OPEN DISK PACKINGS OF A DISK

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(received January 20, 1967)

1. Introduction. It is an old problem to find how a collection of congruent plane figures should be arranged without overlapping to cover the largest possible fraction of the plane or some region of the plane. If similar figures of arbitrary different sizes are permitted, Vitali's theorem ([7], p.109) guarantees that packings which cover almost all points are possible. It is natural to study the diameters of figures used in such a packing and we will investigate this for the case of a closed disk packed with smaller open disks.

The following notation is used throughout. U denotes the closed disk of radius 1 . $\mathbb{C} = \{D_n\}$ is a non-overlapping arrangement of smaller open disks D_n of radius r_n within U leaving uncovered the residual set $R(\mathbb{C}) = U - \bigcup_{n=1}^{\infty} D_n$. In case $R(\mathbb{C})$ has plane Lebesgue measure 0 , $\mathbb{C}$ is called a packing of U .

The distribution of diameters in a packing $\mathbb{C}$ of U may be measured by the convergence properties of the series $\sum_{n=1}^{\infty} r_n^{\alpha}$ where α is a real number. For any packing, the series converges to 1 if $\alpha = 2$; Mergelyan [6] and Wesler [8] have shown that it diverges if $\alpha = 1$. We are led to

DEFINITION 1.1. The exponent of a packing $\mathbb{C}$ is defined to be

$$e(\mathbb{C}) = \inf\{\alpha : \sum_{n=1}^{\infty} r_n^{\alpha} < \infty\} .$$

Canad. Math. Bull., vol. 10, no. 3, 1967

https://doi.org/10.4153/CMB-1967-038-2 Published online by Cambridge University Press

This has been extended by Melzak [5] to the local exponent of a packing at a point in its residual set.

DEFINITION 1.2. Let $\mathbb{C}$ be a packing and x a point in $R(\mathbb{C})$. Let $D(x, r)$ be the open disk of radius r about x. Let

$$e(\mathbb{C}, x, r) = \inf\{\alpha: \sum_{D_n \subset D(x,r)} r_n^{\alpha} < \infty\} .$$

The <u>local exponent</u> of $\mathbb{C}$ at x is defined to be

$$e(\mathbb{C}, x) = \lim_{r \to 0} e(\mathbb{C}, x, r) .$$

In [5] it is argued that the infimum exists: since there are infinitely many terms in the sum, $\{\alpha: \sum_{D_n \subset D(x,r)} r_n^{\alpha} < \infty\}$ is bounded below by 0. The limit exists because $e(\mathbb{C}, x, r)$ is non-increasing in r.

In the next section it is shown that the exponent of a packing is the supremum of its local exponents. Then the special class of "osculatory packings" is introduced and it is shown that all these have the same exponent and constant local exponent. Reasons are given for believing this exponent to be the minimum over all packings, and a lower bound of 1.059 is derived for it. A family of osculatory packings is modified without change of exponent to produce a family of packings which solve a certain obstacle problem. In the final sections, several unsolved problems on packings and exponents are reviewed.

This paper is derived from the author's Master's thesis written at the University of British Columbia. It is a pleasure to record thanks to Dr. Z.A. Melzak for suggesting the problem and offering his assistance and encouragement, to David Kennedy for giving the assurance of a computer check on numerical calculations and to Wayne Welsh and Dr. M. Sion for examining the proofs. I am also grateful to the National Research Council whose financial assistance made this work possible.

2. <u>Exponents and Local Exponents</u>. Before demonstrating that the exponent of a packing is the supremum of its local exponents, it is convenient to compare 1.2 with another possible definition of local exponent.

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DEFINITION 2.1.

$$\tilde{e}(\mathbb{C}, x, r) = \inf\{\alpha : \sum_{D_n \cap D(x,r) \neq \phi} r_n^{\alpha} < \infty\},$$

$$\tilde{e}(\mathbb{C}, x) = \lim_{r \to 0} \tilde{e}(\mathbb{C}, x, r).$$

LEMMA 2.2.

$\tilde{e}(\mathbb{C}, x)$ exists and $\tilde{e}(\mathbb{C}, x) = e(\mathbb{C}, x)$.

Proof. For any $r > 0$, consider $D(x, r)$. Only finitely many disks from $\mathbb{C}$ not contained in $D(x, r)$ can overlap $D(x, r/3)$. (An easy geometric consideration shows there are fewer than six such disks.) It is therefore possible to choose $r_o < r$ so that no disks from $\mathbb{C}$ can meet $D(x, r_o)$ unless they are contained in $D(x, r)$. Then

$$\sum_{D_n \subset D(x,r)} r_n^{\alpha} \geq \sum_{D_n \cap D(x, r_o) \neq \phi} r_n^{\alpha} \geq \sum_{D_n \subset D(x, r_o)} r_n^{\alpha}$$

Hence $e(\mathbb{C}, x, r) \geq \tilde{e}(\mathbb{C}, x, r_o) \geq e(\mathbb{C}, x, r_o)$.

