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ON REGULATING SETS AND THE DISPARITY OF PLANAR CUBIC GRAPHS

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I. **Introduction and preliminary statements.** The goal of this paper is to relate plane cubic graphs which are not bipartite to plane bipartite cubic graphs which have properties (face-coloring with three colors, existence of two-factors whose cycles are boundaries of faces) characterizing this class of graphs. Using such characteristic properties in connection with the concept of regulating sets of plane cubic graphs, we find conditions under which a considered cubic graph has a specific property. Also, the concept of regulating sets generalizes the concept of 1-factors for planar cubic graphs (see Theorem 1). Furthermore, the concept of the disparity is a certain measure for a planar cubic graph G which determines "how far G is from being bipartite."

The concepts used in this paper are identical with those used in [2] if not defined in another way. The boundary of a face L is denoted by ∂L.

The short and simple proof of the following lemma was related to the author by C. St. J. A. Nash-Williams.

LEMMA 1. *Let G be a connected graph with m points and let $V_0=\{v_i, w_i \mid i=1,\ldots,n\}$ be a set of $2n$ arbitrarily chosen distinct points of G, $2\leq 2n\leq m$. Then G contains an acyclic subgraph H with $V_0\subset V(H)$, and $v\in V(H)$ has odd degree in H if and only if $v\in V_0$, and H has no isolated points.*

Proof. It suffices to prove the lemma for a tree G with $m\geq 2n\geq 2$ points. For every $i=1,\ldots,n$, let p_i be a path joining v_i and w_i in G. Denote $P=\bigcup_{i=1}^n p_i$, and denote by H the subgraph of P consisting of all lines of P which belong to an odd number of paths p_i, and all points of P incident to one such line. Denote by $\deg_k x$ the degree of $x\in V(P)$ in the path p_k. Then $\Delta(x)=\sum_{k=1}^n \deg_k x$ is odd if and only if $x\in V_0$. On the other hand, if m_e denotes the number of paths p_i which contain the line e, then $\Delta(x)=\Sigma m_e$, where the sum is taken over all lines e incident to x. By construction, H has no isolated points, and since $e\notin E(H)$ for $e\in E(P)$ if and only if m_e is even, therefore, $\Delta(v)\equiv \deg_H v \bmod 2$ for any $v\in V(H)$ ($\deg_H v$ denotes the degree of v in H). This proves the lemma.

Clearly, for a disconnected graph, G, Lemma 1 is true as long as one assumes that each component of G contains an even number of points of V_0.

DEFINITION 1. Let G be a plane cubic graph containing a 2-factor TF such that a cycle in TF or the disjoint union of such cycles (in case G is disconnected) is a

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boundary of a face of G. Then we call this TF a BTF (*b*oundary-*t*wo-*f*actor). If every cycle of a BTF is bipartite, then we call this BTF a $BBTF$ (*b*ipartite-*b*oundary-*t*wo-*f*actor). These two concepts have been defined already in [1] and are denoted there by LQF, $PLQF$ respectively.

By [1, Theorem 2], a plane cubic graph is bipartite if and only if it contains a $BBTF$; and if a connected plane cubic graph G has a $BBTF$, then it has exactly three such 2-factors which are induced by the color classes of the three-face-coloring of G (in fact, if G is a planar bipartite cubic graph, then different embeddings of G in the plane in general yield different three-face-colorings, even if G is connected).

We note that a regular bipartite graph cannot have a cutpoint since it is 1-factorable; therefore, a bipartite cubic graph is bridgeless.

Of course, in general a plane cubic graph has no BTF (e.g., the tetrahedron or the dodecahedron). However, if a connected, non-bipartite, plane, cubic graph has a BTF, then this BTF is unique. This is expressed in the following lemma. In connection with a BTF F of the plane cubic graph G we call a line e an F-bridge if e is not a line of a cycle of F. The faces of G whose boundaries belong to F, are called the F-islands (of G).

