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ON NORMAL SUBGROUPS OF DIRECT PRODUCTS

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We investigate the equivalence classes of normal subdirect products of a product of free groups $F_{n_1} \times \cdots \times F_{n_k}$ under the simultaneous equivalence relations of *commensurability* and *conjugacy under the full automorphism group*. By abelianisation, the problem is reduced to one in the representation theory of quivers of free abelian groups. We show there are infinitely many such classes when $k \geqq 3$, and list the finite number of classes when $k=2$.

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0. Introduction

Two subgroups H_1, H_2 of a group G are said to be *conjugate in the generalised sense* when $\alpha(H_1)=H_2$ for some automorphism α of G. Similarly, we may consider *generalised commensurability*, by which we mean that $\alpha(H_1) \cap H_2$ has finite index in both $\alpha(H_1)$ and H_2. In this paper, we investigate the generalised conjugacy and commensurability classes of normal subgroups in a direct product of free groups; let F_k denote the free group with basis $(X_i)_{1 \leqq i \leqq k}$, and for $0 \leqq d \leqq \min\{n,m\}$, let $N(n,m,d)$ be the subgroup of $F_n \times F_m$

$$\text{generated by} \quad \begin{cases} (X_i, X_i) & 1 \leqq i \leqq d \\ (X_i, 1) & d+1 \leqq i \leqq n \\ \\ (1, X_j) & d+1 \leqq j \leqq m \\ (X_{ij}, 1) & 1 \leqq i < j \leqq d \end{cases}$$

where X_{ij} is the commutator $X_{ij} = X_i X_j X_i^{-1} X_j^{-1}$. Then $N(n,m,d)$ is a finitely generated normal subgroup, with infinite index when $d \geqq 1$; it is, moreover, a subdirect product; that is, it projects epimorphically onto each factor.

We will show that, up to generalised commensurability, these are the only normal subdirect products.

Theorem A. *Let N be a normal subdirect product of $F_n \times F_m$ where n, $m \geqq 2$. Then there exists a unique integer d with $0 \leqq d \leqq \min\{n,m\}$ such that, for some automorphism α of $F_n \times F_m$, $\alpha(N)$ has finite index in $N(n,m,d)$.*

By contrast, matters become more complicated when the number of factors exceeds three.

Theorem B. *The set of generalised commensurability classes of normal subdirect products in* $F_{n_1} \times \cdots \times F_{n_k}$ *is infinite when* $k \geqq 3$ *and each* $n_i \geqq 2$.

We note that the presence of the automorphism α in Theorem A is essential; the set of *commensurability classes* is infinite when $k \geqq 2$ (see, for example, Remark (2.5) below). The number of generalised commensurability classes is also infinite for a product of two Surface groups [3].

Theorem A also allows us to give the following description of finitely generated normal subgroups of $F_{n_1} \times F_{n_2}$.

Corollary C. *A nontrivial finitely generated normal subgroup* H *of* $F_{n_1} \times F_{n_2}$ *has either finite index in one of the factors, or finite index in a subgroup isomorphic to* $N(m_1, m_2, d)$, *where* $m_r = 1 + j_r(n_r - 1)$, *and* j_r *is the index of the projection of* H *in* F_{n_r}.

The following consequence of Theorem A seems, despite its naturality, to have gone previously unremarked.

Corollary D. *Let* $G_1, \ldots, G_k$ *be an arbitrary sequence of groups and let* $H \lhd G_1 \times \cdots \times G_k$ *be a normal subgroup. Then* H *is finitely generated if and only if each* $\pi_i(H)$ *is finitely generated where* $\pi_i: G_1 \times \cdots \times G_k \to G_i$ *is the projection onto the ith factor.*

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1. Product structures and abelianisation

Let $G_1, \ldots, G_k$ be groups and let $\pi_i: G_1 \times \cdots \times G_k \to G_i$ denote the ith projection. We denote by $S(G_1, \ldots, G_k)$ the set of normal subdirect products of $G_1 \times \cdots \times G_k$; that is

$$S(G_1, \ldots, G_k) = \left\{ \begin{array}{c} N: N \text{ is a normal subgroup of } \prod_{i=1}^{k} G_i \\ \text{and } \pi_i(N) = G_i \text{ for each } i \end{array} \right\}.$$

The following proposition is elementary.