And so $\lim_{r \to 0} \tilde{e}(\mathbb{C}, x, r) = \lim_{r \to 0} e(\mathbb{C}, x, r)$.

THEOREM 2.3.

$$e(\mathbb{C}) = \sup_{x \in R(\mathbb{C})} e(\mathbb{C}, x).$$

Proof. Let $s = \sup_{x \in R(\mathbb{C})} e(\mathbb{C}, x)$. It is clear that $e(\mathbb{C}) \geq s$ since, for any x in $R(\mathbb{C})$, $e(\mathbb{C}) = e(\mathbb{C}, x, 2) \geq e(\mathbb{C}, x)$.

To prove the opposite inequality, take an arbitrary number $\varepsilon > 0$. For any x in $R(\mathbb{C})$, $s + \varepsilon > e(\mathbb{C}, x)$. Hence, there exists $r(x)$ so that

$$\sum_{D_n \cap D(x, r(x)) \neq \phi} r_n^{s+\varepsilon} < \infty .$$

Now $\{D_n\} \cup \{D(x, r(x)) : x \varepsilon R(\mathbf{C})\}$ is an open cover of U. As U is compact, there is a finite subcover

$$\{D_{n_1}, \ldots, D_{n_k}, D(x_1, r(x_1)), \ldots, D(x_\ell, r(x_\ell))\} .$$

Any disk D_m of the packing must either coincide with one of $D_{n_1}, \ldots, D_{n_k}$ or intersect one of $D(x_1, r(x_1)), \ldots, D(x_\ell, r(x_\ell))$; for its centre lies in U and hence in one of the disks covering U. At any rate, r_m is included, perhaps more than once, in the convergent series

$$\sum_{i=1}^{k} r_{n_i}^{s+\varepsilon} + \sum_{j=1}^{\ell} \sum_{D_n \cap D(x_j, r(x_j)) \neq \phi} r_n^{s+\varepsilon} .$$

Hence $\sum_{n=1}^{\infty} r_n^{s+\varepsilon} < \infty$ and $e(\mathbf{C}) \leq s+\varepsilon$. Since ε was arbitrary, $e(\mathbf{C}) \leq s$ and the proof is complete.

It is interesting to see how this theorem relates to what is known about certain packings. Melzak [5] has constructed a packing whose local exponent is everywhere 2. This shows that the supremum may be attained in a rather spectacular way. In the same paper he demonstrated that an osculatory packing has exponent less than 1.999971. By replacing one of the disks in the first packing by a scaled down copy of this packing one obtains a packing whose local exponent is not constant.

3. Osculatory Packings. A certain class of packings of U will be called osculatory. We will show that they all have the same exponent. As the definition of osculatory packings of U given here is slightly more general than that in [5], it requires some preliminary discussion.

There are two ways in which three circles can be tangent in pairs to bound a curvilinear triangle. In the first way, the

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circles are all externally tangent to produce a curvilinear triangle of the F-type. In the second way, two of the circles are externally tangent to one another and internally tangent to the third to produce a curvilinear triangle of the G-type lying inside the third circle.

Given a curvilinear triangle of either type there is a unique disk fitting into it tangent to the three bounding circles. This disk divides the rest of the curvilinear triangle into three new curvilinear triangles. These uniquely determine a second generation of disks leaving a total of nine curvilinear triangles in the residual set. We can proceed by induction introducing 3^{n-1} disks in the $\underline{n}$th generation to produce an osculatory packing of the curvilinear triangle.

Having described the osculatory packing of a curvilinear triangle, it is possible to define an osculatory packing of U .

DEFINITION 3.1. $\mathbb{C} = \{D_n\}$ is an <u>osculatory packing</u> of U if there is a positive integer N so that $U - \bigcup_{n=1}^{N} D_n$ consists of curvilinear triangles packed in the osculatory fashion by $\{D_n\}_{n=N}^{\infty}$.

In [5], it is verified that "osculatory packings" really are packings in the sense that they leave uncovered a residual set of plane Lebesgue measure 0 . That proof still holds with the revised definition of osculatory packings. In what follows, operations will be performed on certain osculatory packings to derive new collections of disks. These new collections will always be packings either because they are again osculatory or because a countable union of sets of measure 0 is a set of measure 0 .

In the osculatory packing of a curvilinear triangle the radii of the packing disks are uniquely determined by the radii of the circles bounding it. It is convenient to adopt a notation which shows that the sum of the αth power of these packing radii $\{r_n\}_{n=1}^{\infty}$ depends only on α and the radii a, b, c of the bounding circles. It is necessary to distinguish between the two types of curvilinear triangles.

DEFINITION 3.2. If disks of radii $\{r_n\}_{n=1}^{\infty}$ form the osculatory packing of a curvilinear triangle bounded by circles of radii a, b, c then $\sum_{n=1}^{\infty} r_n^{\alpha}$ is denoted by $F(\alpha, a, b, c)$ in the F case and by $G(\alpha, a, b, c)$ in the G case. In the G case we adopt the convention that a is the radius of the circle containing the curvilinear triangle.