LEMMA 2. *If the connected, plane, cubic graph G contains two different BTF, F_1 and F_2, then G is bipartite.*

Proof. Let B_0 be a cycle of F_1, denote by $e_1, \ldots, e_k$ the F_1-bridges which have a point in common with B_0, and denote by $B_1, \ldots, B_m$, $m \leq k$, the boundaries (of faces of G) distinct from B_0 which do not contain any e_i, $i=1, \ldots, k$, but have a point with at least one e_i in common. Then $B_1, \ldots, B_m$ belong to F_1. Now we consider each of the $B_1, \ldots, B_m$ as we did for B_0, a.s.o. Since G is connected and finite this procedure ends at the boundary of an F_1-island from which every F_1-bridge yields to an F_1-island whose boundary has been determined before as belonging to F_1. That is, knowing one F_1-island, one knows the whole BTF F_1. Therefore, by hypothesis, F_1 and F_2 have no cycle in common. By this and the fact that G is connected we conclude that G is 2-connected. Therefore, if the boundary of a face of G does not belong to F_1, then it is an even cycle (see the proof of Theorem 2 in [1] and Figure 1 there). That is, both F_1 and F_2 are $BBTF$.

COROLLARY 1. *If G is a disconnected plane cubic graph containing a BTF and if no component of G is bipartite, then this BTF is unique and induces a (unique) BTF in the components of G.*

The proof of Corollary 1 is trivial.

REMARK 1. A simple example for a plane cubic graph with a bridge and a BTF is constructed as follows: Take two copies of the cube and embed them in the plane in such a way that the exterior face has a disconnected boundary B. Now subdivide in each cycle of B exactly one line by exactly one point x_1, x_2 respectively, and draw the line $[x_1, x_2]$. This plane cubic graph G has the bridge $[x_1, x_2]$ and obviously, it has a (unique) *BTF*. If we embed in the exterior face of G a third copy of the cube, then we obtain a nonbipartite, plane, cubic graph having exactly two *BTF*. This shows the necessity for no component being bipartite as stated in Corollary 1.

II. Regulating sets of plane, cubic graphs.

DEFINITION 2. Let G be a plane cubic graph, $e=[x, y]$ a line of G. Denote the points different from $y(x)$ and adjacent to $x(y)$ by $x_i(y_i)$, $i=1, 2$, corresponding to their cyclic order (possibly $x_2=y_1$ or $x_1=y_2$). Now we form $G_0=G-\{x, y\}$ and obtain the plane cubic graph $G(e)$ from G_0 by introducing in G_0 four new points v_j, $j=1, \ldots, 4$, and the lines $[v_i, x_i]$, $[v_{2+i}, y_i]$, $i=1, 2$, $[v_j, v_{j+1}]$, $j=1, \ldots, 4$, letting $v_5=v_1$ (we place these points and lines in the face containing x_i, y_i). We say $G(e)$ is obtained from G by a Q-extension of e (quadrangular extension). Analogously, if S is an independent set of lines of the plane cubic graph G, then $G(S)$ denotes the graph obtained by applying the Q-extension to all elements of S.

DEFINITION 3. An independent set of lines R of the plane cubic graph G is said to be a regulating set of G if $G(R)$ is bipartite. In such a case, we call $G(R)$ the associated graph of G (with respect to R) and denote it shortly by G^*.

REMARK 2. Although Definitions 2 and 3 are related to a plane graph G, i.e., to an actual embedding of a planar graph H, a regulating set S of an embedding of H is a regulating set for any embedding of H since $G(S)$ is bipartite, independent of the fact of its being an actual embedding of a planar, bipartite graph. However, since we connect in the following regulating sets with 3-face-colorings (for which we need plane graphs), we shall speak of regulating sets of plane graphs.

The existence of a regulating set for a large class of plane cubic graphs is provided by:

THEOREM 1. *If the plane cubic graph G contains a linear factor L, then $R=L$ is a regulating set of G.*

Proof. Form $G^*=G(L)$ which has a *BBTF* consisting of the quadrangles obtained by Q-extensions from the lines of L. By [1, Theorem 2], G^* is bipartite, i.e. G^* is the associated graph of G with respect to the regulating set L.

The following lemma is obviously true.

LEMMA 3. *A plane cubic graph contains a regulating set if and only if each of its components does.*

Because of Lemma 3 we restrict ourselves from now on to connected graphs without losing generality.

Another similarity between linear factors and regulating sets is expressed in the next theorem.

THEOREM 2. *If R is a regulating set of the plane cubic graph G, then R contains all bridges of G.*

Proof. As stated before, a regular bipartite graph has no cutpoints. Since the associated graph G^* of G (with respect to R) could have a bridge b if and only if b is a bridge of G, and since G^* is bipartite, the truth of Theorem 2 is proven.

