldots \otimes \Psi'(v_{i,2n-2})\Psi(v_{i,2n-1}).$$

Proof of (i). For fixed n, the elements that can be represented in the given form constitute a Z-subspace M of Z_{m^n}; we need only verify that M has dimension m^{2n}.

Let $e_1, \ldots, e_m$ be a fixed basis for $\mathfrak{D}$ and let

$$w(r_1, \ldots, r_{2n}) = \Psi(e_{r_1})\Psi'(e_{r_2}) \otimes \ldots \otimes \Psi(e_{r_{2n-1}})\Psi'(e_{r_{2n}})$$

where each r_j varies independently over $1, \ldots, m$. It is sufficient to show that the w's are independent. Suppose that

$$\sum z(r_1, \ldots, r_{2n})w(r_1, \ldots, r_{2n}) = 0 \qquad \text{with all } z \in Z.$$

By applying Lemma 1 we obtain:

$$\text{for } \textit{fixed } r_1, r_2 : \sum z(r_1, \ldots, r_{2n})w(r_1, \ldots, r_{2n}) = 0.$$

By repeated applications of Lemma 1 we obtain that for every fixed choice of $r_1, \ldots, r_{2n}$; $z(r_1, \ldots, r_{2n})w(r_1, \ldots, r_{2n}) = 0$; hence $z(r_1, \ldots, r_{2n}) = 0$. This shows that the w's are independent and proves (i).

Proof of (ii). The elements that can be represented in the given form constitute a subspace of $\mathfrak{D}_{m^{n-1}}$, of dimension (by an argument like that used to prove (i)) equal to m^{2n-1}; hence this subspace is all of $\mathfrak{D}_{m^{n-1}}$.

LEMMA 4. *For each* $n \geqslant 1$

(i) *the rule*

$$\sum_{i=1}^{N} v_{i,1} \otimes \Psi'(v_{i,2})\Psi(v_{i,3}) \otimes \ldots \otimes \Psi'(v_{i,2n-2})\Psi(v_{i,2n-1})$$

$$\mapsto \sum_{i=1}^{N} \Psi(v_{i,1})\Psi'(v_{i,2}) \otimes \ldots \otimes \Psi(v_{i,2n-1})$$

determines an injective Z-algebra homomorphism

$$\alpha_n : \mathfrak{D}_{m^{n-1}} \rightarrow Z_{m^n}$$

(when $n = 1$ the given rule is to be read $v_i \mapsto \Psi(v_i)$; it has been pointed out previously that this determines an injective Z-algebra homomorphism $\mathfrak{D} \rightarrow Z_m$);

(ii) *the rule*

$$\sum_{i=1}^{N} \Psi(v_{i,1})\Psi'(v_{i,2}) \otimes \ldots \otimes \Psi(v_{i,2n-1})\Psi'(v_{i,2n})$$
$$\mapsto \sum_{i=1}^{N} v_{i,1} \otimes \Psi'(v_{i,2})\Psi(v_{i,3}) \otimes \ldots \otimes \Psi'(v_{i,2n})$$

determines an injective Z-algebra homomorphism

$$\beta_n : Z_{m^n} \rightarrow \mathfrak{D}_{m^n};$$

(iii) $$\beta_n \alpha_n A = A \otimes m \quad \textit{and} \quad \alpha_{n+1} \beta_n A = A \otimes m.$$

Proof of (i) *and* (ii). From Lemma 2 and Lemma 3(ii) it follows that α_n, as described, is an injective Z-linear mapping. Similarly, from the Corollary to Lemma 2 and Lemma 3 (i) it follows that β_n, as described, is an injective Z-linear mapping. Thus we need only verify that α_n and β_n preserve multiplication.