Notice that F is symmetric in its last three arguments while G is symmetric in its last two arguments only.

Now let $\mathbb{C} = \{D_n\}$ be an osculatory packing of U. Let $U - \bigcup_{n=1}^{N} D_n$ consist of N_F curvilinear triangles of the F type and N_G curvilinear triangles of the G type. Let $F_i(\alpha)$ $(i = 1, \ldots, N_F)$, $G_j(\alpha)$ $(j = 1, \ldots, N_G)$ be the sums corresponding to these curvilinear triangles. Then we obtain

$$(3.3) \qquad \sum_{n=1}^{\infty} r_n^{\alpha} = \sum_{n=1}^{N} r_n^{\alpha} + \sum_{i=1}^{N_F} F_i(\alpha) + \sum_{j=1}^{N_G} G_j(\alpha) .$$

It is clear that the convergence of $\sum_{n=1}^{\infty} r_n^{\alpha}$ depends only on the convergence of the series represented by $F_i(\alpha)$ and $G_j(\alpha)$. To show that all osculatory packings of U have the same exponent, it is sufficient to show that the convergence of the series represented by $F(\alpha, a, b, c)$ and $G(\alpha, a, b, c)$ is independent of their last three arguments.

Let us begin by considering $F(\alpha, a, b, c)$. Two simple results are collected in

LEMMA 3.4.

(i) $F(\alpha, a, b, c)$ is monotone increasing in a, b and c.

(ii) $F(\alpha, \mu a, \mu b, \mu c) = \mu^{\alpha} F(a, b, c)$.

Proof. (i) By symmetry it suffices to show what happens when a is increased. Increasing a, while b and c are held constant, increases the radius of the disk that will fit into the curvilinear triangle they determine. The effect of increasing radii is felt the same way down through all generations of disks. A term by term comparison of the series for $F(\alpha, a, b, c)$ and $F(\alpha, a+ \Delta a, b, c)$ proves the result.

(ii) Changing a, b and c by the same factor μ amounts to a similarity transformation of the whole configuration. Since each radius is changed by the factor μ, each term in the series for $F(\alpha, a, b, c)$ is multiplied by the factor μ^{α}, and the result follows.

Now it is possible to show

LEMMA 3.5. The convergence of the series represented by $F(\alpha, a, b, c)$ is independent of a, b and c. $F(\alpha, a, b, c)$ converges if and only if $F(\alpha, 1, 1, 1)$ converges.

Proof. Let $m = \min\{a, b, c\}$, $M = \max\{a, b, c\}$. By Lemma 3.4 (i), $F(\alpha, m, m, m) \leq F(\alpha, a, b, c) \leq F(\alpha, M, M, M)$.

By Lemma 3.4 (ii), $m^{\alpha}F(\alpha, 1, 1, 1) \leq F(\alpha, a, b, c) \leq M^{\alpha}F(\alpha, 1, 1, 1)$. That is, $F(\alpha, a, b, c)$ converges if and only if $F(\alpha, 1, 1, 1)$ converges.

In order to consider $G(\alpha, a, b, c)$, it is necessary to recall several facts about the involutory transformation called inversion. Here and in later sections where no confusion can arise, the same symbol will be used for a disk or circle as for its radius.

Consider inversion with respect to a circle k of radius k centered at O. It maps a point P onto P' where O, P and P' are collinear and $OP \times OP' = k^2$. The circumference of k is invariant pointwise. Lines through O are mapped into themselves; other lines, into circles through O. Circles through O are mapped into lines; other circles into circles.

LEMMA 3.6. Let r be a circle of radius r centered at distance $d > r$ from the centre of inversion. Then r inverts in a circle of radius k to the circle r' of radius r' where

$$r' = \frac{k^2 r}{d^2 - r^2} .$$

Proof. The line through the centres of k, r, r' cuts r' at points whose distances from the centre of inversion are

$$\frac{k^2}{d+r} \quad \text{and} \quad \frac{k^2}{d-r} \; .$$

Hence $r' = \frac{1}{2}\left[\frac{k^2}{d-r} - \frac{k^2}{d+r}\right] = \frac{k^2 r}{d^2 - r^2}$.

COROLLARY. Let T be a set packed with disks. If T is contained in the circle of inversion k of radius k, if the distance from T to the centre of inversion is bounded below by d_o, and if the i^{th} packing disk r_i of radius r_i inverts into a disk r_i' of radius r_i', then

$$1 < \frac{r_i'}{r_i} \leq \frac{k^2}{d_o^2} \; .$$

Proof. Let d_i be the distance from the centre of the i^{th} disk to the centre of inversion. By the lemma, $\frac{r_i'}{r_i} = \frac{k^2}{d_i^2 - r_i^2}$. Since T is contained in the circle of inversion, $k \geq d$; and $\frac{k^2}{d_i^2 - r_i^2} > 1$. Since each disk r_i is contained in T, $d_i - r_i \geq d_o$. Hence $\frac{k^2}{d_i^2 - r_i^2} = \frac{k^2}{(d_i - r_i)(d_i + r_i)} \leq \frac{k^2}{(d_i - r_i)^2} \leq \frac{k^2}{d_o^2}$.

