ON THE DECOMPOSITION OF A CLASS OF FUNCTIONS OF BOUNDED VARIATION

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1. Introduction. Let $F_1(x)$ and $F_2(x)$ be two distribution functions, that is, non-decreasing, right-continuous functions such that $F_j(-\infty) = 0$ and $F_j(+\infty) = 1 (j = 1, 2)$. We denote their convolution by $F(x)$ so that

$$F(x) = F_1(x) * F_2(x) = \int_{-\infty}^{\infty} F_1(x-y)dF_2(y) = \int_{-\infty}^{\infty} F_2(x-y)dF_1(y),$$

the above integrals being defined as the Lebesgue-Stieltjes integrals. Then it is easy to verify (**2**, p. 189) that $F(x)$ is a distribution function. Let $f_1(t)$, $f_2(t)$, and $f(t)$ be the corresponding characteristic functions, that is,

$$f_j(t) = \int_{-\infty}^{\infty} e^{itx}dF_j(x), \qquad j = 1, 2,$$

$$f(t) = \int_{-\infty}^{\infty} e^{itx}dF(x).$$

Then, according to the Convolution Theorem (**2**, pp. 188–190), the relation

$$f(t) = f_1(t)f_2(t)$$

holds for all real t. Here $f(t)$ is said to be a decomposable characteristic function and $f_1(t)$ and $f_2(t)$ are called the factors of $f(t)$. In order to exclude the trivial decompositions we consider only the case when both the factors $f_1(t)$ and $f_2(t)$ are characteristic functions of some non-degenerate distributions, that is, the distributions which have at least two points of increase. If the function $f(z)$ as a function of the complex variable z ($z = t + iv$, t and v both real) is regular in every finite region of the complex plane, it is an entire characteristic function. Entire characteristic functions have many interesting decomposition (factorization) properties. For example, the factors of an entire characteristic function are also entire characteristic functions (**5**). The factors of a normal distribution can be only normal (**1**). A similar property also holds for a Poisson distribution (**5**).

In the present paper we study some decomposition properties of a class of functions of bounded variation. We denote by B the class of all real-valued functions $G(x)$ which are of bounded variation in $x(-\infty < x < \infty)$ having at least two points of increase and for which $G(-\infty) = 0$. We also denote the

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total variation of $G(x)$ in the interval $[a, b]$ by $VG(x)]_a^b$. Then clearly we have $VG(x)]_{-\infty}^{\infty} < \infty$ for all $G \in B$.

We now consider two functions $G_1(x)$ and $G_2(x)$, where $G_j(x) \in B(j = 1, 2)$. Let S_i denote the countable set of points where $G_i(x)$ is discontinuous; $i = 1, 2$. Denote by S_{12} the set of points x which have co-ordinates of the form $x = x_1 + x_2$, where x_1 and x_2 are the co-ordinates of the points of the sets S_1 and S_2 respectively. We also make the convention that the set S_{12} is empty if at least one of the sets S_1 and S_2 is empty. Then **(6**, pp. 248–250**)** the convolution $G(x)$ of $G_1(x)$ and $G_2(x)$ given by

$$(1) \qquad G(x) = \int_{-\infty}^{\infty} G_1(x - y)dG_2(y) = \int_{-\infty}^{\infty} G_2(x - y)dG_1(y)$$

exists for all x which are not in the set S_{12}. Moreover, $G(x)$ as defined in (1) as a Lebesgue-Stieltjes integral may be defined for all x in the set S_{12} so as to be of bounded variation in $(-\infty, \infty)$. It is easy to verify **(6**, p. 250**)** that the total variation of $G(x)$ cannot exceed the product of the total variations of $G_1(x)$ and $G_2(x)$ and, further, that $G(-\infty) = 0$, while $G(+\infty) = G_1(+\infty) \times G_2(+\infty)$ so that $G(x) \in B$. Let $g_1(t)$, $g_2(t)$, and $g(t)$ be the corresponding Fourier-Stieltjes transforms, that is

$$g_j(t) = \int_{-\infty}^{\infty} e^{itx} dG_j(x), \qquad j = 1, 2,$$

$$g(t) = \int_{-\infty}^{\infty} e^{itx} dG(x).$$

The above integrals are defined as the Lebesgue-Stieltjes integrals and these are uniformly and absolutely convergent for all real t. According to the Convolution Theorem **(6**, p.254**)** the relation

$$(2) \qquad g(t) = g_1(t)g_2(t)$$

holds for all real t. Here we say that both $g_1(t)$ and $g_2(t)$ are factors of $g(t)$. In this case the fact that $g(z)$ (z complex) is an entire function does not ensure that each of its factors must also be an entire function. A simple example will make it clear. Let

$$g(t) = e^{-t^2}; \qquad g_1(t) = (1 + t^2)e^{-\frac{1}{2}t^2}; \qquad g_2(t) = \frac{1}{1 + t^2} e^{-\frac{1}{2}t^2}.$$

We can easily verify that both $g(t)$ and $g_2(t)$ are characteristic functions, while $g_1(t)$ is the Fourier-Stieltjes transform of the function

$$G_1(x) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{x} (2 - u^2)e^{-\frac{1}{2}u^2} du$$

so that $G_1(x) \in B$. However, the functions $g(z)$ as well as $g_1(z)$ are entire functions, while the factor $g_2(z)$ has singularities at the points $z = \pm i$ so that it is regular only in the strip $|\mathrm{Im}\, z| < 1$.

