On γ-transformations of series

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The theorems given here deal with the efficiency, mutual consistency, and regularity of γ-transformations[1].

§1. THEOREM 1. *The necessary and sufficient conditions that γ-matrices are efficient for all divergent series whose partial sums are bounded, are*

(1) $$\lim_{k \to \infty} g_k(a) = 0, \quad (a \text{ fixed})$$

(2) $$\lim_{a \to \infty} \sum_{k=1}^{\infty} |\Delta g_k(a)| = 0,$$

where $\Delta g_k(a) \equiv g_k(a) - g_{k+1}(a)$.

Since $(g_k(a))$ is a γ-matrix we have

(3) $$\sum_{k=1}^{\infty} |\Delta g_k(a)| \leqq M \text{ independently of } a,$$

(4) $$\lim_{a \to \infty} g_k(a) = 1 \text{ for each fixed } k.$$

The proof of the theorem is based on the following lemma:—

The necessary and sufficient conditions that $\Sigma\, d_k c_k$ may be convergent whenever $s_k = \sum_{i=1}^{k} c_i$ is bounded are

(5) $$\lim_{k \to \infty} d_k = 0,$$

(6) $$\sum_{k=1}^{\infty} |\Delta d_k| \text{ is convergent.}$$

Since

$$\sigma_n \equiv \sum_{k=1}^{n} d_k c_k = \sum_{k=1}^{n-1} s_k (\Delta d_k) + s_n d_n,$$

the conditions are obviously sufficient. The necessity of (6) is established as in Abel's Lemma[2]. If we take $c_k = (-1)^k$, we see

[1] All these terms and others used here are explained in P. Dienes, *The Taylor Series* (Oxford), 1931, Chapter 12. A knowledge of this chapter is assumed. The book will be referred to as D.

[2] See p. 394 of D.

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that $\sum_{k=1}^{\infty} d_k(-1)^k$ is divergent when $\lim_{k \to \infty} d_k \neq 0$. (5) is therefore necessary.

Thus

$$\sigma \equiv \lim_{n \to \infty} \sigma_n = \sum_{k=1}^{\infty} [\Delta d_k] s_k.$$

Hence for particular values of a we have

$$(7) \qquad \gamma(a) \equiv \sum_{k=1}^{\infty} g_k(a) c_k = \sum_{k=1}^{\infty} [\Delta g_k(a)] s_k.$$

$\lim_{a \to \infty} \gamma(a)$ exists by (2), s_k being bounded. The conditions are therefore sufficient.

(1) is necessary in view of (5) already proved necessary. We use (7) to prove the necessity of (2). If (2) does not hold, we can select a real increasing sequence $\{a_n\}$ such that $\sum_{k=1}^{\infty} |\Delta g'_k(a_n)|$ or $\sum_{k=1}^{\infty} |\Delta g''_k(a_n)|$ (or both) is greater than $4\lambda > 0$, where $g_k(a) = g'_k(a) + ig''_k(a)$. Suppose the first.

Put $\Delta g'_k(a_n) = a_{nk}$. Then $\sum_{k=1}^{\infty} |a_{nk}| > 4\lambda$. Choose a value n_1 of n and determine, by (3), p_1 such that

$$\sum_{k=p_1+1}^{\infty} |a_{n_1 k}| < \lambda/2.$$

We will construct a real sequence s_k such that $|s_k| \leqq 1$. Suppose $s_k = sgn(a_{n_1 k})$, $(1 \leqq k \leqq p_1)$. Then $|T_{n_1}| > 3\lambda$ where $T_n = R\{\gamma(a_n)\}$. Next choose, by (4), $n_2 > n_1$ such that $\sum_{k=1}^{p_1} |a_{n_2 k}| < \lambda/2$ and determine, as before, $p_2 > p_1$ such that $\sum_{k=p_2+1}^{\infty} |a_{n_2 k}| < \lambda/2$.

