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REVERSIBLE SKEW GENERALIZED POWER SERIES RINGS

A. R. NASR-ISFAHANI

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Abstract

In this note we show that there exist a semiprime ring R, a strictly ordered artinian, narrow, unique product monoid $(S, \leq)$ and a monoid homomorphism $\omega : S \longrightarrow \mathrm{End}(R)$ such that the skew generalized power series ring $R[[S, \omega]]$ is semicommutative but $R[[S, \omega]]$ is not reversible. This answers a question posed in Marks *et al*. ['A unified approach to various generalizations of Armendariz rings', *Bull. Aust. Math. Soc.* **81** (2010), 361–397].

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1. Introduction

Let $(S, \leq)$ be a partially ordered set. Then $(S, \leq)$ is called artinian if every strictly decreasing sequence of elements of S is finite, and $(S, \leq)$ is called narrow if every subset of pairwise order-incomparable elements of S is finite. A monoid S equipped with an order $\leq$ is called an ordered monoid if for any $s_1, s_2, t \in S$, $s_1 \leq s_2$ implies $s_1 t \leq s_2 t$ and $t s_1 \leq t s_2$. Moreover, if $s_1 < s_2$ implies $s_1 t < s_2 t$ and $t s_1 < t s_2$, then $(S, \leq)$ is said to be strictly ordered. Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. For $s \in S$, let ω_s denote the image of s under ω. Let A be the set of all functions $f : S \longrightarrow R$ such that the support $\mathrm{supp}(f) = \{s \in S : f(s) \neq 0\}$ is artinian and narrow. Then for any $s \in S$ and $f, g \in A$ the set

$$X_s(f, g) = \{(x, y) \in \mathrm{supp}(f) \times \mathrm{supp}(g) : s = xy\}$$

is finite. Thus one can define the product $fg : S \longrightarrow R$ of $f, g \in A$ as follows:

$$(fg)(s) = \sum_{(x,y) \in X_s(f,g)} f(x)\omega_s(g(y))$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, A becomes a ring, called the ring of skew generalized

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power series with coefficients in R and exponents in S, denoted by $R[[S, \omega, \leq]]$ (or by $R[[S, \omega]]$ when there is no ambiguity concerning the order) (for more details, see [2]). Special cases of the skew generalized power series construction include skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew group rings and Mal'cev–Neumann Laurent series rings.

Let R be a ring, $(S, \leq)$ a strictly ordered monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. A ring R is called (S, ω)-Armendariz if whenever $fg = 0$ for $f, g \in R[[S, \omega]]$, then $f(s)\omega_s(g(t)) = 0$ for all $s, t \in S$. Marks *et al.* in [1] introduced and investigated the notion of (S, ω)-Armendariz rings and studied some property of this class of rings.

A ring R is called reduced if it contains no nonzero nilpotent elements, reversible if for all $a, b \in R$, $ab = 0$ implies $ba = 0$, and semicommutative if $ab = 0$ implies $aRb = 0$ for each $a, b \in R$. It is known that each reduced ring is reversible and each reversible ring is semicommutative, but the converse not true in general. Marks *et al.* in [1] characterized when a skew generalized power series ring is reduced or semicommutative, and obtained a partial characterization for it to be reversible. They proved that for a strictly ordered monoid $(S, \leq)$, a monoid homomorphism $\omega : S \longrightarrow \mathrm{End}(R)$ and an (S, ω)-Armendariz S-compatible ring R, $R[[S, \omega]]$ is semicommutative if and only if R is semicommutative. They also showed that for a strictly ordered monoid $(S, \leq)$ which is an artinian, narrow, unique product (a.n.u.p., see [1, Definition 4.11]) and a monoid homomorphism $\omega : S \longrightarrow \mathrm{End}(R)$, $R[[S, \omega]]$ is reduced if and only if R is semiprime and the ring $R[[S, \omega]]$ is reversible. Marks *et al.* in [1] posed the following question (Question 4.14): 'Suppose R is a semiprime ring, $(S, \leq)$ is a strictly ordered a.n.u.p. monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ is a monoid homomorphism. If the skew generalized power series ring $R[[S, \omega]]$ is semicommutative, must $R[[S, \omega]]$ be reversible (and therefore reduced)?'.

In this note we provide a semiprime ring R, a strictly ordered a.n.u.p. monoid $(S, \leq)$ and a monoid homomorphism $\omega : S \longrightarrow \mathrm{End}(R)$ such that the skew generalized power series ring $R[[S, \omega]]$ is semicommutative but $R[[S, \omega]]$ is not reversible. This gives a negative answer to the question posed by Marks *et al.* We also prove that for a semiprime ring R, a strictly ordered a.n.u.p. monoid $(S, \leq)$ and a monoid homomorphism $\omega : S \longrightarrow \mathrm{End}(R)$, $R[[S, \omega]]$ is reversible if and only if $R[[S, \omega]]$ is semicommutative and ω_s is injective for each $s \in S$.