This example clearly indicates that we have to impose additional conditions on the functions belonging to the class B in order to obtain some significant results. Recently Linnik and Skitovich **(3)** have derived a generalization of Cramér's theorem **(1)** for the case of functions of bounded variation under some conditions. In this paper we investigate the Fourier-Stieltjes transforms of a particular class of functions belonging to B, which has the form $g(t) = \exp P(t)$, where $P(t)$ is a polynomial in t. Finally, we derive a decomposition theorem for this particular class of functions.

2. A class of functions belonging to *B*. Before proceeding further, we prove a lemma which is instrumental for our subsequent investigation.

LEMMA 1. *Let a function* $G(x)$ *satisfy the following conditions*:

(i) $G(x) \in B$,

(ii) $\begin{cases} VG(x)]_y^{+\infty} = O(e^{-y^{1+\delta}}), \\ VG(x)]_{-\infty}^{-y} = O(e^{-y^{1+\delta}}), \end{cases} \qquad$ *as* $y \to \infty$,

where δ *is a fixed positive number and* $y > 1$. *Let* $g(t)$ *be the Fourier-Stieltjes transform of* $G(x)$. *Then* $g(z)$ *as a function of the complex variable* $z(z = t + iv$, t *and* v *both real) is an entire function of some finite order* $\rho \leqslant 1 + 1/\delta$.

Proof. From Condition (ii), it follows easily that the integral

$$\int_{-\infty}^{\infty} e^{|v||x|}|dG|$$

exists and is finite for every real v. Then, using the inequality

$$|g(z)| = \left| \int_{-\infty}^{\infty} e^{izx} dG(x) \right| \leqslant \int_{-\infty}^{\infty} e^{|v||x|}|dG|, \tag{3}$$

we conclude that $g(z)$ is an entire function. We fix a value of v sufficiently large, say $|v| > \frac{1}{2}e^{\delta}$, and then select a value X_0 given by the relation $X_0 = (2|v|)^{1/\delta}$. Then, using Condition (ii), we obtain

$$\int_{X_0}^{X_0+1} e^{|v|x}|dG| \leqslant C_1 e^{|v|} e^{-\frac{1}{2}X_0^{1+\delta}}$$

so that

$$\int_{X_0}^{\infty} e^{|v|x}|dG| \leqslant C_1 e^{|v|} \int_{X_0-1}^{\infty} e^{-\frac{1}{2}x^{1+\delta}} \leqslant 2C_1 e^{|v|},$$

where $C_1 > 0$ is independent of X_0. Proceeding in the same manner, we can show that

$$\int_{-\infty}^{-X_0} e^{|v||x|}|dG| \leqslant 2C_2 e^{|v|},$$

where $C_2 > 0$ is independent of X_0. Finally, we note that

$$\int_{-X_0}^{+X_0} e^{|v||x|}|dG| \leqslant C_3 e^{|v|X_0} = C_3 \exp(2^{1/\delta}|v|^{1+1/\delta}),$$

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where $C_3 > 0$ is independent of X_0. Thus, combining these three estimates, we obtain

$$\int_{-\infty}^{\infty} e^{|v||x|}|dG| \leqslant C_4 \exp(C_5|z|^{1+1/\delta}), \tag{4}$$

where $C_4 > 0$, $C_5 > 0$. Then the proof of the lemma follows immediately from (3) and (4).

Next we study the Fourier-Stieltjes transform of a particular class of functions belonging to B. Let B_0 be the class of functions $G(x)$ which satisfy the following three conditions:

$G(x) \in B$,

$$\int_{-\infty}^{\infty} e^{vx}|dG| < \infty \qquad \text{for all real } v,$$

The Fourier-Stieltjes transform $g(t)$ of $G(x)$ is of the form $g(t) = \exp P(t)$, where $P(t)$ is a polynomial in t of degree $\geqslant 2$. Now we prove the following lemma.

LEMMA 2. *Let $G(x)$ be a function belonging to the class B_0. Then the polynomial $P(t)$ must satisfy the following conditions:*

(i) *$P(t)$ is of an even degree;*

(ii) *$P(t)$ is of the form*

$$P(t) = \alpha_0 + i\alpha_1 t + \alpha_2 t^2 + i\alpha_3 t^3 + \ldots + \alpha_{2m} t^{2m},$$

where the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_2$ are all real and $\alpha_{2m} < 0$.