Proposition 1.1. *Let* $\phi_i: G_i \twoheadrightarrow H_i$ *be surjective group homomorphisms for* $1 \leqq i \leqq k$. *If* N

is a normal subdirect product of $H_1 \times \cdots \times H_k$, *then* $(\phi_1 \times \cdots \times \phi_k)^{-1}(N)$ *is a normal subdirect product of* $G_1 \times \cdots \times G_k$.

In particular, when $\natural_i: G_i \twoheadrightarrow G_i^{ab}$ is the abelianisation map, we get a function $\natural^{-1}: S(G_1^{ab}, \ldots, G_k^{ab}) \to S(G_1, \ldots, G_k)$.

Proposition 1.2. *For any groups* $G_1, \ldots, G_k$, *the map*

$$\natural^{-1}: S(G_1^{ab}, \ldots, G_k^{ab}) \to S(G_1, \ldots, G_k) \text{ is bijective.}$$

Proof. Since $\natural^{-1}$ is clearly injective, we show that it is also surjective. Write $G_1 \times \cdots \times G_k$ as an "internal direct sum" $G = \tilde{G}_1 \oplus \cdots \oplus \tilde{G}_k$, in which $\tilde{G}_i$ centralises $\tilde{G}_j$ for $i \neq j$. Then the commutator subgroup is also an internal direct sum $[G, G] = [\tilde{G}_1, \tilde{G}_1] \oplus \cdots \oplus [\tilde{G}_k, \tilde{G}_k]$. When H is a normal subdirect product in G, we claim that each $[\tilde{G}_i, \tilde{G}_i] \subset H$. To see this, fix i: let x_i, $y_i \in \tilde{G}_i$, and choose $h \in H$ such that $\pi_i(h) = y_i$; that is, h is a product $h = h_1, \ldots, h_k$ with $h_j \in \tilde{G}_j$ for $j \neq i$, and $h_i = y_i$.

Since $\tilde{G}_i$ centralises $\tilde{G}_j$ for $i \neq j$, and H is normal in G, $x_i y_j x_i^{-1} y_j^{-1} = x_i h x_i^{-1} h^{-1} \in H$, so that $[\tilde{G}_i, \tilde{G}_i] \subset H$. Hence also $[G, G] \subset H$. Thus

$$H = H[G, G] = \natural^{-1}\natural(H)$$

so that $\natural^{-1}$ is surjective, and hence also bijective. □

Put $\mathrm{Aut}(G_1, \ldots, G_k) = \prod_{i=1}^k \mathrm{Aut}(G_i)$, considered as the group of product-preserving automorphisms of $G_1 \times \cdots \times G_k$ in the obvious way. We consider the relation of conjugacy of subdirect products under $\mathrm{Aut}(G_1, \ldots, G_k)$, strengthened by taking commensurability into account. Recall that two subgroups A, B are commensurable when $A \cap B$ has finite index in both A and B: when A and B are normal subgroups, this is equivalent to saying that both A and B have finite index in AB. We obtain an equivalence relation '$\approx$' on $S(G_1, \ldots, G_k)$ as follows;

$N_1 \approx N_2$ if and only if $\alpha(N_1)$ is commensurable with N_2 for some $\alpha \in \mathrm{Aut}(G_1, \ldots, G_k)$;

we write

$$\mathscr{C}(G_1, \ldots, G_k) = S(G_1, \ldots, G_k)/\approx.$$

Let F_n denote the free group of rank n. Abelianisation gives a map

$$\natural: F_{n_1} \times \cdots \times F_{n_k} \to \mathbf{Z}^{n_1} \oplus \cdots \oplus \mathbf{Z}^{n_k}$$

and a bijection

$$\natural: S(F_{n_1},\ldots,F_{n_k}) \to S(\mathbf{Z}^{n_1},\ldots,\mathbf{Z}^{n_k}),$$

namely, the inverse of the map of (1.2). Let $\langle N\rangle$ denote the class of $N\in S(G_1,\ldots,G_k)$ in $\mathscr{C}(G_1,\ldots,G_k)$. If $\alpha\in\operatorname{Aut}(F_{n_1},\ldots,F_{n_k})$, let $\alpha^{ab}\in\operatorname{Aut}\mathbf{Z}^{n_1},\ldots,\mathbf{Z}^{n_k})$ denote its abelianisation. If H and K are subgroups of $F_{n_1}\times\cdots\times F_{n_k}$, then

$$[\natural(H);\natural(H)\cap\alpha^{ab}\natural(K)]\leqq[H;H\cap\alpha(K)]$$

so that the mapping $\Psi:\mathscr{C}(F_{n_1},\ldots,F_{n_k})\to\mathscr{C}(\mathbf{Z}^{n_1},\ldots,\mathbf{Z}^{n_k})$

$$\Psi\langle N\rangle=\langle\natural(N)\rangle$$

is well defined.