Since $\Psi(v)$ and $\Psi'(v)$ are matrices with entries in the centre Z, we have:

$$\begin{aligned}
&(v_1 \otimes \Psi'(v_2)\Psi(v_3) \otimes \ldots \otimes \Psi'(v_{2n-2})\Psi(v_{2n-1}))(u_1 \otimes \Psi'(u_2)\Psi(u_3) \otimes \ldots \\
&\qquad \otimes \Psi'(u_{2n-2})\Psi(u_{2n-1})) \\
&= v_1 u_1 \otimes \Psi'(v_2)\Psi(v_3)\Psi'(u_2)\Psi(u_3) \otimes \ldots \\
&\qquad \otimes \Psi'(v_{2n-2})\Psi(v_{2n-1})\Psi'(u_{2n-2})\Psi(u_{2n-1}) \\
&= v_1 u_1 \otimes \Psi'(u_2 v_2)\Psi(v_3 u_3) \otimes \ldots \otimes \Psi'(u_{2n-2} v_{2n-2})\Psi(u_{2n-1} v_{2n-1})
\end{aligned}$$

and

$$\begin{aligned}
&(\Psi(v_1)\Psi'(v_2) \otimes \ldots \otimes \Psi(v_{2n-1}))(\Psi(u_1)\Psi'(u_2) \otimes \ldots \otimes \Psi(u_{2n-1})) \\
&= \Psi(v_1)\Psi'(v_2)\Psi(u_1)\Psi'(u_2) \otimes \ldots \otimes \Psi(v_{2n-1})\Psi(u_{2n-1}) \\
&= \Psi(v_1 u_1)\Psi'(u_2 v_2) \otimes \ldots \otimes \Psi(v_{2n-1} u_{2n-1}),
\end{aligned}$$

which shows that α_n preserves multiplication. A similar argument shows that β_n preserves multiplication. This proves (i) and (ii).

Proof of (iii)

We have:

$$\beta_1 \alpha_1(v) = \beta_1(\Psi(v)\Psi'(1)) = v \otimes \Psi'(1) = v \otimes m;$$

for $n > 1$ we have:

$$\begin{aligned}
&\beta_n \alpha_n(v_1 \otimes \ldots \otimes \Psi'(v_{2n-2})\Psi(v_{2n-1})) \\
&= \beta_n(\Psi(v_1)\Psi'(v_2) \otimes \ldots \otimes \Psi(v_{2n-1})) \\
&= \beta_n(\Psi(v_1)\Psi'(v_2) \otimes \ldots \otimes \Psi(v_{2n-1})\Psi'(1)) \\
&= (v_1 \otimes \ldots \otimes \Psi'(v_{2n-2})\Psi(v_{2n-1})) \otimes m;
\end{aligned}$$

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for $n > 1$ we have:

$$\begin{aligned}\alpha_{n+1}\,\beta_n(\Psi(v_1)\Psi'(v_2) \otimes \ldots \otimes \Psi(v_{2n-1})\Psi'(v_{2n}))\\ = \alpha_{n+1}(v_1 \otimes \Psi'(v_2)\Psi(v_3) \otimes \ldots \otimes \Psi'(v_{2n}))\\ = \alpha_{n+1}(v_1 \otimes \Psi'(v_2)\Psi(v_3) \otimes \ldots \otimes \Psi'(v_{2n})\Psi(1))\\ = (\Psi(v_1)\Psi'(v_2) \otimes \ldots \otimes \Psi(v_{2n-1})\Psi'(v_{2n})) \otimes m.\end{aligned}$$

This proves (iii) and completes the proof of Lemma 4 and hence of Theorem 1.

COROLLARY TO THEOREM 1. *If $\mathfrak{D}$ and $\mathfrak{D}'$ are division rings each of finite order over the same centre, then $\mathfrak{D}_\infty\hat{}$ and $(\mathfrak{D}')_\infty\hat{}$ are isomorphic.*

Proof. This follows directly from Theorem N(ii) of von Neumann; hence from our Theorem 1 combined with N(i).

4. Division rings of infinite order over the centre.

4.1. We shall now assume the hypotheses of Theorem 2 and we shall use the notation: if $\mathfrak{A} \subset \bigcup_{n\in\mu}\mathfrak{D}_n$, then $\mathfrak{A}(\mu)$ denotes $\bigcup_{A\in\mathfrak{A}} A(\mu)$.