The graphs in Figure 1 and Figure 2 demonstrate that the question, which plane cubic graphs have regulating sets and which have not, will not have a simple answer. Using Theorem 2, one can prove easily that the graph of Figure 1 has no regulating set, and by the same theorem, one easily finds a regulating set in the graph of Figure 2. In addition, the block-cutpoint-graphs of these two graphs are isomorphic.

Figure 2 also shows that the converse of Theorem 1 is not true. But if we assume that G has a linear factor, is then every regulating set a subset of some linear factor? A negative answer is given by the choice of $R=\{[v_i, w_i]/i=1, \ldots, 6\}$ in the dodecahedron (see Figure 3). Since all points adjacent to c are incident to lines of R, there is no linear factor L with $R \subset L$. However, the following theorem gives a sufficient condition to answer the above question positively.

THEOREM 3. *A regulating set R is a subset of some linear factor L of the plane cubic graph G if the quadrangular faces obtained from the elements of R belong to at most two color classes in a given 3-face-coloring of G^*.*

Proof. Suppose the quadrangular faces obtained from the elements of R belong to at most two color classes in a given 3-face-coloring C_F of G^*. Denote the color classes of C_F by (1), (2), (3). We construct a 3-line-coloring C_E of G^* with the color classes (1′), (2′), (3′) ($i=i'$, $i=1, 2, 3$) by coloring the line e of G^* with the color k' if the faces for which e is a boundary line, have color i, j respectively, where $\{i, j, k'\}=\{1, 2, 3\}$. By this, for any face B of G^*, the lines of ∂B are alternatingly colored with $2'$ and $3'$ if $B \in (1)$, and so on. Without loss of generality, none of the considered quadrangular faces belongs to (3). Then, by construction, any such quadrangle is line-colored with $1'$ and $3'$ or $2'$ and $3'$ in C_E. Following the construction of these quadrangles from lines of R, we obtain a 3-line-coloring C_Q from C_E such that the lines $[v_1, v_4]$ and $[v_2, v_3]$ always belong in C_Q to (3′), by eventually commuting the line-coloring of the considered quadrangles. By identifying these lines $[v_1, v_4]$, $[v_2, v_3]$ for each quadrangle, the linear factor (3′) of G^* is transformed into a linear factor L of G.

However, for Theorem 3, the converse is not true. This is shown in the graph G of Figure 4, where the lines $[v_i, w_i]$, $i=1, \ldots, 5$, and the lines $[p, q]$, $[r, s]$ form a linear factor L for which the subset $R=\{[v_i, w_i]/i=1, 3, 5\}$ is a regulating set of G. But in G^*, each of the three quadrangular faces obtained from the elements of R, belongs to a different color class in the 3-face-coloring of G^*.

DEFINITION 4. Let R be a regulating set of the plane cubic graph G. Similar to the above we declare a face B of G to be an R-island if ∂B contains an endpoint of

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a line e of R but $e \notin \partial B$. Now we define the graph G' whose points are the R-islands of G, and two points of G' are joined by exactly so many lines as there are lines in R joining the corresponding R-islands (by this, G' can have multiple lines and/or loops). This graph G' is called the regulator of G (with respect to R) and denoted by $\text{reg}_R(G)$.

THEOREM 4. *The 2-connected plane, cubic graph G has a BTF if and only if G contains a regulating set R such that the R-islands of G become faces of G* belonging to the same color class of the 3-face-coloring of G*.*

Proof. (A) Suppose G has a *BTF* F. Then we form the half-reduced graph G/F by contracting the cycles of F to the points of G/F. The lines of the linear factor $L_F = E(G) - E(F)$ become the lines of G/F. (For an exact definition of G/F, see [1, Definition 1]). Since G is connected, so is G/F. By Lemma 1, G/F contains an acyclic subgraph T in which exactly those points have odd degree which have odd degree in G/F. Denote $R' = E(T)$, and let $R \subset L_F$ be the line set corresponding to R'. If S is a face of G with $\partial S \in F$ such that ∂S contains k endpoints of lines of R, and if $|E(\partial S)| = \gamma$, then we have for the corresponding face S^* in $G(R)$ the equation $|E(\partial S^*)| = \gamma + k$ (since for any $\partial S \in F$, no line of ∂S belongs to R); i.e., F becomes a *BBTF* F^* of $G(R)$, i.e., $G(R) = G^*$ is the associated graph of G with respect to the regulating set R since k is odd if and only if γ is odd. By our introductory statement on the relation between a *BBTF* and a 3-face-coloring C of G^*, and since by construction of G^* the boundaries of the R-islands of G become elements of F^*, the R-islands of G necessarily become faces belonging to the same color class of C.