Theorem 1.3. $\Psi:\mathscr{C}(F_{n_1},\ldots,F_{n_k})\to\mathscr{C}(\mathbf{Z}^{n_1},\ldots,\mathbf{Z}^{n_k})$ *is bijective.*

Proof. For $N\in\mathscr{C}(\mathbf{Z}^{n_1},\ldots,\mathbf{Z}^{n_k})$, $\Psi\natural^{-1}(N)=\langle N\rangle$, so that Ψ is surjective. If N_1, N_2 are commensurable in $\mathbf{Z}^{n_1}\oplus\cdots\oplus\mathbf{Z}^{n_k}$, then $\natural^{-1}(N_1)$, $\natural^{-1}(N_2)$ are commensurable in $F_{n_1}\times\cdots\times F_{n_k}$; if N_1 is conjugate to N_2 under $\operatorname{Aut}(\mathbf{Z}^{n_1},\ldots,\mathbf{Z}^{n_k})$, it follows from the Nielsen–Magnus Theorem ([7, 8]) on lifting automorphisms from $\mathbf{Z}^n$ to F_n, that $\natural^{-1}(N_1)$ is conjugate to $\natural^{-1}(N_2)$ under $\operatorname{Aut}(F_{n_1},\ldots,F_{n_k})$. Thus Ψ is injective. □

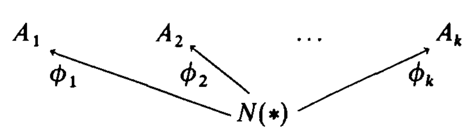
2. Subdirect products of abelian groups

In this section, we consider generalised commensurability classes of subdirect products in $A_1\oplus\cdots\oplus A_k$, where $(A_i)_{1\leqq i\leqq k}$ are finitely generated free abelian groups. We work in the category $Ab\langle k\rangle$ of diagrams of homomorphisms of abelian groups over the diagram scheme $\langle *,k\rangle$.

$\langle *,k\rangle$: the directed graph with vertex set $\{*,1,\ldots,k)$, having arrows

$$\phi_r: *\to r$$

for $1\leqq i\leqq k$. $\langle *,k\rangle$ is the dual of the "k-subspace quiver" ([2, 4]). Subdirect products N of $A_1\oplus\cdots\oplus A_k$ may be considered as objects in $Ab\langle k\rangle$ of the form



in which each ϕ_r is surjective, and the canonical morphism $N(*)\to\bigoplus_{i=1}^k A_i$ is injective. When $A_1,\ldots,A_k$ are understood, we confuse N with $N(*)$. Such a diagram is said to be *nondegenerate* when $\bigcap_{i\neq j}\operatorname{Ker}(\phi_j)=0$ for each i. We write $\operatorname{Supp}(N)=\{j: A_j\neq 0\}$.

$Ab\langle k\rangle$ has coproducts, defined by taking coproducts at each vertex. Similarly, one

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gets exact sequences in $Ab\langle k\rangle$ by taking sequences which are exact at each vertex. Thus for subdirect products in particular, one obtains a coproduct pairing

$$\square: S(A_1,\dots,A_k)\times S(B_1,\dots,B_k)\to S(A_1\oplus B_1,\dots,A_k\oplus B_k)$$

by means of $N\,\square\,M=\psi(N\oplus M)$ where

$$\psi:\left(\bigoplus_{i=1}^{k}A_i\right)\oplus\left(\bigoplus_{i=1}^{k}B_i\right)\to\bigoplus_{i=1}^{k}(A_i\oplus B_i)$$

is the obvious shuffling isomorphism.

An abelian subdirect product $N\in S(A_1,\dots,A_k)$ gives rise to a canonical exact sequence as follows; let $N_i=N\cap A_i$, and let $N(i)$ be the subdirect product with $N(i)(*)=N_i=N(i)_i$, $\phi_i=Id$, and $N(i)_j=0$ for $i\neq j$; let $\tilde{N}$ be the subdirect product in which $\tilde{N}(*)=N/(N_1+\cdots+N_k)$; $\tilde{N}_i=A_i/N_i$; and where $\tilde{\phi}_i:\tilde{N}(*)\to\tilde{N}_i$ is the map induced from $\phi_i:N\to A_i$. The following is clear.