This result enables us to show

LEMMA 3.7. The convergence of the series represented by $G(\alpha, a, b, c)$ is independent of a, b and c. $G(\alpha, a, b, c)$ converges if and only if $F(\alpha, 1, 1, 1)$ converges.

Proof. Consider $G(\alpha, a, b, c)$ and the related packed G-type curvilinear triangle. Choose a circle k which surrounds the largest disk in this configuration and has its centre in this disk but bounded away from the curvilinear triangle by some distance d_o. Invert in k to transform the packed G-type curvilinear triangle $a\,b\,c$ into a packed F-type curvilinear triangle $a'b'c'$.

https://doi.org/10.4153/CMB-1967-038-2 Published online by Cambridge University Press

Under this inversion, the disks in the packing all grow but the growth ratio r_i'/r_i does not exceed $\gamma = k^2/d_o^2$. Thus

$$F(\alpha, a', b', c') \geq G(\alpha, a, b, c) .$$

But

$$\gamma^{\alpha} G(\alpha, a, b, c) \geq F(\alpha, a', b', c') .$$

Hence

$$F(\alpha, a', b', c') \geq G(\alpha, a, b, c) \geq \gamma^{-\alpha} F(\alpha, a', b', c') .$$

That is, $G(\alpha, a, b, c)$ converges if and only if $F(\alpha, a', b', c')$ converges. By Lemma 3.5, this occurs if and only if $F(\alpha, 1, 1, 1)$ converges.

Now it is possible to prove

THEOREM 3.8. *All osculatory packings of* U *have the same exponent. It is equal to* $\inf\{\alpha: F(\alpha, 1, 1, 1) < \infty\}$.

Proof. The result is immediate from Equation 3.3, Lemma 3.5 and Lemma 3.7.

The number $\inf\{\alpha : F(\alpha, 1, 1, 1) < \infty\}$ occurs often in what follows. It is convenient to have a special symbol for it.

DEFINITION 3.9. $\sigma = \inf\{\alpha : F(\alpha, 1, 1, 1) < \infty\}$.

4. *Local Properties of Osculatory Packing.* Osculatory packings of U can be characterized by their local behaviour. After doing this we show that all osculatory packings have constant local exponent equal to σ .

DEFINITION 4.1. A packing $\mathbb{C} = \{D_n\}$ of U is called *osculatory* at the point x in U in case there exists a positive integer N and a radius $r > 0$ such that $(U - \bigcup_{n=1}^{N} D_n) \cap D(x, r)$ is covered by a finite number of curvilinear triangles and these are packed in the osculatory fashion.

DEFINITION 4.2. A packing of U is called *locally osculatory* in case it is osculatory at every point in U .

THEOREM 4.3. <u>A packing of</u> U <u>is osculatory if and only if it is locally osculatory.</u>

<u>Proof.</u> Necessity is clear. Sufficiency follows from the compactness of U . For if $\mathbb{C}$ is a locally osculatory packing then to every point x in U there corresponds a positive integer N(x) and a radius $r(x) > 0$ such that $(U - \bigcup_{n=1}^{N(x)} D_n) \cap D(x, r(x))$ meets finitely many curvilinear triangles and these are packed in the osculatory fashion. The open disks $\{D(x, r(x)) : x \in U\}$ cover U so there is a finite subcover

$$\{D(x_1, r(x_1)) \ldots D(x_k, r(x_k))\} .$$

Let $N = \max\{N(x_1) \ldots N(x_k)\}$. Then $U - \bigcup_{n=1}^{N} D_n$ is the union of finitely many curvilinear triangles packed in the osculatory fashion. That is, $\mathbb{C}$ is an osculatory packing.

The highest possible exponent is 2 , and in Section 1 a reference was given to the construction of a packing whose local exponent is everywhere 2 . In the next section we will discuss reasons for believing that σ , the exponent of the osculatory packings, is the lowest possible exponent. Accordingly, it is interesting to observe

THEOREM 4.4. <u>The local exponent of an osculatory packing is constant and equal to</u> σ <u>at every point.</u>