2. Main results

Let R be a ring and α be a ring endomorphism. We denote by $R[x; \alpha]$ the skew polynomial ring whose elements are the polynomials over R, addition is defined as usual, and multiplication is subject to the relation $xa = \alpha(a)x$ for any $a \in R$.

Example 2.1. Let K be a field, $R = K[x]$, $S = \mathbb{N} \cup \{0\}$ with the usual addition and trivial order. $\alpha : R \to R$ given by $\alpha(f(x)) = f(0)$ is a ring homomorphism and so $\omega : S \longrightarrow \mathrm{End}(R)$ given by $\omega(1) = \alpha$ is a monoid homomorphism. We have $R[[S, \omega]] \cong R[y; \alpha]$.

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We show that $R[y;\alpha]$ is semicommutative but not reversible. Assume that $f = f_0 + f_1y + \cdots + f_ny^n$, $g = g_0 + g_1y + \cdots + g_my^m \in R[y;\alpha]$ is such that $fg = 0$. By induction on $\deg(g) = m$ we show that $fR[y;\alpha]g = 0$. If $m = 0$ then $f_n\alpha^n(g_0) = 0$. Since R is a domain, we have $\alpha^n(g_0) = 0$ and so $g_0 \in (x)$, where (x) is the ideal generated by x in R. We also have $f_0g_0 = 0$. If $g_0 = 0$ then $fR[y;\alpha]g = 0$. If $g_0 \neq 0$ then $f_0 = 0$ and so $fR[y;\alpha]g = 0$.

Now assume inductively that the assertion is true for all $g \in R[y;\alpha]$ with $\deg(g) < m$. Since $fg = 0$, we have $f_n\alpha^n(g_m) = 0$ and so $g_m \in (x)$. Also we have $f_0g_0 = 0$. If $f_0 \neq 0$ then $g_0 = 0$ and so $f_0g_1 = 0$. Thus $g_1 = 0$ and, by the same argument, in this case we have, for each i, $g_i = 0$. Then $fR[y;\alpha]g = 0$ and the result follows.

Now assume that $f_0 = 0$. Since $g_m \in (x)$ and $f_0 = 0$ then $fR[y;\alpha]g_my^m = 0$ and so $f(g_0 + g_1y + \cdots + g_{m-1}y^{m-1}) = 0$. By the induction hypothesis,

$$fR[y;\alpha](g_0 + g_1y + \cdots + g_{m-1}y^{m-1}) = 0.$$

Thus we have $fR[y;\alpha]g = 0$ and the result follows. In $R[y;\alpha]$ we have $yx = \alpha(x)y = 0$ but $xy \neq 0$. Thus $R[y;\alpha]$ is not reversible.

Let R be a semiprime ring. In the next theorem we provide a necessary and sufficient condition for a skew generalized power series ring $R[[S,\omega]]$ to be reversible.

Theorem 2.2. *Let R be a semiprime ring, $(S,\leq)$ a strictly ordered a.n.u.p. monoid and $\omega : S \longrightarrow \mathrm{End}(R)$ a monoid homomorphism. Then $R[[S,\omega]]$ is reversible if and only if $R[[S,\omega]]$ is semicommutative and ω_s is injective for each $s \in S$.*

Proof. If $R[[S,\omega]]$ is reversible, then by [1, Theorem 4.12] ω_s is injective for each $s \in S$. Now assume that $R[[S,\omega]]$ is semicommutative and ω_s is injective for each $s \in S$. Let $s \in S$, $\omega_s \in \mathrm{End}(R)$ and $a \in R$ such that $a\omega_s(a) = 0$. Since R is semiprime and semicommutative, then R is a reduced ring and so $\omega_s(a)a = 0$. Thus by [1, Lemma 4.4], we have $\omega_s(a)\omega_s(a) = 0$. Then $a^2 = 0$ and so $a = 0$. Thus for each $s \in S$, ω_s is a rigid endomorphism. Then, by [1, Theorem 4.12], $R[[S,\omega]]$ is reversible and the result follows. □

References

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A. R. NASR-ISFAHANI, Department of Mathematics, University of Isfahan,
PO Box 81746-73441, Isfahan, Iran
and
School of Mathematics, Institute for Research in Fundamental Sciences (IPM),
PO Box 19395-5746, Tehran, Iran
e-mail: a_nasr_isfahani@yahoo.com

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