The hypothesis

$$\mathfrak{D} = \bigcup_{n=1}^{\infty} S^{(n)}$$

clearly implies that

$$\mathfrak{D}_\mu = \bigcup_{n=1}^{\infty} (S^{(n)})_{(m_1\ldots m_n)^{n-1}}(\mu).$$

Hence, to prove Theorem 2 it is sufficient to exhibit injective ring homomorphisms for each $n \geqslant 1$:

$$\gamma_n\colon\ (S^{(n)})_{(m_1\ldots m_n)^{n-1}} \longrightarrow Z_{(m_1\ldots m_n)^n},$$

$$\delta_n\colon\ Z_{(m_1\ldots m_n)^n} \longrightarrow (S^{(n+1)})_{(m_1\ldots m_{n+1})^n}$$

such that

$$\delta_n\,\gamma_n\,A = A \otimes m_1\ldots m_n\, m_{n+1}{}^n \qquad \text{for } A \in (S^{(n)})_{(m_1\ldots m_n)^{n-1}},$$

$$\gamma_{n+1}\,\delta_n\,A = A \otimes m_1\ldots m_n\, m_{n+1}{}^{n+1} \qquad \text{for } A \in Z_{(m_1\ldots m_n)^n}.$$

4.2. For this purpose we shall prove a generalization of Lemmas 2, 3, and 4. We shall make use of the mappings Ψ_n, Ψ_n' which would be defined in § 3.1 if the ring $\mathfrak{D}$ in § 3.1 were replaced by $\mathfrak{D}^{(n)}$. Thus for each $v \in \mathfrak{D}^{(n)}$ the elements $\Psi_n(v)$, $\Psi'_n(v)$ are $m_n \times m_n$ matrices with entries in Z. However, if v in $\mathfrak{D}^{(n)}$ is branded by a superscript n (for example, v^n, v_i^n, $v_{i,j}{}^n$), we may without ambiguity write Ψ, Ψ' in place of Ψ_n, Ψ'_n respectively.

We are going to describe certain functions f_i^n, g_i^n, h_i^n, k_i^n (of given elements $v_{i,j}{}^t$ in $\mathfrak{D}$, with $v_{i,j}{}^t \in \mathfrak{D}^{(t)}$ for all i, j, t) such that the rules

$$\sum_{i=1}^{N} f_i^n \longmapsto \sum_{i=1}^{N} g_i^n, \qquad \sum_{i=1}^{N} h_i^n \longmapsto \sum_{i=1}^{N} k_i^n$$

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will determine mappings γ_n, δ_n respectively, with the properties stated in § 4.1.

We define f^n by induction on $n : f^1 = v_1^1$ and if $n > 1$,

$$f^n = f^{n-1} \otimes v_1^n \otimes \Psi'(v_2^n)\Psi(v_3^n) \otimes \ldots \otimes \Psi'(v_{2n-4}^n)\Psi(v_{2n-3}^n) \otimes \Psi'(v_{2n-2}^1)\Psi(v_{2n-1}^1) \otimes \ldots \otimes \Psi'(v_{2n-2}^n)\Psi(v_{2n-1}^n).$$

Next we define h^n by induction on $n : h^1 = \Psi(v_1^1)\Psi'(v_2^1)$ and if $n > 1$,

$$h^n = h^{n-1} \otimes \Psi(v_1^n)\Psi'(v_2^n) \otimes \ldots \otimes \Psi(v_{2n-3}^n)\Psi'(v_{2n-2}^n) \otimes \Psi(v_{2n-1}^1)\Psi'(v_{2n}^1) \otimes \Psi(v_{2n-1}^2)\Psi'(v_{2n}^2) \otimes \ldots \otimes \Psi(v_{2n-1}^n)\Psi'(v_{2n}^n).$$

Then we define g^n to coincide with h^n when all $v_{2n}{}^t$ are replaced by 1; we define k^n to coincide with f^{n+1} when all $v_{2n+1}{}^t$ are replaced by 1.