<u>Proof.</u> In Theorem 2.3 it was shown that the exponent of a packing is the supremum of its local exponents. Because of this and Theorem 3.8, the local exponent of an osculatory packing can nowhere exceed σ . Now suppose that $\mathbb{C}$ is an osculatory packing, x is a point in the residual set $R(\mathbb{C})$, and the local exponent $e(\mathbb{C}, x)$ is less than σ . As $e(\mathbb{C}, x) = \lim_{r \to 0} e(\mathbb{C}, x, r)$, there exists an $r_o > 0$ such that $e(\mathbb{C}, x, r_o) < \sigma$. As $\mathbb{C}$ is osculatory, there exists a positive integer N such that $U - \bigcup_{n=1}^{N} D_n$ consists of finitely many curvilinear triangles packed in the osculatory fashion. One of these contains x , and as each succeeding generation of disks is packed into this triangle, the residual set is fragmented into smaller and smaller curvilinear triangles, one

of which always contains x. Eventually a stage is reached when the diameter of the curvilinear triangle containing x is less than r_o. This curvilinear triangle is contained in $D(x, r_o)$. As the exponent for the packing of this curvilinear triangle is σ, $e(\mathbb{C}, x, r_o) < \sigma$ is impossible and $e(\mathbb{C}, x) < \sigma$ is impossible.

5. Is σ the minimum exponent? In addition to considering the exponent of a packing $e(\mathbb{C})$, it is possible to consider the Hausdorff dimension of the residual set, $\dim R(\mathbb{C})$. ([3], p.107.)

D.G. Larman [4] has shown that for the packing of an n-dimensional cube by balls,

$$\inf_{\mathbb{C}} \{\dim R(\mathbb{C})\} \leq \inf_{\mathbb{C}} \{e(\mathbb{C})\} .$$

H.G. Eggleston [2] has shown that, for the packing of an equilateral triangle by oppositely oriented equilateral triangles,

$$\inf_{\mathbb{C}} \{\dim R(\mathbb{C})\} = \frac{\log 3}{\log 2} .$$

Moreover, this is attained by a packing $\mathbb{C}_o$ which uses the largest possible triangle at each stage. Defining the exponent for triangle packings in terms of the side lengths, we would have in this case

$$e(C_o) = \inf \{\alpha : \sum_{i=1}^{\infty} d_i^\alpha < \infty\} = \inf\{\alpha : \sum_{j=1}^{\infty} 3^{j-1} (2^{-j})^\alpha < \infty\} = \frac{\log 3}{\log 2} .$$

If Larman's result holds for triangle packings,

$$\inf_{\mathbb{C}} \{e(\mathbb{C})\} = e(\mathbb{C}_o) .$$

One might then expect that for disk packings by disks, $\inf_{\mathbb{C}}\{e(\mathbb{C})\}$ would be attained by a packing which, by analogy with Eggleston's construction, uses the largest possible disk at each stage. However, such a packing is osculatory when D_1 is inserted tangent to U.

This evidence is entirely circumstantial but points strongly to the conjecture that the osculatory exponent, σ, is minimal.

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6. A lower Bound for σ. In this section, several lower bounds are derived for σ. The best lower bound so far established is 1.059. However, before establishing this number, we shall obtain the lower bound 1.035 by two methods which motivate the final analysis. In all cases we work from the fact that

$$\sigma = \inf \{\alpha : F(\alpha, a, b, c) < \infty\} .$$

This follows from Definition 3.9 and Lemma 3.5.

As geometric progressions are easy to test for convergence, it would seem desirable to find radii a, b, and c so that $F(\alpha, a, b, c)$ could be bounded from below by a geometric progression. We are led to ask if there exists a curvilinear triangle which can be packed in the osculatory fashion so that each generation of disks includes one whose radius belongs to a geometric progression. An affirmative answer is given in

LEMMA 6.1. *If* $\lambda = \tau - \sqrt{\tau}$ *(where* $\tau = \frac{1+\sqrt{5}}{2}$, *the golden ratio), then there exists a curvilinear triangle bounded by disks of radii* $1, \lambda, \lambda^2$ *and such that the first-generation packing disk has radius* λ^3.

Proof. Assume such an arrangement of disks is possible and apply Soddy's formula ([1], pp. 13-15) for the radius of the first-generation disk in order to get an equation for λ. One obtains $1/\lambda^3 = 1 + \frac{1}{\lambda} + \frac{1}{\lambda^2} + 2\sqrt{1/\lambda + 1/\lambda^2 + 1/\lambda^3}$. The substitution $u = \lambda + \lambda^{-1}$ leads to the equation $u^2 - 2u - 4 = 0$. As λ must be positive, the only relevant root is $u = 1 + \sqrt{5} = 2\tau$. This gives rise to $\lambda = \tau \pm \sqrt{\tau}$, and the choice $\lambda = \tau - \sqrt{\tau}$ is forced by $\lambda < 1$.

LEMMA 6.2.

$$\sigma \geq \frac{\log 3}{\log(\tau + \sqrt{\tau})} > 1.035 .$$

First Proof. Applying the Soddy formula again and again, there is always a disk of radius λ^{N+2} in the N^{th} generation of the osculatory packing of the curvilinear triangle described in the previous lemma. Moreover, this is the smallest of the 3^{N-1} disks in the N^{th} generation.