Take $s_k = 0$ for $p_1 < k \leqq p_2$. Then $|T_{n_2}| < \lambda$. Next choose $n_3 > n_2$ such that

$$\sum_{k=1}^{p_2} |a_{n_3 k}| < \lambda/2,$$

and determine $p_3 > p_2$ such that

$$\sum_{k=p_3+1}^{\infty} |a_{n_3 k}| < \lambda/2.$$

Take $s_k = sgn(a_{n_3 k})$ for $p_2 < k \leqq p_3$. Now

$$T_{n_3} = \sum_{k=1}^{p_2} a_{n_3 k} s_k + \sum_{k=p_2+1}^{p_3} |a_{n_3 k}| + \sum_{k=p_3+1}^{\infty} a_{n_3 k} s_k.$$

Therefore

$$|T_{n_3}| \geqq \sum_{k=1}^{\infty} |a_{n_3k}| - 2\sum_{k=1}^{p_3} |a_{n_3k}| - 2\sum_{k=p_3+1}^{\infty} |a_{n_3k}| > 2\lambda.$$

Continuing in this way we have $|T_{n_r}| > 2\lambda$ for odd r and $|T_{n_r}| < \lambda$ for even r. Therefore $\gamma(a_n)$ diverges and (2) is necessary.

THEOREM 2. *Every indefinitely divergent series of complex constants has for its generalised sum any given complex constant.*

By an indefinitely divergent series we mean a series for which there are at least 2 distinct limiting values of its partial sums s_k. Suppose L' and L'' are any two limiting values of s_k.

Let the given complex constant be s. Take suffixes k'_n, $k''_n\,(> k'_n)$ such that $s_{k'_n} \to L'$ and $s_{k''_n} \to L''$, where $s_{k''_n} \neq s_{k'_n}$, and determine the γ-matrix by putting

$$\begin{aligned} g_{nk} &= 1 && (1 \leqq k \leqq k'_n) \\ &= (s - s_{k'_n})/(s_{k''_n} - s_{k'_n}) && (k'_n < k \leqq k''_n) \\ &= 0 && (k > k''_n). \end{aligned}$$

Then

$$\sum_{k=1}^{\infty} g_{nk}\,c_k = s_{k'_n} + \{(s - s_{k'_n})/(s_{k''_n} - s_{k'_n})\}\,(s_{k''_n} - s_{k'_n}) = s.$$

As an example of a definitely divergent series that can be evaluated to any complex constant s we take the series

$$1^r + 2^r + 3^r + \ldots,\quad r = 0, 1, 2, \ldots.$$

and the γ-matrix

$$\begin{aligned} g_{nk} &= 1 - \frac{k\,S_{n-1}^{(r)}}{S_{n-1}^{(r+1)}} && (1 \leqq k < n) \\ &= \frac{s}{n^r} && (k = n) \\ &= 0 && (k > n), \end{aligned}$$

where $S_n^{(r)} = 1^r + 2^r + 3^r + \ldots + n^r$.

§ 2. Let G and G' be two γ-matrices which evaluate the divergent series $\Sigma\, c_k$.

Suppose that

$$\gamma_n = \sum_{k=1}^{\infty} g_{nk}\,c_k \to \gamma \tag{8}$$

$$\gamma'_n = \sum_{k=1}^{\infty} g'_{nk}\,c_k \to \gamma'. \tag{9}$$

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Let us find conditions that $\gamma = \gamma'$.

I. If (g_{nk}) has an inverse with respect to c_k, that is, if (8) can be solved for c_k,

$$c_k = \sum_m \bar{g}_{km} \gamma_m.$$

(9) becomes

(10) $$\gamma'_n = \sum_k g'_{nk} \sum_m \bar{g}_{km} \gamma_m.$$

If the order of summation can be changed, we have

$$\gamma'_n = \sum_m (\sum_k g'_{nk} \bar{g}_{km}) \gamma_m,$$

and so we have constructed a sequence transformation leading from one of the transforms to the other. Now the necessary and sufficient condition that the convergent sequence $\{\gamma_m\}$ should be transformed into another tending to the same value is that $a_{nm} = \sum_k g'_{nk} \bar{g}_{km}$ be a T-matrix.

