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SEMIREGULAR GROUP DIVISIBLE DESIGNS WITH DUAL PROPERTIES

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A construction method for group divisible designs is employed to construct (i) infinitely many non-symmetric semiregular group divisible designs whose duals are semiregular group divisible designs, and (ii) infinitely many transversal designs whose duals are group divisible 3-associate designs. A construction method for affine α-resolvable balanced incomplete block designs is also given and illustrated.

1. INTRODUCTION

In this paper a construction method for group divisible designs (given in Section 3) is employed to construct semiregular group divisible designs having various sorts of uniform block intersections. (We refer to such group divisible designs as having "dual properties".) In particular, in Section 4 we construct infinitely many non-symmetric semiregular group divisible designs whose duals are semiregular group divisible designs and also infinitely many transversal designs whose duals are group divisible 3-associate designs [7]. Excluding constructions for self-dual transversal designs, constructions for infinite classes of semiregular group divisible designs whose duals are semiregular group divisible designs were largely unknown prior to the author's paper [8]. (The present paper, in fact, should be read as a sequel to [8]. Further details about what was known prior to [8] concerning the existence of semiregular group divisible designs with semiregular group divisible duals can be found in the introductory section of [8].) In Section 3 of this paper we also give a construction method for semiregular group divisible designs $\mathcal{G}^*$ whose duals are BIBDs (see Theorem 3(a)). The dual $\mathcal{G}^{*d}$ of $\mathcal{G}^*$ is then an affine α-resolvable BIBD. This construction method is illustrated in Section 4.

2. GROUP DIVISIBLE DESIGNS

A tactical configuration ([3], p.4) with v points, b blocks, r blocks on each point and k points on each block is called a (v,b,r,k)-*configuration*. A (v,b,r,k)-configuration $(\mathcal{P},\mathcal{B},\mathcal{I})$ is said to be a *group divisible design* (GDD) if there is a partition

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of $\mathcal{P}$ into "groups" $\mathcal{P}_1, \ldots, \mathcal{P}_{m_2}$, where $m_2 \geqslant 2$, such that there are integers $m_1 \geqslant 2$ and λ_1 and λ_2 such that

(a) $|\mathcal{P}_i| = m_1$ for all $i = 1, \ldots, m_2$,
(b) any two points common to a group are on λ_1 blocks of $\mathcal{B}$,
(c) any two points in different groups are on λ_2 blocks of $\mathcal{B}$, and
(d) $\lambda_1 \neq \lambda_2$.

We say that such a GDD $\mathcal{G}$ "has parameters v, b, r, k; m_1, m_2; λ_1, λ_2" and that $\mathcal{P}_1, \ldots, \mathcal{P}_{m_2}$ is a "group division" of $\mathcal{G}$.

The parameters of a GDD satisfy the following equations

and

(1) $$vr = bk$$

(2) $$v = m_1 m_2$$

(3) $$(m_1 - 1)\lambda_1 + m_1(m_2 - 1)\lambda_2 = r(k - 1).$$

It is well-known that group divisions can be exhaustively classified into the following mutually exclusive types:

1. Singular, for which $r = \lambda_1$.
2. Semiregular, for which $r > \lambda_1$ and

(4) $$rk = v\lambda_2.$$

3. Regular, for which $r > \lambda_1$ and $rk > v\lambda_2$.

Since a GDD has a unique group division we can apply the terms "singular", "semiregular" and "regular" to GDDs as well as to group divisions.

For a semiregular GDD with parameters v, b, r, k; m_1, m_2; λ_1, λ_2 we have, from (2), (3) and (4), that

(5) $$r + (m_1 - 1)\lambda_1 = m_1\lambda_2.$$

REMARK. The basic results we have thus far mentioned concerning GDDs are given in [2].

A semiregular GDD with $\lambda_1 = 0$ is called a *transversal design*. For a transversal design we must have $k = m_2$.

It is possible for a GDD to possess two points which are incident with precisely the same blocks. Such points are said to be "repeated". It is easy to show that a GDD $\mathcal{G}$ has repeated points if and only if $\mathcal{G}$ is singular.