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Hence

$$F(\alpha, 1, \lambda, \lambda^2) \geq \sum_{N=1}^{\infty} 3^{N-1} \left(\lambda^{N+2}\right)^{\alpha} .$$

The right side is a geometric progression of common ratio $3\lambda^{\alpha}$. The geometric progression diverges unless $3\lambda^{\alpha} < 1$. That is,

$$\alpha > -\frac{\log 3}{\log \lambda} = \frac{\log 3}{\log 1/\lambda} = \frac{\log 3}{\log (\tau + \sqrt{\tau})} .$$

Calculation shows

$$\frac{\log 3}{\log (\tau + \sqrt{\tau})} > 1.035 .$$

Second Proof. We may break up $F(\alpha, 1, \lambda, \lambda^2)$ into series for the packing of the curvilinear triangles which result from the insertion of the first-generation disk. We obtain

$$F(\alpha, 1, \lambda, \lambda^2) = F(\alpha, \lambda, \lambda^2, \lambda^3) + F(\alpha, 1, \lambda^2, \lambda^3) + F(\alpha, 1, \lambda, \lambda^3) + (\lambda^3)^{\alpha} .$$

By Lemma 3.4 ,

(a) $F(\alpha, \lambda, \lambda^2, \lambda^3) = \lambda^{\alpha} F(\alpha, 1, \lambda, \lambda^2)$

(b) $F(\alpha, 1, \lambda^2, \lambda^3) > F(\alpha, \lambda, \lambda^2, \lambda^3) = \lambda^{\alpha} F(\alpha, 1, \lambda, \lambda^2)$

(c) $F(\alpha, 1, \lambda, \lambda^3) > F(\alpha, \lambda, \lambda^2, \lambda^3) = \lambda^{\alpha} F(\alpha, 1, \lambda, \lambda^2)$.

Hence

$$F(\alpha, 1, \lambda, \lambda^2) > 3\lambda^{\alpha} F(\alpha, 1, \lambda, \lambda^2) + \lambda^{3\alpha}$$

And so

$$F(\alpha, 1, \lambda, \lambda^2) > \frac{\lambda^{3\alpha}}{1 - 3\lambda^{\alpha}} .$$

This shows that $F(\alpha, 1, \lambda, \lambda^2)$ diverges if $1 - 3\lambda^{\alpha} = 0$. A lower bound for σ is again obtained from the equation $3\lambda^{\alpha} = 1$.

This bound is unnecessarily low because too much was sacrificed in estimate (c) , $F(\alpha, 1, \lambda, \lambda^3) > \lambda^{\alpha} F(\alpha, 1, \lambda, \lambda^2)$. The method which we will use to improve this estimate comes

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from the inversion proof in Lemma 3.7 that $G(\alpha, a, b, c)$ converges if and only if $F(\alpha, a', b', c')$ converges. We invert the packed curvilinear triangle bounded by disks of radii $1, \lambda, \lambda^3$ into the packed curvilinear triangle bounded by disks of radii $1, \lambda, \lambda^2$. If no radius in the packing is thereby increased by more than the factor γ, then

$$\gamma^{\alpha} F(\alpha, 1, \lambda, \lambda^3) > F(\alpha, 1, \lambda, \lambda^2)$$

$$F(\alpha, 1, \lambda, \lambda^3) > \gamma^{-\alpha} F(\alpha, 1, \lambda, \lambda^2) .$$

Proceeding as in the second proof of Lemma 6.2

$$F(\alpha, 1, \lambda, \lambda^2) > \frac{\lambda^{3\alpha}}{1-2\lambda^{\alpha}-\gamma^{-\alpha}} .$$

The resulting equation for the lower bound on σ is

$$(6.3) \qquad 2\lambda^{\alpha} + \gamma^{-\alpha} = 1 .$$

Computation will show that $\gamma^{-1} > \lambda$; so this equation gives a larger value of the lower bound than the equation $3\lambda^{\alpha} = 1$.

It remains to compute the maximum growth ratio, γ, which we know by the corollary of Lemma 3.6 to be of the form k^2/d_o^2, where k is the radius of the circle of inversion and d_o is the distance from the curvilinear triangle to the centre of inversion. Our first task is to find the circle k which inverts the disks $1, \lambda, \lambda^3$ into the disks $1, \lambda, \lambda^2$.

Since the inversion in k is to leave 1 and λ invariant, these circles must be orthogonal to k. Let 1 and λ be centred on the positive and negative x-axis respectively of a cartesian coordinate system, and let them be tangent at the origin. Then k must pass through the origin tangent to the x-axis. The centre of k may be chosen on the negative y-axis so that λ^3 lies in the lower half plane and λ^2 in the upper half plane.

The point of tangency of λ^3 and 1 inverts into the point

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of tangency of λ^2 and 1. It follows that these points are collinear with the centre of inversion. The centre of inversion is therefore determined as the intersection of this line with the y-axis. The radius k is the distance from the centre of inversion to the origin, the distance d_o is the distance from the centre of inversion to the point of tangency of λ^3 and 1. (The assertion regarding d_o requires an easy computation to verify that the centre of inversion is closer to the point of tangency of λ^3 and 1 than to the point of tangency of λ^3 and λ.)

