

FACTORIZING ABSOLUTELY SUMMING OPERATORS THROUGH HILBERT-SCHMIDT OPERATORS

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Introduction. Let K be a compact Hausdorff space, and let $C(K)$ be the corresponding Banach space of continuous functions on K . It is well-known that every 1-summing operator $S: C(K) \rightarrow l_2$ is also nuclear, and therefore factors $S = S_1 S_2$, with $S_1: l_2 \rightarrow l_2$ a Hilbert-Schmidt operator and $S_2: C(K) \rightarrow l_2$ a bounded operator. It is easily seen that this latter property is preserved when $C(K)$ is replaced by any quotient, and that a Banach space X enjoys this property if and only if its second dual, X^{**} , does. This led A. Pełczyński [15] to ask if the second dual of a Banach space X must be isomorphic to a quotient of a $C(K)$ -space if X has the property that every 1-summing operator $X \rightarrow l_2$ factors through a Hilbert-Schmidt operator. In this paper, we shall first of all reformulate the question in an appropriate manner and then show that counter-examples are available among super-reflexive Tsirelson-like spaces as well as among quasi-reflexive Banach spaces.

Preliminaries. We shall use standard notation and terminology for Banach space theory. In particular, by a subspace of a Banach space we always mean a closed linear submanifold, and an operator will always be a continuous linear mapping between Banach spaces. Given a Banach space X , we shall write as usual B_X for its closed unit ball, X^* for its dual, and X^{**} for its second dual. Further unexplained notation and terminology is as in [12] and [13].

As for the theory of operator ideals, we shall follow A. Pietsch [16] with only minor changes in notation. By Γ_p we denote the ideal of all L_p -factorable operators ($1 \leq p \leq \infty$) and by Π_r the ideal of all (absolutely) r -summing operators ($1 \leq r < \infty$). We say that a Banach space X has the *property GL* if $\Pi_1(X, Y)$ is contained in $\Gamma_1(X, Y)$ for every Banach space Y . Every Banach lattice and, in particular, every Banach space with an unconditional basis has this property. By trace duality, X has the property *GL* if and only if X^* has it. The concept goes back to Y. Gordon and D. R. Lewis [5]. For more details, see G. Pisier [18, Ch. 8], [17] and [20].

Let (g_n) be a sequence of independent standard Gaussian random variables. Following W. Linde and A. Pietsch [11] we say that an operator $S: X \rightarrow Y$ between Banach spaces X and Y is γ -*summing* if there is a constant C such that $(\mathbb{E} \|\sum_{i=1}^n g_i Sx_i\|^2)^{1/2} \leq C \cdot \sup\{(\sum_{i=1}^n |\langle u, x_i \rangle|^2)^{1/2} \mid u \in B_{X^*}\}$ for all finite sequences $(x_i)_{i=1}^n$ in X . These operators form an ideal Π_γ which contains all the ideals Π_p ($1 \leq p < \infty$).

The concept of a γ -summing operator is closely related to the notion of type and cotype. Recall that $S: X \rightarrow Y$ is said to be of *type* p ($1 < p \leq 2$), respectively *cotype* q

($2 \leq q < \infty$), if there is a constant C such that

$$\left(\mathbb{E} \left\| \sum_{i=1}^n g_i S x_i \right\|^2 \right)^{1/2} \leq C \cdot \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p},$$

respectively

$$\left(\sum_{i=1}^n \|S x_i\|^q \right)^{1/q} \leq C \cdot \left(\mathbb{E} \left\| \sum_{i=1}^n g_i x_i \right\|^2 \right)^{1/2},$$

again for all finite sequences $(x_i)_{i \leq n}$ in X . If the identity map of a Banach space X has this property then we say that X has *type p* , respectively *cotype q* . It is known that X has cotype 2 if and only if $\Pi_\gamma(Z, X) = \Pi_2(Z, X)$ holds for every Banach space Z (or only for $Z = l_2$). Similarly, X has type 2 if and only if $\Gamma_1(Z, X)$ is contained in $\Pi_\gamma(Z, X)$ for all Z . Instead of using Gaussian variables, we could also have worked with independent Bernoulli variables, alias Rademacher functions. It is well-known, however, that this would lead to an equivalent definition of type and cotype, and also of a γ -summing operator (result by T. Figiel, communicated by A. Pietsch). We refer to B. Maurey and G. Pisier [14], A. Pietsch [16], and G. Pisier [18] for further information.