Proposition 2.1. *Each $N\in S(A_1,\dots,A_k)$ decomposes functorially as an exact sequence $0\to N(1)\square\cdots\square N(k)\to N\to\tilde{N}\to 0$ in which $\tilde{N}$ is nondegenerate, and* $\operatorname{Supp}(N(i))=\{i\}$.

We define

$$r_i(N)=rk_{\mathbf{Z}}(N(i)(*))\quad(=rk_{\mathbf{Z}}(N_i)),$$

$$d(N)=rk_{\mathbf{Z}}(\tilde{N}(*))\quad(=rk_{\mathbf{Z}}(N/(N_1+\cdots+N_k)).$$

When $k=2$, the triple (r_1,r_2,d) is a complete set of invariants for the generalised commensurability class of N; to clarify the statement, consider the following subdirect product diagrams:

$$\tau_1=\left(\begin{array}{ccc}\mathbf{Z} & & 0\\ & \nwarrow^{1}\ \ \nearrow & \\ & \mathbf{Z} & \end{array}\right);\quad \tau_2=\left(\begin{array}{ccc}0 & & \mathbf{Z}\\ & \nwarrow\ \ \nearrow^{1} & \\ & \mathbf{Z} & \end{array}\right);$$

$$\Delta=\left(\begin{array}{ccc}\mathbf{Z} & & \mathbf{Z}\\ & \nwarrow^{1}\ \ \nearrow^{1} & \\ & \mathbf{Z} & \end{array}\right).$$

Theorem 2.2. *Let A_1, A_2 be free abelian groups of finite rank, and let $N\subset A_1\oplus A_2$ be* *a subdirect product: if N is maximal in its commensurability class then there is an isomorphism in $Ab\langle 2\rangle$*

$$N \cong \tau_1^{r_1(N)} \square \tau_2^{r_2(N)} \square \Delta^{d(N)}.$$

Proof. The hypothesis that N be maximal in its commensurability class is clearly equivalent to requiring that $(A_1 \oplus A_2)/N$ be torsion free. Hence $\tilde{N}_i = A_i/N_i$, and $\tilde{N}(*) = N/(N_1 + N_2)$ are free abelian. Moreover, $\tilde{\phi}_i : \tilde{N}(*) \to \tilde{N}_i$ is surjective; since $\tilde{N}$ is nondegenerate and $k=2$, each $\tilde{\phi}_i$ is also injective. (This fails for $k \geqq 3$.) Hence $\tilde{N}$ is a diagram of isomorphisms of free abelian groups;

$$\tilde{N} = \left(\tilde{N}_1 \xleftarrow{\ \phi_1\ } \tilde{N}(*) \xrightarrow{\ \phi_2\ } \tilde{N}_2 \right)$$

from which it follows immediately that $\tilde{N} \cong \Delta^{d(N)}$, since $d(N) = rk_{\mathbf{Z}}(\tilde{N}(*))$.

Since $\tilde{N}$ is torsion free, we may find a complementary subgroup $\Delta(N)$ to $N_1 \oplus N_2$ in N; $N = N_1 \oplus N_2 \oplus \Delta(N)$. The restriction of each ϕ_i to $\Delta(N)$ is injective. Writing $C_i = \phi_i(\Delta(N))$, we see that $A_i = N_i \oplus C_i$, and that N decomposes as a direct sum thus;

$$N = N(1) \oplus N(2) \oplus \tilde{\Delta}$$

where

$$\tilde{\Delta} = \left(C_1 \xleftarrow{\ \phi_1\ } \Delta(N) \xrightarrow{\ \phi_2\ } C_2 \right) \cong \tilde{N} \cong \Delta^{d(N)}.$$

The result follows, since $N(i) \cong \tau_i^{r_i(N)}$. □

If $N \subset A_1 \oplus A_2$ is a subdirect product, $rk_{\mathbf{Z}}(A_i) = r_i(N) + d(N)$. We may rephrase things in the following way.