The main problem is to find the above points of tangency. It is no harder to do this in full generality and so we prove

LEMMA 6.4. If a and b are circles of radii a and b, centred at $(a, 0)$ and $(-b, 0)$ respectively and if c is a circle of radius c lying in the upper half plane and tangent to a and b, then $c.a$, the point of tangency of c and a, has coordinates

$$\left(\frac{2bc}{a+b+c+bc/a} , \frac{2\sqrt{abc(a+b+c)}}{a+b+c+bc/a} \right).$$

Proof. Inverting the circles a, b, and c in the unit circle centred at the origin, we obtain the following images:

- a', the line $x = 1/2a$;
- b', the line $x = -1/2b$;
- c', a circle in the upper half plane tangent to a' and b'.

$c.a$ will be determined as the inverse of $c'.a'$.

The coordinates of $c'.a'$ are $(1/2a, y_o)$ where y_o is the y coordinate of the centre of c'. As c' is tangent to a' and b', its radius is $\frac{1}{2}(\frac{1}{2a} - \frac{-1}{2b}) = \frac{b+a}{4ab}$ and the x coordinate of its centre is $\frac{1}{2}(\frac{1}{2a} + \frac{-1}{2b}) = \frac{b-a}{4ab}$. y_o may be determined from the fact that c' inverts into c, a circle of radius c. Applying the formula from Lemma 3.6 for the radius of the image of a given circle under inversion,

$$c = \frac{1^2 \frac{b+a}{4ab}}{[y_o^2 + (\frac{b-a}{4ab})^2] - (\frac{b+a}{4ab})^2}.$$

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Simplifying and solving for y_o,

$$y_o = \sqrt{\frac{a+b+c}{4abc}} .$$

Thus c'. a' has coordinates

$$\left(\frac{1}{2a}, \sqrt{\frac{a+b+c}{4abc}}\right) .$$

In the unit circle centred at the origin, the point (x, y) inverts into the point $\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$. Thus c'. a' inverts into the point with coordinate

$$\left(\frac{\frac{1}{2a}}{\frac{1}{4a^2}+\frac{a+b+c}{4abc}}, \frac{\sqrt{\frac{a+b+c}{4abc}}}{\frac{1}{4a^2}+\frac{a+b+c}{4abc}}\right) .$$

Simplifying these gives the final form of the coordinates of c.a.

COROLLARY. <u>If</u> c <u>is in the lower half plane, the same formula holds with the second coordinate of</u> c.a <u>taken negative.</u>

Applying these results to the circles $1, \lambda, \lambda^2, \lambda^3$ we find that 1 and λ^2 are tangent at

$$(x_1, y_1) = \left(\frac{2\lambda^3}{1+\lambda^2+\lambda^3}, \frac{-2\sqrt{\lambda^3+\lambda^4+\lambda^5}}{1+\lambda+\lambda^2+\lambda^3}\right)$$

and 1 and λ^3 are tangent at

$$(x_2, y_2) = \left(\frac{2\lambda^4}{1+\lambda+\lambda^3+\lambda^4}, \frac{2\sqrt{\lambda^4+\lambda^5+\lambda^7}}{1+\lambda+\lambda^3+\lambda^4}\right) .$$

If the centre of the circle of inversion is $(0, y_3)$, the collinearity of (x_1, y_1), (x_2, y_2) and $(0, y_3)$ gives

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$$y_3 = y_2 - x_2\left(\frac{y_1 - y_2}{x_1 - x_2}\right).$$

Now it is possible to compute $k^2 = y_3^2$, $d_o^2 = x_2^2 + (y_3 - y_2)^2$, and finally $\gamma^{-1} = d_o^2 / k^2$. Equation 6.3 for the lower bound on σ is

$$2(.34601)^\alpha + (.37203)^\alpha = 1$$

and by computation we obtain

THEOREM 6.5. $\sigma > 1.059$.

7. Other Packings with Exponent σ. There are packings with exponent σ which are not osculatory. We now describe a whole family of such packings.

Around a disk of radius 1/3 it is possible to place six more disks of radius 1/3 so that adjacent disks are tangent. This set of disks may be placed in U leaving a residual set which is the union of curvilinear triangles, six of the F-type and six of the G-type. Each of these curvilinear triangles may then be packed in the osculatory fashion to complete the packing of U. This packing clearly has the exponent σ.

Moreover, corresponding to every integer $N \geq 3$ there is an analogous packing in which the central disk has radius a_N and the N surrounding disks each having radius b_N. The values of a_N and b_N are obtained from the equations

$$a_N + 2b_N = 1,$$

$$\frac{b_N}{a_N + b_N} = \sin\frac{\pi}{N}.$$

The first is derived from the requirement that the disks fit into U; the second is read off from the right-angled triangle whose vertices are the centre of U, the centre of a b_N disk and the point of tangency of this b_N disk with either of the two

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adjacent b_N disks.