Factorization through Hilbert–Schmidt operators. We say that an operator $S : X \rightarrow Y$ *factors through a Hilbert–Schmidt operator* if $S = S_1 S_2 S_3$ with operators $S_1 : l_2 \rightarrow Y$, $S_3 : X \rightarrow l_2$, and a Hilbert–Schmidt operator $S_2 : l_2 \rightarrow l_2$. These operators form an ideal which we denote by \mathfrak{X} . It consists of course of 1-summing operators only. As was mentioned before, we wish to characterize the Banach spaces X such that $\mathfrak{X}(X, l_2) = \Pi_1(X, l_2)$. We have the following result.

PROPOSITION 1. *For every Banach space X , the following are equivalent.*

- (a) $\mathfrak{X}(X, l_2) = \Pi_1(X, l_2)$.
- (b) Every 1-summing operator with domain X has a 2-summing adjoint.
- (c) Every 2-summing operator with domain X^* is L_1 -factorable.
- (d) X^* has cotype 2 and the property GL .

Proof. (a) \Leftrightarrow (b) is due to the known fact that an operator $S : Y \rightarrow Z$ has a 2-summing adjoint iff US is in $\mathfrak{X}(Y, l_2)$ for every $U \in \mathcal{L}(Z, l_2)$.

(b) \Leftrightarrow (c) By trace duality, (b) is equivalent to saying that every operator from Y into X with 2-summing adjoint belongs to $\Gamma_\infty(Y, X)$, for any Banach space Y . But this is (c), by transposition and localization.

Since (d) \Rightarrow (c) is obvious, all we need to show is (c) \Rightarrow (d). Note that (c) implies that X^* has the property GL and satisfies $\Pi_1(X^*, Y) = \Pi_2(X^*, Y)$ for all Y . Every Banach space Z of cotype 2 is known to satisfy $\Pi_1(Z, Y) = \Pi_2(Z, Y)$, but it is an open question if conversely this property implies that Z has cotype 2. No problems arise, however, if Z has in addition the property GL ; therefore the proof is completed by the following known result.

LEMMA 2. *If a Banach space Z has the property GL and satisfies $\Pi_1(Z, l_2) = \Pi_2(Z, l_2)$, then Z has cotype 2.*

A proof is given in [18, Proposition 8.16]. For the sake of completeness, we include another proof which is apparently new. Suppose Z has the announced properties. We wish to show that every $U \in \Pi_\gamma(l_2, Z)$ is 2-summing; equivalently, SU is nuclear for every $S \in \Pi_2(Z, l_2)$. By hypothesis, $S = BA$ for some $A: Z \rightarrow L_1(\mu)$, $B: L_1(\mu) \rightarrow l_2$, where μ is an appropriate measure. Since $L_1(\mu)$ has cotype 2, AU is 2-summing. By Grothendieck's theorem, B is 1-summing, and so $SU = BAU$ is a trace class operator on l_2 . Since $S \in \Pi_2(Z, l_2)$ was arbitrary, $U \in \Pi_2(l_2, Z)$ follows by trace duality.

Some examples. It is clear that the equivalent properties listed in the preceding proposition are shared by all L_∞ -spaces and that they are preserved under the formation of quotients. It is also clear that a Banach space X has this property if and only if X^{**} does. There are, however, Banach spaces X satisfying $\mathfrak{X}(X, l_2) = \Pi_1(X, l_2)$ such that X^{**} is not isomorphic to a quotient of an L_∞ -space. We prove first of all the following result.