Proof. First we consider the case when the degree of the polynomial $P(t)$ is even, say $2m$ (m a positive integer). We note that $g(z)$ is an entire function of the complex variable z ($z = t + iv$, t and v both real) and that

$$g(iv) = \int_{-\infty}^{\infty} e^{-vx} dG(x)$$

exists for all real v and is real. It is then easy to verify that $P(t)$ is of the form (ii), where the coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{2m}$ are all real. We also note that $g(t)$ must be bounded for all real t so that $\alpha_{2m} < 0$.

Next we show that the degree of the polynomial $P(t)$ cannot be odd. Let us suppose that $P(t)$ has an odd degree, say $2m + 1$ (m a positive integer). Then, proceeding as above, we can show that $P(t)$ must be necessarily of the form

$$P(t) = \alpha_0 + i\alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_{2m} t^{2m} + i\alpha_{2m+1} t^{2m+1}, \tag{5}$$

where the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{2m}, \alpha_{2m+1}$ are all real and $\alpha_{2m} < 0$. We also note that for an arbitrary real V, the function

$$h(z) = g(z + iV) = \int_{-\infty}^{\infty} e^{izx} dH(x) \tag{6}$$

is the Fourier-Stieltjes transform of a function

$$H(x) = \int_{-\infty}^{x} e^{-Vy} dG(y),$$

which again belongs to the class B. Therefore, the function $h(t)$ must be bounded for all real t. Using (5) and (6) we obtain

$$(7) \quad h(t) = g(t + iV) = \exp[\alpha_0 + i\alpha_1(t + iV) + \ldots + i\alpha_{2m+1}(t + iV)^{2m+1}].$$

We can verify easily that the coefficient of t^{2m} in the polynomial in (7) is $\alpha_{2m} - (2m + 1)V\alpha_{2m+1}$, which can always be made positive for a suitable choice of V, and consequently $h(t)$ becomes unbounded for real t. Hence we conclude that the degree of the polynomial $P(t)$ cannot be odd.

It is also easy to verify that all the functions belonging to the class B_0 are absolutely continuous. We are now in a position to prove the following decomposition theorem for this special class of functions belonging to B_0.

THEOREM. *Let a function $G(x)$ belong to the class B_0. Assume that $G(x)$ admits the decomposition*

$$G(x) = G_1(x) * G_2(x),$$

where the functions $G_j(x)$ $(j = 1, 2)$ satisfy the conditions

(i) $G_j(x) \in B$,

$$\text{(ii)} \begin{cases} VG_j(x)]_y^{\infty} = O(e^{-y^{1+\delta}}), & \\ VG_j(x)]_{-\infty}^{-y} = O(e^{-y^{1+\delta}}), & \text{as } y \to \infty, \end{cases}$$

where δ is some positive number and $y > 0$. Then

$$G_j(x) \in B_0 \qquad (j = 1, 2).$$

Proof. From Lemma 1 it follows that both the functions $g_1(z)$ and $g_2(z)$ are entire functions of some finite orders not exceeding $1 + 1/\delta$. We further note that the relation

$$(8) \qquad g_1(z)g_2(z) = \exp[P(z)]$$

holds for all complex values of z. From (8) we see easily that the functions $g_1(z)$ and $g_2(z)$ cannot have any zeros throughout the entire complex plane. Therefore, we conclude from Hadamard's factorization theorem that

$$g_j(z) = \exp[P_j(z)], \qquad j = 1, 2,$$

where $P_j(z)$ is a polynomial in z of degree $\leqslant 1 + 1/\delta$. Therefore, each of $G_1(x)$ and $G_2(x)$ belongs to the class B_0. We then conclude from Lemma 2 that $P_j(t)$ is a polynomial in t of some even degree $2m_j < \min(2m, 1 + 1/\delta)$ and is given by

$$P_j(t) = \alpha_{0j} + i\alpha_{1j}t + \alpha_{2j}t^2 + \ldots + \alpha_{2m_j,j}t^{2m_j},$$

where the coefficients $\alpha_{0j}, \alpha_{1j}, \ldots, \alpha_{2m_j,j}$ are all real and $\alpha_{2m_j,j} < 0$ $(j = 1, 2)$. We have also the relation

$$P_1(t) + P_2(t) = P(t)$$

holding for all real t.

The following corollary is an immediate consequence of the theorem.

COROLLARY. *In addition to the conditions of the theorem, let $G(x)$ be a distribution function. Then the components $G_1(x)$ and $G_2(x)$ are normal distribution functions.*

Proof. From the theorem of Marcinkiewicz **(4)** we conclude that $G(x)$ is normal and the rest of the proof is similar to that of the above theorem. This result has been obtained earlier by Linnik and Skitovich **(3)**.

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