Let us denote the resulting packing by C_N. For the corresponding sum let us write

$$\sum_{n=1}^{\infty} r_n^{\alpha} = a_N^{\alpha} + \delta_N(\alpha) ,$$

so that $\delta_N(\alpha)$ denotes the sum which arises from the osculatory packing of the annulus $U - a_N$.

Now the a_N disk in $\mathbb{C}_N$ may be replaced by a copy of $\mathbb{C}_N$ scaled down by the ratio $1:a_N$. When this process is iterated infinitely often, a packing $\mathbb{C}_N^*$ is obtained for which

$$\sum_{n=1}^{\infty} r_n^{\alpha} = \delta_N(\alpha) + a_N^{\alpha} \delta_N(\alpha) + a_N^{2\alpha} \delta_N(\alpha) + \dots$$

$$= \frac{\delta_N(\alpha)}{1-a_N^{\alpha}} .$$

As $a_N^{\alpha} < 1$ for $\alpha \geq 1$, this series is convergent if and only if the series for $\mathbb{C}_N$ is convergent. Thus the family of packings $\mathbb{C}_N^* (N \geq 3)$ all have exponent σ.

It is clear that the packings $\mathbb{C}_N^*$ are not osculatory because they fail to be osculatory at the centre of U.

Now consider the following obstacle problem. A point is defined to be <u>strongly avoided</u> by a packing if it is left uncovered not only by the open disks of the packing but also by their closures. Is there a packing with exponent σ which strongly avoids an arbitrary given point in the interior of U ?

It is clear that the packings $\mathbb{C}_N^*$ solve this problem when the point in question is the centre of U. Let us fix on some $\mathbb{C}_N^*$ and produce a packing which strongly avoids an arbitrary given point X in the interior of U .

The construction of such a packing depends on the fact that a nest of concentric disks converging to their common centre can be inverted into a nest of disks converging to any point within the largest disk, the inversion leaving the largest disk fixed.

Invert X in the circumference of U to X' . The circle centred at X' and orthogonal to U inverts X to the centre of U and leaves U invariant. This is a special case of the theorem ([1], pp. 84, 85) that a circle and a pair of inverse points invert (in another circle) into a circle and a pair of inverse points. The inversion described above must take the centre of U to X and any disk containing the centre of U to a disk containing X .

Consider the action of this inversion on any $\mathbb{C}_N^*$ to produce a new packing $\mathbb{C}_N^{*'}$. Topologically the role of X relative to $\mathbb{C}_N^{*'}$ is the same as the role of the centre of U relative to $\mathbb{C}_N^*$. Hence X is strongly avoided by $\mathbb{C}_N^{*'}$. Disks grow and shrink under this inversion but the maximum growth and shrinkage ratios are bounded as the interior of U is bounded away from the centre of inversion. It follows that the exponent σ is preserved.

8. Unsolved Problems. In this section, a number of unsolved problems are listed.

The leading question is whether σ is the minimum of all exponents for the packing of a disk by disks.

Even if this is not the case, an exact determination of σ is of interest. Combining the upper bound proved in [5] with the lower bound of Theorem 5.5 we have $1.059 < \sigma < 1.999971$.

M. Sion has asked what figures other than curvilinear triangles and disks can be packed with disks so that the packing has exponent σ . This opens the whole question of the relevance of boundary conditions to packing problems. Does it matter if we pack disks, squares, or arbitrary bounded regions?

In [5] it is shown that the exponent of osculatory packings, σ , is less than 2 and that there exists a packing whose exponent is 2 . But these are the only two exponents known to exist. It would be interesting to know which numbers can be exponents.

Intuitively one feels that any number between two exponents should be an exponent. Yet the fact proved in Theorem 1.5, that the exponent of a packing is equal to the supremum of its local exponents, suggests that there is no obvious way of combining two packings to obtain one of intermediate exponent.

In [4] it is shown that the infimum of exponents is not less than the infimum of the Hausdorff dimensions of residual sets. This result prompts a number of questions. Are these infima minima and if so, are they attained for the same packing? Is there a connection between the sum of the series $\sum_{n=1}^{\infty} r_n^{e(\mathbb{C})}$ and the measure of the residual set in its Hausdorff dimension ([3], pp. 102-104)?

Finally, there is a question which is likely to be easier and which bears directly on the obstacle problem of Section 7. It is pure geometry. Can a packing which begins with an arbitrary finite collection of disks be completed to an osculatory packing? That is, can a finite number of disks be added to $U - \bigcup_{n=1}^{N} D_n$ to triangulate it?

If this can be done, then there is a packing with exponent σ which strongly avoids any finite number of points in the interior of U. For these points may each be covered by separate non-overlapping disks centred over them and this finite collection of disks completed to an osculatory packing. Then the disks covering the points may be replaced by suitably scaled down packings of the type $\mathbb{C}_N^*$.

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