We notice that (8) can be solved for c_k if (g_{nk}) is row-finite and has a row-finite left reciprocal. If (g'_{nk}) is also row-finite, the order of summation in (10) can be changed. Hence

THEOREM 3. *If* (i) (g_{nk}) *and* (g'_{nk}) *are row-finite and* (ii) (g_{nk}) *has row-finite left reciprocal* $(\bar{g}_{km})$, *then* (g_{nk}) *and* (g'_{nk}) *are mutually consistent (for every* c_k*) if and only if* $(\sum_{k=1}^{\infty} g'_{nk} \bar{g}_{km})$ *is a* T*-matrix.*

COROLLARY 3.1. *When* (g'_{nk}) *is a reciprocal of* (g_{nk}), *the theorem shows that the matrix square of* (g'_{nk}) *should be a* T*-matrix.*

COROLLARY 3.2. *The necessary and sufficient condition that* G' *includes* G *is that* $(\sum_{k=1}^{\infty} g'_{nk} \bar{g}_{km})$ *is a* T*-matrix,* G' *and* G *being subject to the conditions of the theorem.*

COROLLARY 3.3. *When* G *and* G' *satisfy the conditions of the theorem, they are equivalent if and only if* $(\sum_{k=1}^{\infty} g'_{nk} \bar{g}_{km})$ *and* $(\sum_{k=1}^{\infty} g_{nk} \bar{g}'_{km})$ *are* T*-matrices.*

II. Whether or not (g_{nk}) has an inverse with respect to c_k, we proceed as follows:—

We transform the sequence $\{\gamma'_n\}$ by a T-matrix A. The transformed sequence is

$$\gamma''_n = \sum_m a_{nm} \gamma'_m \to \gamma''$$

(11) $$= \sum_m a_{nm} \sum_k g'_{mk} c_k.$$

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If the order of summation can be changed, we have

$$\gamma''_n = \sum_k (\sum_m a_{nm}\, g'_{mk})\, c_k.$$

Now γ'' is necessarily equal to γ' and $\gamma = \gamma''$ if

(12) $$(g_{nk}) = (\sum_m a_{nm}\, g'_{mk}).$$

The order of summation in (11) can be changed if A and G' are row-finite and therefore, from (12), G must also be row-finite. Hence

THEOREM 4. *If G, G' are row-finite, G and G' are mutually consistent if $G = AG'$, where the product is of the matrix kind and A is a row-finite T-matrix.*

COROLLARY 4.1. *G includes G' if $G = AG'$, G, G' and A being subject to the conditions of the theorem.*

COROLLARY 4.2. *G and G' are equivalent if $G = AG'$ and $G' = BG$, where G and G' are row-finite γ-matrices, A and B row-finite T-matrices.*

If we transform γ_n and γ'_n by T-matrices B and B' respectively, we easily get as in II above, the following theorem:—

THEOREM 5. *If G and G' are row-finite γ-matrices and B and B' are row-finite T-matrices, G and G' are mutually consistent if $BG = B'G'$, the product on each side being a matrix one.*

Since $(g_{nk} - g_{n,k+1})$ and $(g'_{nk} - g'_{n,k+1})$ are T-matrices[1], we have

COROLLARY 5.1. *If two row-finite γ-matrices (g_{nk}) and (g'_{nk}) satisfy*

$$\sum_l (g_{nl} - g_{n,l+1})\, g'_{lk} = \sum_l (g'_{nl} - g'_{n,l+1})\, g_{lk},$$

they are mutually consistent for every series they evaluate.

THEOREM. 6. *The necessary and sufficient conditions that a row-finite γ-matrix (g_{nk}) should be regular are that* (i) $(\sum_{k=1}^{\infty} g_{nk}\, \bar{g}_{k,m+1})$, *and* (ii) $(\sum_{k=1}^{\infty} g_{n,k+1}\, \bar{g}_{k+1,m+1})$ *are T-matrices.*

This follows from Corollary 3.3.

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[1] See VII, p. 399 of D after noting that G and G' are row-finite.

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