(B) Suppose the faces of G^* corresponding to the R-islands of G belong to the color class (1) of the 3-face-coloring C of G^*. Denote by F^* the *BBTF* of G^* corresponding to (1). Then F^* corresponds to a set F of boundaries of faces in G since G is 2-connected. Clearly $V(F) = V(G)$ since $V(F^*) = V(G^*)$, and for δS_1, $\partial S_2 \in F$, $\partial S_1 \cap \partial S_2 = \phi$ must hold; otherwise, $\partial S_1 \cap \partial S_2$ would contain a line $e = [x, y] \in R$ for which each of $\partial S(x)$, $\partial S(y)$ of the R-islands $S(x)$, $S(y)$ has a line in common with each of ∂S_1 and ∂S_2. Since $S^*(x)$ and $S^*(y)$—the faces of G^* corresponding to $S(x)$, $S(y)$ respectively—belong to (1), the contradiction to the 3-face-coloring C is obtained. That is, F is a *BTF* of G.

The following result is an application of Theorem 4.

THEOREM 5. *Let G be a 2-connected plane, cubic graph containing at most three triangles. If* $\text{reg}_R(G)$ *is connected for some regulating set R, then G is 4-face-colorable.*

Proof. Consider G^* and a 3-face-coloring C of G^*, and let S^* be a face of G^* corresponding to an R-island S of G. Without loss of generality $S^* \in (1) \subset C$. Consider a face S_1^* of G^* which corresponds to an R-island S_1 of G such that S_1 corresponds in $\text{reg}_R(G)$ to a neighbor of the point representing S. Then also S_1^*

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belongs to (1), and so on; i.e., the R-islands of G become faces of G^* belonging to the same color class (1) of C. By Theorem 4, G contains a *BTF* F which clearly must contain all triangles of G. Now consider $D(G)$, the dual graph of G, and let D_F be the subgraph of $D(G)$ which is induced exactly by the points corresponding to the faces (of G) whose boundaries do not belong to F. Since G contains at most three triangles, so does D_F. By Grünbaum's Theorem [2, Theorem 12.8], D_F has a 3-point-coloring C_D with the color classes (1′), (2′), (3′). By coloring in G the faces whose boundaries form F, with color 4 and by coloring the other faces with 1, 2, 3 corresponding to C_D, a 4-face-coloring of G is given.

COROLLARY 2. *If H is a component of* $\mathrm{reg}_R(G)$, *then the faces of G^* corresponding to the points of H, belong to the same color class of the 3-face-coloring of G^*.*

(The proof of Corollary 2 is immediate from the proof of Theorem 5.)

From Corollary 2 and Theorem 4 the following corollary is immediate.

COROLLARY 3. *If G is a 2-connected, plane, cubic graph and if* $\mathrm{reg}_R(G)$ *is connected for some regulating set R, then G has a BTF.*

The following theorem is the basis for another definition.

THEOREM 6. *For every regulating set R of the plane cubic graph G, there is a subset R_a of R such that R_a is a regulating set and* $\mathrm{reg}_{R_a}(G)$ *is acyclic.*

Proof. Suppose $\mathrm{reg}_R(G)$ contains a cycle K with the points $b_1, \ldots, b_p$, $p \geq 1$, corresponding to their cyclic order in K. Then, in G there is a line $[v_1, w_2]$ joining points of ∂B_1 and $\partial B_2, \ldots$, and $[v_p, w_1]$ joining points of ∂B_p and ∂B_1. Denote $R_K = \{[v_i, w_{i+1}]/i=1, \ldots, p; p+1=1\}$ and $R_0 = R - R_K$. By definition, R_0 also is a regulating set. Since R_0 is a proper subset of R and since G is finite, the proof is complete.

DEFINITION 5. A regulating set R of the plane cubic graph G is said to be a minimally regulating set if no proper subset of R is a regulating set of G.

THEOREM 7. *Given the plane cubic graph G; then a regulating set R is a minimally regulating set if and only if* $\mathrm{reg}_R(G)$ *is acyclic.*

The proof of Theorem 7 is obvious.