PROPOSITION 3. *Given $2 \leq q < \infty$, let X be an infinite-dimensional Banach space which is of cotype $q + \varepsilon$ for all $\varepsilon > 0$ but no quotient of which is isomorphic to l_q . Then X^{**} cannot be isomorphic to a quotient of any L_∞ -space.*

Proof. Suppose we are wrong: X^{**} is isomorphic to a quotient space of an L_∞ -space, say Z . Let $Q: Z \rightarrow X^{**}$ be the quotient map. We shall now use known results relating p -summing operators and cotype; cf. [14] and [16] for details. By the local reflexivity principle, X^{**} has also cotype $q + \varepsilon$ for all $\varepsilon > 0$. Fix $r > q$. Since Z is an L_∞ -space, Q is r -summing and even r -integral. Consequently, X^{**} appears as a quotient of some $L_r(\mu)$. In particular, we get $X = X^{**}$, and this space is even super-reflexive. It follows that X^* has type $q^* - \varepsilon$ for all $\varepsilon > 0$, with $q^* := q \cdot (q - 1)^{-1}$. Being isomorphic to a subspace of the \mathcal{L}_1 -space Z^* (via Q^*), X^* contains an isomorphic copy of l_{q^*} , by the result of S. Guerre and M. Lévy [6]. Equivalently, l_q must be isomorphic to a quotient of X , a contradiction.

REMARK. In case $q = 2$ we could have applied as well the forerunner, due to D. Aldous [2], of the Guerre–Lévy result.

EXAMPLE 1. In [9], W. B. Johnson constructed a super-reflexive Banach space, \mathcal{T} say, with monotone unconditional basis, which is of type 2 and of cotype $2 + \varepsilon$ for all $\varepsilon > 0$, and which has the property that every subspace of each of its quotients has a basis but fails to be isomorphic to l_2 . \mathcal{T} belongs to the class of “Tsirelson-like” Banach spaces; detailed information on these spaces may be found in the forthcoming lecture notes [3] by P. G. Casazza and T. J. Shura.

By the preceding two propositions, \mathcal{T} satisfies $\mathfrak{X}(\mathcal{T}, l_2) = \Pi_1(\mathcal{T}, l_2)$ and is not isomorphic to a quotient of any L_∞ -space.

EXAMPLE 2. A variant of Johnson's construction appears as Example 5.3 in [4]. The spaces $\tilde{X} = \tilde{X}(q, \eta)$ ($2 \leq q < \infty$, $\eta > 1$) constructed there have again a monotone unconditional basis, are of type 2 and of cotype $q + \varepsilon$ for all $\varepsilon > 0$, and have the property that no quotient of \tilde{X} (in fact, of any subspace of \tilde{X}) can contain a copy of l_q . I am grateful to P.

G. Casazza for pointing out to me this latter fact. Again the preceding two propositions show that $\mathfrak{X}(\tilde{X}, l_2) = \Pi_1(\tilde{X}, l_2)$ holds but \tilde{X} cannot be the range of any L_∞ -operator.

REMARK. We take this opportunity to point out that each of the above spaces \tilde{X} and \mathcal{T} provides a negative answer to the following question raised by T. Kühn [10]. Let X be a Banach space of type 2 such that $\Pi_r(l_2, X) = \Pi_r(l_2, X)$ for $2 < r < \infty$. Is X then isomorphic to a subspace of L_r ? Certainly $X = \tilde{X}(q, \eta)$ cannot be a subspace of any L_r , for $q < r < \infty$, but we have $S \in \Pi_r(l_2, X) \Leftrightarrow S^* \in \Pi_1(X^*, l_2) \Leftrightarrow S^* \in \Gamma_1(X^*, l_2) \Leftrightarrow S \in \Gamma_\infty(l_2, X) \Leftrightarrow S \in \Pi_r(l_2, X)$.

EXAMPLE 3. Recently G. Pisier [20] has shown that the classical James space \mathcal{J} (of codimension one in \mathcal{J}^{**} , cf. R. C. James [7]) has the properties discussed in Proposition 1: \mathcal{J}^* has cotype 2 and the property GL . Actually the same holds for the analogously defined spaces v_p^0 , $2 \leq p < \infty$, obtained from completing $\mathbb{R}^{(\mathbb{N})}$ (or $\mathbb{C}^{(\mathbb{N})}$) with respect to the norm $\|(x_i)\| = \sup_i (\sum |x_{n_i} - x_{n_{i-1}}|^p)^{1/p}$, the supremum being extended over all increasing sequences (n_i) of positive integers. \mathcal{J} is then just the space v_2^0 . By Proposition 1, we have $\mathfrak{X}(v_p^0, l_2) = \Pi_1(v_p^0, l_2)$. It is easy to see that $(v_p^0)^{**}$ does not contain a copy of c_0 . Hence it cannot appear as a quotient even of any C^* -algebra since otherwise it would be reflexive [1] (and even super-reflexive [8]).