We shall write f_i^n, g_i^n, h_i^n, k_i^n to indicate that the $v_j{}^t$ have been replaced by $v_{i,j}{}^t$.

We now prove a generalization of Lemma 1.

LEMMA 5. *Suppose that $\mathfrak{D}$ is a division ring of centre Z. Suppose that U, V are division rings contained in $\mathfrak{D}$, each of centre Z such that*

(i) *for each $v \in \mathfrak{D}$, $uv = vu$ for all $u \in U$ if and only if $v \in V$;*

(ii) $\sum_{i=1}^{N} u_i v_i = 0,\qquad u_1, \ldots, u_N$ *Z-independent, all $u_i \in U$ and all $v_i \in V$,*

together imply all $v_i = 0$;

(iii) *U has finite order m over Z.*

Suppose that Ψ, Ψ' are defined as in § 3.1 *but with U in place of $\mathfrak{D}$, that all $u_i^1, u_i^2 \in U$, all $v_i \in V$, and*

$$\sum_{i=1}^{N} u_i^1 x u_i^2 v_i = 0, \qquad \text{for all } x \in U$$

(*equivalently*

$$\sum_{i=1}^{N} \Psi(u_i^1)\Psi'(u_i^2)v_i = 0).$$

Then if the u_i^1, $i = 1, \ldots, N$, are Z-independent (*except for repetitions*), *it follows that*

$$\textstyle\sum_{\{i \mid u_i^1 = u\}} u_i^2 v_i = 0 \text{ for each } u \in U;$$

if the u_i^2, $i = 1, \ldots, N$, are Z-independent (*except for repetitions*) *it follows that*

$$\textstyle\sum_{\{i \mid u_i^2 = u\}} u_i^1 v_i = 0 \text{ for each } u \in U.$$

Proof. If $N = 1$, the hypothesis implies $u_1^1 u_1^2 v_1 = 0$ and since u_1^1 is independent (hence different from zero, hence invertible), it follows that $u_1^2 v_1 = 0$ as required to prove Lemma 5.

We complete the proof by induction on N.

Suppose that the lemma has been established for all $N \leqslant n$ for some $n \geqslant 1$ and that $u_1^1, \ldots, u_{n+1}^1$ are (except for repetitions) Z-independent and that $\sum(u_i^1 x u_i^2)v_i = 0$ for all $x \in U$. We may suppose that $u_{n+1}^2 \neq 0$ and we need only prove that $\sum_{\{i|u_i^1 = u_{n+1}^1\}} u_i^2 v_i = 0$.

As in the proof of Lemma 1 (since all y in U and all v in V are permutable) we obtain for all x, y in U

$$\sum_{i=1}^{n} (u_i^1 x((u_{n+1}^2)^{-1} u_i^2 y - y(u_{n+1}^2)^{-1} u_i^2)) v_i = 0.$$

Because of the inductive hypothesis, we have for every $u \in U$,

$$\sum_{\{i|u_i^1 = u\}} ((u_{n+1}^2)^{-1} u_i^2 y - y(u_{n+1}^2)^{-1} u_i^2) v_i = 0.$$

Thus $(u_{n+1}^2)^{-1}(\sum_{\{i|u_i^1=u\}} u_i^2 v_i)$ and y are permutable for every y in U. Hence for each $u \in U$,

$$(u_{n+1}^2)^{-1} (\sum_{\{i|u_i^1=u\}} u_i^2 v_i) = w(u) \in V,$$
$$\sum_{\{i|u_i^1=u\}} u_i^2 v_i = u_{n+1}^2 w(u).$$

Hence if we let u vary over the Z-independent elements in the set

$$\{u_i^1 | \, i = 1, \ldots, N\},$$

we obtain (letting $x = 1$ in the hypothesis of Lemma 5)

$$\sum_u (u u_{n+1}^2 w(u)) = 0.$$

Since the $u u_{n+1}^2$ are Z-independent along with the elements u, it follows from the hypothesis (ii) that each $w(u) = 0$, in particular,

$$\sum_{\{i|u_i^1 = u_{n+1}^1\}} u_i^2 v_i = 0,$$

as required.