REMARK 3. It can happen that—starting from the same regulating set R of G—one obtains minimally regulating sets of different cardinality. This fact is demonstrated in the following graph: Replace each point in the tetrahedron Δ with a triangle, thus obtaining the cubic graph G which contains a *BTF* of four triangles. Take as R the lines not in the *BTF*. Let $R_1 \subset R$ consist of three lines which correspond to a spanning star of Δ; and $R_2 \subset R$ consists of two lines which correspond to a linear factor of Δ. Both R_1, R_2 are minimally regulating sets of G, but $|R_2| < |R_1|$. Also, Figure 3 shows that a minimally regulating set does not have to be a subset of a linear factor.

III. The disparity of a plane cubic graph and open questions.

DEFINITION 6. We say the plane cubic graph G has disparity m if G contains a regulating set of m elements, and if any other regulating set of G contains at least m elements; we write shortly disp $G=m$. For plane cubic graphs G having no regulating set we define disp $G=\infty$.

From Definition 6 and the previous results we obtain immediately the next theorem.

THEOREM 8. *For any plane cubic graph G with disp $G<\infty$,*

$$\frac{F_0}{2} \leq \operatorname{disp} G \leq \frac{|V(G)|}{2}$$

where F_0 is the number of faces S for which $|E(\partial S)|$ is odd.

The tetrahedron shows that these bounds for disp G are sharp.

In view of Remark 3, the problem of determining the exact value of disp G seems to be rather difficult. However, for graphs containing a *BTF*, the following theorem expresses a better upper bound for disp G than Theorem 8.

THEOREM 9. *Let G be a plane, cubic graph containing a BTF which has n cycles. Then* disp $G \leq n-1$.

Proof. Theorem 9 is an immediate consequence of part (A) of the proof of Theorem 4.

Concerning regulating sets and disparity, the following problems seem to be of interest.

(1) Find a necessary and sufficient condition for a plane cubic graph to have a regulating set.

(2) Find a necessary and sufficient condition for a regulating set (of the plane, cubic graph G) to be a subset of some linear factor (of G).

(3) Describe the class of plane, cubic graphs such that every regulating set is a subset of some linear factor.

(4) Describe the class of plane, cubic graphs G such that every minimally regulating set R of G fulfills $|R|=$disp G.

(5) Is there an integer M such that disp $G \leq M$ for any plane, 2-connected, cubic graph G? If not, describe the function $f(n)=|V(G)|/(2 \cdot \operatorname{disp} G)$, where G is chosen among the plane, 2-connected, cubic graphs with $2n$ points such that its disparity is maximal.

(6) Does a plane, 2-connected, cubic graph G exist different from the tetrahedron, with disp $G=|V(G)|/2$? If so, does there exist another G with $F_0/2=$ disp $G=|V(G)|/2$?

IV. An application to plane triangular graphs. It is a well-known fact that a graph $D \neq K_3$ is a plane triangular graph if and only if D is the dual graph of a plane, cubic 3-connected graph.

DEFINITION 7. For a plane triangular graph D, we obtain the plane triangular graph $D_4(e)$, $e \in E(D)$, by subdividing e with exactly one point v and joining v by a line to x, y respectively, which are the boundary points opposite to e in the boundaries containing e. We define analogously $D_4(S)$ for $S \subset E(D)$, where no boundary of D contains more than one line of S.

THEOREM 10. *Let D be a plane triangular graph. Then one can find $R'=\{e'_1, \ldots, e'_d\} \subset E(D)$ with no two lines of R' belonging to a boundary of D, such that $D_4(R')$ is Eulerian; and $d=\text{disp } G$, where D is the dual graph of G. Furthermore, if we denote by e_i the line of G corresponding to $e'_i \in R'$, then $R=\{e_1, \ldots, e_d\}$ is a (minimally) regulating set of G.*

Proof. Consider the graph G. If G is bipartite, then $R'=\phi$ and $D=D_4(R')$. Therefore, assume disp $G>0$.

Since G is 3-connected, we have $d=\text{disp } G<\infty$; i.e., G contains a regulating set $R=\{e_1, \ldots, e_d\}$; and there is a 1-1-correspondence between the lines of G and the lines of D. Denote the line set corresponding to R by $R'=\{e'_1, \ldots, e'_d\}$. Since R is an independent set of lines, no two lines of R' belong to a boundary of D.

In $G^*=G(R)$ we have for every $e_i \in R$ a quadrangle. Forming D^*—the dual graph of G^*—we see immediately that $D^* \simeq D_4(R')$. This proves the theorem.

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