Corollary 2.3. *If N is a subdirect product of $\mathbf{Z}^m \oplus \mathbf{Z}^n$, its class in $\mathscr{C}(\mathbf{Z}^m, \mathbf{Z}^n)$ is completely specified by the single integer-valued invariant $d(N)$ which takes arbitrary values in the range $0 \leqq d(N) \leqq \min\{m, n\}$; in particular, $\mathscr{C}(\mathbf{Z}^m, \mathbf{Z}^n)$ is finite.*

By contrast, we have:

Proposition 2.4. *If $k \geqq 3$, and each $n_i \geqq 1$, then $\mathscr{C}(\mathbf{Z}^{n_1}, \ldots, \mathbf{Z}^{n_k})$ is infinite.*

Proof. Let (a,b): $\mathbf{Z}^2 \to \mathbf{Z}$ be the mapping $(a,b)(x,y) = ax + by$, where a, $b \in \mathbf{Z}$. For each integer $n \geqq 1$, the diagram

$$D(n) = \left(\begin{array}{ccccc} \mathbf{Z} & & \mathbf{Z} & & \mathbf{Z} \\ & \nwarrow^{(1,0)} & \uparrow^{(0,1)} & \nearrow^{(1,n)} & \\ & & \mathbf{Z}^2 & & \end{array} \right).$$

describes a nondegenerate subdirect product, maximal in its commensurability class, and an easy computation shows that $D(n)$ is not isomorphic to $D(m)$ in $Ab\langle 3\rangle$ unless $n = m$. Thus $\mathscr{C}(\mathbf{Z}, \mathbf{Z}, \mathbf{Z})$ is infinite: by imbedding $\langle *, 3\rangle$ in $\langle *, k\rangle$, for $k \geqq 3$, and adding suitable degenerate summands, one sees that $\mathscr{C}(\mathbf{Z}^{n_1}, \ldots, \mathbf{Z}^{n_k})$ is infinite for $k \geqq 3$. $\square$

Remark 2.5. Observe that the set of *commensurability classes* obtained from $S(\mathbf{Z}^{n_1}, \ldots, \mathbf{Z}^{n_k})$ is always infinite, even for $k = 2$: for example, two maximal subgroups of rank 1 in $\mathbf{Z} \oplus \mathbf{Z}$ typically intersect in $\{0\}$; one obtains a corresponding statement for $S(F_{n_1}, \ldots, F_{n_k})$ from (1.2). Thus the presence of the automorphism α in Theorem A is essential.

One may compare this with the analogous problem for finite dimensional rational vector spaces; then "commensurability" is the same as "identity". When $k \geqq 4$, $\langle *, k\rangle$ is not the Dynkin diagram of any simple Lie group, and $\mathscr{C}(\mathbb{Q}^{n_1}, \ldots, \mathbb{Q}^{n_k})$ is infinite [2, 4]. However, $\mathscr{C}(\mathbb{Q}^{n_1}, \ldots, \mathbb{Q}^{n_k})$ is finite when $k \leqq 3$. $\mathscr{C}(\mathbb{Q}^{n_1}, \mathbb{Q}^{n_2})$ is described by a triple in a manner analogous to $\mathscr{C}(\mathbf{Z}^{n_1}, \mathbf{Z}^{n_2})$, on replacing '$rk_{\mathbf{Z}}$' by '$\dim_{\mathbb{Q}}$'; similarly $\mathscr{C}(\mathbb{Q}^{n_1}, \mathbb{Q}^{n_2}, \mathbb{Q}^{n_3})$ is described by an 8-tuple, corresponding to the multiplicities of various indecomposable diagrams over $\langle *, 3\rangle$. We note, however, that $(-)\otimes_{\mathbf{Z}}\mathbb{Q}: \mathscr{C}(\mathbf{Z}^{n_1}, \ldots, \mathbf{Z}^{n_k}) \to \mathscr{C}(\mathbb{Q}^{n_1}, \ldots, \mathbb{Q}^{n_k})$ fails to be injective for $k \geqq 3$.

3. Normal subgroups of a direct product of free groups

Our results so far depend for their expression on a particular product structure. In a direct product of free groups $F_{n_1} \times \cdots \times F_{n_k}$ with each $n_k \geqq 2$, however, the product structure is unique up to permutation of factors with the same rank ([5, Theorems 2.1, 2.7]), so for these groups, the notion of subdirect product is an intrinsic one. Indeed, the following is true (see, for example, [5, Corollary 2.9]).

Proposition 3.1. *The group* $\prod_{i=1}^{k} \mathrm{Aut}(F_{n_i})$ *of product-preserving automorphisms of* $F_{n_1} \times \cdots \times F_{n_k}$ *is a normal subgroup of finite index in the full automorphism group* $\mathrm{Aut}(F_{n_1} \times \cdots \times F_{n_k})$ *when each* $n_k \geqq 2$.