By induction, this proves part of Lemma 5; the rest of Lemma 5 is proved in the same way.

LEMMA 6.

(i) $$\sum_{i=1}^{N} f_i^n = 0 \Leftrightarrow \sum_{i=1}^{N} g_i^n = 0;$$

(ii) $$\sum_{i=1}^{N} h_i^n = 0 \Leftrightarrow \sum_{i=1}^{N} k_i^n = 0;$$

(iii) *each element in* $(S^{(n)})_{(m_1 \cdots m_n)^{n-1}}$ *can be represented in the form*

$$\sum_{i=1}^{N} f_i^n;$$

(iv) *each element in* $Z_{(m_1 \cdots m_n)^n}$ *can be represented in the form*

$$\sum_{i=1}^{N} h_i^n;$$

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(v) *the rules*

$$\sum_{i=1}^{N} f_i^n \mapsto \sum_{i=1}^{N} g_i^n, \qquad \sum_{i=1}^{N} h_i^n \mapsto \sum_{i=1}^{N} k_i^n$$

determine injective Z-algebra homomorphisms γ_n, δ_n *respectively, with the properties listed in* § 4.1.

Proof of (i).

The method used to prove Lemma 2 can be applied, but we need Lemma 5 in place of Lemma 1.

When $n = 1$ the assertion (i) becomes:

$$\sum_{i=1}^{N} v_{i,1}{}^1 = 0 \Leftrightarrow \sum_{i=1}^{N} \Psi_1(v_{i,1}{}^1) = 0,$$

which holds since Ψ_1 is additive and injective.

Now we shall prove (i) for the case $n > 1$. We first express $v_{1,1}{}^1, \ldots, v_{N,1}{}^1$ in terms of independent elements. Then we apply Lemma 5 (with $\mathfrak{D} = S^{(n)}$, $U = \mathfrak{D}^{(1)}$, and V = set of finite sums of products $u^2 \ldots u^n$, with all $u^t \in \mathfrak{D}^{(t)}$) to the right side of (i); this shows that it is sufficient to prove (i)′, which is (i) with all $v_{1,1}{}^1, v_{2,1}{}^1, \ldots, v_{N,1}{}^1$ omitted.

Then we express $v_{1,2}{}^1, \ldots, v_{N,2}{}^1$ in terms of independent elements and apply Lemma 5 to the left side of (i)′; this shows that it is sufficient to prove (i) with all $v_{i,1}{}^1$ and $v_{i,2}{}^1$, $i = 1, \ldots, N$ omitted. We repeat this reduction procedure until we reach the equivalent assertion

$$\sum_{i=1}^{N} \Psi(v_{2n-1}{}^n) = 0 \Leftrightarrow \sum_{i=1}^{N} \Psi(v_{2n-1}{}^n) = 0.$$

This completes the proof of (i).

Proof of (ii). (ii) is proved in the same way as (i), by repeatedly applying Lemma 5.

Proof of (iii). We need only verify that the Z-subspace $(S^n)_{(m_1 \cdots m_n)^{n-1}}$ has the same dimensionality as the subspace of finite sums

$$\sum_{i=1}^{N} f_i^n,$$

(namely, $(m_1 \ldots m_n)^{2n-1}$). The method used to prove Lemma 3 is applicable here.

Proof of (iv). The method used to prove (iii) applies to prove (iv).

Proof of (v). The method used to prove Lemma 4 applies here to show that γ_n and δ_n preserve multiplication and have the other properties stated in Lemma 6.

This proves Lemma 6 and completes the proof of Theorem 2.

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ERRATUM

The paragraphs "Added in proof" appearing above the reference lists on pp. 918 and 949 of Vol. XVIII, No. 5, should be interchanged.

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