Of course, this last statement prompts us to ask the following questions.

QUESTION 1. Can any of the spaces \mathcal{T} or $\tilde{X}(q, \eta)$ ($2 \leq q < \infty, \eta > 1$) mentioned before be isomorphic to a quotient of a C^* -algebra?

\mathcal{T} is the first genuine example of what is now called a weak Hilbert space [19].

QUESTION 2. Suppose that X is a weak Hilbert space and isomorphic to a quotient of an L_∞ -space (or a C^* -algebra). Is X isomorphic to a Hilbert space?

The spaces \mathcal{T} and $\tilde{X}(q, \eta)$ are super-reflexive, whereas the spaces v_p^0 are non-reflexive but quasi-reflexive of order one. This leads to our final question.

QUESTION 3. Are there Banach spaces X which are (a) reflexive but not super-reflexive, or (b) not of finite codimension in X^{**} , such that $\mathfrak{X}(X, l_2) = \Pi_1(X, l_2)$ and X^{**} is not isomorphic to a quotient of any L_∞ -space?

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REFERENCES

1. C. A. Akemann, P. G. Dodds, and J. L. B. Gamlen, Weak compactness in the dual space of a C^* -algebra. *J. Functional Analysis* **10** (1972), 446–450.
2. D. Aldous, Subspaces of L^1 via random measures. *Trans. Amer. Math. Soc.* **258** (1981), 445–463.
3. P. G. Casazza and T. J. Shura, *Tsirelson’s space*. Lecture Notes in Mathematics, to appear.

4. T. Figiel, J. Lindenstrauss and V. D. Milman, The dimension of almost spherical sections of convex bodies. *Acta Math.* **139** (1977), 53–94.
5. Y. Gordon and D. R. Lewis, Absolutely summing operators and local unconditional structure. *Acta Math.* **133** (1974), 27–48.
6. S. Guerre and M. Lévy, Espaces l^p dans les sous-espaces de L^1 . *Trans. Amer. Math. Soc.* **279** (1983), 611–616.
7. R. C. James, A non-reflexive Banach space isometric with its second conjugate space. *Proc. Nat. Acad. Sci. U.S.A.* **37** (1951), 174–177.
8. H. Jarchow, On weakly compact operators on C^* -algebras. *Math. Ann.* **273** (1986), 341–343.
9. W. B. Johnson, Banach spaces all of whose subspaces have the approximation property. *Special Topics of Applied Mathematics, Bonn 1979*; (North-Holland, 1980), 15–26. See also Séminaire d'Analyse Fonctionnelle 79/80, Ecole Polytechn. Palaiseau, Exp. no 16.
10. T. Kühn, γ -summing operators in Banach spaces of type p , $1 < p \leq 2$, and cotype q , $2 \leq q < \infty$. *Theory Prob. Appl.* **26** (1981), 118–129.
11. W. Linde and A. Pietsch, Mappings of Gaussian measures of cylindrical sets in Banach spaces. *Theory Prob. Appl.* **19** (1974), 445–460.
12. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I. Sequence spaces* (Springer-Verlag, 1977).
13. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II. Function spaces* (Springer-Verlag, 1979).
14. B. Maurey and G. Pisier, Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach. *Studia Math.* **58** (1976), 45–90.
15. A. Pelczyński, Compactness of absolutely summing operators. *Linear and Complex Analysis Problem Book*; Lecture Notes in Mathematics **1043** (1984), 7–9.
16. A. Pietsch, *Operator Ideals*. (VEB Deutscher Verlag der Wissenschaften 1978/North-Holland 1980).
17. G. Pisier, Some results on Banach spaces without local unconditional structure. *Compositio Math.* **37** (1978), 3–19.
18. G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, Amer. Math. Soc. CBMS **60** (1986).
19. G. Pisier, Weak Hilbert spaces, *Proc. London Math. Soc.* **56** (1988), 547–579.
20. G. Pisier, The dual J^* of the James space has cotype 2 and the Gordon-Lewis property. Preprint.

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