Our results thereby assume an absolute character:

Theorem 3.2. *The set $\mathcal{G}(n_1,\dots,n_k)$ of generalised commensurability classes of normal subdirect products of $F_{n_1}\times\cdots\times F_{n_k}$ is infinite when $k\geqq 3$ and finite when $k=2$, provided that each $n_k\geqq 2$.*

Proof. Assume that each $n_i\geqq 2$. Then $\mathcal{G}(n_1,\dots,n_k)$ is a quotient of $\mathcal{C}(F_{n_1},\dots,F_{n_k})$ by the finite group

$$\Phi=\operatorname{Aut}(F_{n_1}\times\cdots\times F_{n_k})\Big/\prod_{i=1}^{k}\operatorname{Aut}(F_{n_i}).$$

However, by (1.3), (2.3) and (2.4), $\mathcal{C}(F_{n_1},\dots,F_{n_k})$ is finite when $k=2$, and infinite when $k\geqq 3$, whence the result. □

In the case of two factors, $F_n\times F_m$, we can give representatives for the generalised commensurability classes: we have bijections

$$\mathcal{C}(F_n,F_m)\xrightarrow{\Psi}\mathcal{C}(\mathbf{Z}^n,\mathbf{Z}^m)\xrightarrow{d}\{0,1,\dots,\min\{n,m\}\}.$$

We define the *diagonal rank* $\delta(N)$ of a normal subdirect product, N, of $F_n\times F_m$ by

$$\delta(N)=d(\Psi\langle N\rangle).$$

All possible values of δ in the range $0\leqq\delta\leqq\min\{n,m\}$ can occur. We may see this explicitly as follows; for $0\leqq d\leqq\min\{n,m\}$, let $N(n,m,d)$ be the subgroup of $F_n\times F_m$ generated by:

$$(X_i,X_i)\,(1\leqq i\leqq d);(X_i,1)\,(d+1\leqq i\leqq n);(1,X_j)\,(d+1\leqq j\leqq m);$$

$$(X_{ij},1)\,(1\leqq i<j\leqq d);$$

where $\{X_1,\dots,X_k\}$ denotes a free basis for F_k, and X_{ij} denotes the commutator $X_{ij}=X_iX_jX_i^{-1}X_j^{-1}$.

Proposition 3.3. *$N(n,m,d)$ is a normal subdirect product of $F_n\times F_m$ with $\delta(N(n,m,d)=d$.*

Proof. Write $N=N(n,m,d)$; $\xi_i=\eta_i=(X_i,X_i)$ $1\leqq i\leqq d$;

$$\xi_i=(X_i,1)\,(d+1\leqq i\leqq n);\quad \eta_i=(1,X_j)\,(d+1\leqq j\leqq m);$$

$$\xi^{ij}=(X_{ij},1)\,(1\leqq i<j\leqq n);\quad \eta^{rs}=(1,X_{rs})\,(1\leqq r<s\leqq m);$$

By definition, each ξ_i, $\eta_i \in N$, and it is easy to see that $\xi^{ij}, \eta^{rs} \in N$ for each i,j,r,s with $1 \leqq i<j \leqq n$ and $1 \leqq r<s \leqq m$. If $W=W(X_1,\dots,X_k)$ is a word in $\{X_1,\dots,X_k\}$, let $w=w(\xi_1,\dots,\xi_k)$ denote the corresponding word in $\{\xi_1,\dots,\xi_k\}$. Then

$$w\xi^{ij}w^{-1}=(WX_{ij}W^{-1},1)$$

Since $[F_n,F_n]$ is generated by elements of the form $WX_{ij}W^{-1}$, we see that $[F_n,F_n]\times\{1\} \subset N$. Replacing ξ_i, ξ^{ij} by η_r, η^{rs}, we see, by symmetry, that $\{1\}\times[F_m,F_m]\subset N$. Hence $N=\square^{-1}\square(N)$, where

$$\square : F_n \times F_m \twoheadrightarrow \mathbf{Z}^n \oplus \mathbf{Z}^m$$

is the abelianisation map, so that N is normal. N is obviously a subdirect product, and $\delta(N)$ clearly takes the value d. □

If $n \neq m$, $\mathrm{Aut}(F_n)\times\mathrm{Aut}(F_m)=\mathrm{Aut}(F_n\times F_m)$, whilst $\mathrm{Aut}(F_n)\times\mathrm{Aut}(F_n)$ has index 2 in $\mathrm{Aut}(F_n\times F_n)$, with the swap involution representing the nontrivial coset. Since each $N(n,n,d)$ is invariant under the swap, we obtain:

Theorem 3.4. *If $2\leqq m\leqq n$, $F_n\times F_m$ has precisely $m+1$ generalised commensurability classes of normal subdirect products: in particular, if N is a normal subdirect product of $F_n\times F_m$, there exists a unique integer d with $0\leqq d\leqq m$ such that, for some automorphism α of $F_n\times F_m$, $\alpha(N)$ has finite index in $N(n,m,d)$.*

Proof. Observe that $\square(N(n,m,d))$ is maximal in its commensurability class in $\mathbf{Z}^n\oplus\mathbf{Z}^m$, so that $N(n,m,d)$ is similarly maximal within $F_n\times F_m$. Note also that $N(n,m,d)$ is normal in $F_n\times F_m$. If N is a normal subdirect product of $F_n\times F_m$ it follows from (1.3), (2.3) and (3.3), that one can find $\alpha\in\mathrm{Aut}(F_n\times F_m)$ such that $\alpha(N)$ and $N(n,m,d)$ are commensurable. By maximality of $N(n,m,d)$, $\alpha(N)$ is contained with finite index in $N(n,m,d)$. □

In general, a normal subgroup H of $F_{n_1}\times F_{n_2}$ is a normal subdirect product of $\pi_1(H)\times\pi_2(H)$, where π_r denotes projection onto the rth factor; by the Nielsen–Schreier Theorem ([**6**, p. 104]), $\pi_r(H)$ is either trivial or free; we obtain:

Corollary 3.5. *A nontrival finitely generated normal subgroup H of $F_{n_1}\times F_{n_2}$ has either finite index in one of the factors, or finite index in a subgroup isomorphic to $N(m_1,m_2,d)$, where $m_r=1+j_r(n_r-1)$, and j_r is the index of the projection of H in F_{n_r}.*

Finally, we observe:

Corollary 3.6. *Let $G_1, \dots, G_k$ be an arbitrary sequence of groups and let $H \triangleleft G_1 \times \cdots \times G_k$ be a normal subgroup. Then H is finitely generated $\Leftrightarrow$ each $\pi_i(H)$ is finitely generated where $\pi_i: G_1 \times \cdots \times G_k \to G_i$ is the projection onto the ith factor.*

Proof. The implication "$\Rightarrow$" is trivial. In proving "$\Leftarrow$", one may, by projection, reduce the problem to that for a subdirect product; that is, it suffices to prove the following for each $k \geqq 2$:

$\mathscr{P}(k)$: let $G_1, \dots, G_k$ be finitely generated groups, and let H be a normal subdirect product of $G_1 \times \cdots \times G_k$. Then H is finitely generated.

Suppose that $k=2$; in the special case where G_1, G_2 are free, the conclusion follows from (3.4), since H is isomorphic to a subgroup of finite index in a group of the form $N(n,m,d)$, which is finitely generated by definition. In general, let $\phi_i: F_i \twoheadrightarrow G_i$ be an epimorphism from a finitely generated free group F_i. Then $(\phi_1 \times \phi_2)^{-1}(H)$ is a normal subdirect product of $F_1 \times F_2$ hence is finitely generated by the special case above. Thus H is also finitely generated, being an epimorphic image of $(\phi_1 \times \phi_2)^{-1}(H)$.

Suppose that H is a normal subdirect product of $G_1 \times \cdots \times G_{k+1}$. Let K be the projection of H in $G_2 \times \cdots \times G_{k+1}$; K is a normal k-fold subdirect product, hence is finitely generated, by induction, and H is a normal subdirect product of $G_1 \times K$, so is finitely generated, by $\mathscr{P}(2)$. □

The groups $N(n,m,d)$ are not finitely presented unless $d=0$. This follows easily from Theorem 2 of [1]. They also provide examples of Zariski dense, *algebraically irreducible* discrete subgroups of $SL_2(\mathbf{R}) \times SL_2(\mathbf{R})$, both of whose projections are discrete. Thus the simple picture provided by Borel's Density Theorem ([9, Ch. V]) breaks down for Zariski dense subgroups of infinite covolume.

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