Next, let $k > 0$ and $\overline{m}_2 \geqslant 2$. An *$\alpha$-resolution* of a (v, b, r, k)-configuration $\mathcal{C} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a partition $\mathcal{B}_1, \ldots, \mathcal{B}_{\overline{m}_2}$ of $\mathcal{B}$ such that each point of $\mathcal{P}$ is on precisely α

blocks of each block class $\mathcal{B}_i$. For any α-resolution $\mathcal{B}_1, \ldots, \mathcal{B}_{\overline{m}_2}$ it is straightforward to show that $|\mathcal{B}_i| = b/\overline{m}_2$ for each $i = 1, \ldots, \overline{m}_2$ and that $\alpha = r/\overline{m}_2$. A 1-resolution is called a *parallelism.*

Let $\mathcal{G} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a semiregular GDD with parameters v, b, r, $k; m_1$, $m_2; \lambda_1$, λ_2. Bose and Connor [2] have shown that each block of $\mathcal{B}$ meets each group of $\mathcal{G}$ in k/m_2 points. Clearly, the groups of $\mathcal{G}$ form a (k/m_2)-resolution of $\mathcal{G}^d$ (the dual of $\mathcal{G}$). We also have

RESULT 1. Suppose $\mathcal{G}$ is a GDD with parameters v, b, r, $k; m_1$, $m_2; \lambda_1$, λ_2. If each block of $\mathcal{B}$ meets each group of $\mathcal{G}$ in k/m_2 points (that is, if the groups of $\mathcal{G}$ form a (k/m_2)-resolution of $\mathcal{G}^d$), then $\mathcal{G}$ is semiregular.

PROOF: There are $\Omega_1 = \binom{m_1 m_2}{2} - \binom{m_1}{2} m_2$ pairs of points P, Q that can be chosen such that P and Q are in different groups of $\mathcal{G}$. Each block of G accounts for $\Omega_2 = \binom{k}{2} - \binom{k/m_2}{2} m_2$ such pairs of points. So we have $\Omega_1 \lambda_2 = \Omega_2 b$ from which $(m_1 m_2)^2 \lambda_2 = k^2 b$ follows readily. But then (1) and (2) yield $v\lambda_2 = rk$. □

REMARK. Result 1 is well known. A proof has been included since the author has not found one in the literature.

An α-resolution $\mathcal{B}_1, \ldots, \mathcal{B}_{\overline{m}_2}$ of a (v, b, r, k)-configuration $\mathcal{G}$ is said to be an *affine α-resolution* if $\mathcal{B}_1, \ldots, \mathcal{B}_{\overline{m}_2}$ is a (by Result 1, necessarily semiregular) group division of $\mathcal{G}^d$. A BIBD with an affine parallellism with t blocks in each affine parallel class and each pair of non-disjoint blocks meeting in μ points is called an $ARD(\mu, t)$.

We shall also need the following result.

RESULT 2. (Mitchell, [6]). Let $\mathcal{G}$ be a GDD such that $\mathcal{G}^d$ is also a GDD. Suppose that $\mathcal{G}$ has no repeated points or blocks (equivalently, that neither $\mathcal{G}$ nor $\mathcal{G}^d$ is a singular GDD). Then

(a) $\mathcal{G}$ and $\mathcal{G}^d$ are each semiregular, or

(b) $\mathcal{G}$ and $\mathcal{G}^d$ are each regular and $\mathcal{G}$ is symmetric.

Consider a semiregular GDD $\mathcal{G}$ with parameters $v, b, r, k;\ m_1, m_2;\ \lambda_1, \lambda_2$ whose dual is a semiregular GDD with parameters $b, v, k, r;\ \overline{m}_1, \overline{m}_2;\ \rho_1, \rho_2$. Equations (2), (4) and (5) for $\mathcal{G}^d$ yield $b = \overline{m}_1 \overline{m}_2$, $rk = b\rho_2$ and

(6) $$k + (\overline{m}_1 - 1)\rho_1 = \overline{m}_1 \rho_2.$$

There is one other equation which will be of use to us. Assuming $\mathcal{G}$ and $\mathcal{G}^d$ are semiregular GDDs with parameters as just given, let A be an incidence matrix for $\mathcal{G}$. Now AA^t has a unique eigenvalue distinct from 0 and rk (namely $r - \lambda_1$) and A^tA also has a unique such eigenvalue (namely $k - \rho_1$). Since the non-zero eigenvalues of AA^t and A^tA are the same, we have

(7) $$r - \lambda_1 = k - \rho_1.$$

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REMARKS. (a) Equation (7) has been derived by Mitchell [**6**]. For a more general context in which such an equation applies see [**1**].

(b) We also note that if $\mathcal{G}^d$ is a BIBD with index ρ, then an argument similar to that which led to (7) yields

$$r - \lambda_1 = k - \rho. \tag{8}$$

A GDD is said to be *self-dual* if $\mathcal{G}^d$ is a GDD with the same parameters as $\mathcal{G}$. (Note that this is a weaker notion of self-duality than that which requires $\mathcal{G}$ and $\mathcal{G}^d$ to be isomorphic.)

3. A Construction Method

Suppose $\mathcal{G} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a semiregular GDD with parameters $v, b, r, k;\ m_1, m_2;\ \lambda_1, \lambda_2$ and groups $\mathcal{P}_i$, $i = 1, \ldots, m_2$. Suppose also that $\mathcal{G}' = (\mathcal{P}', \mathcal{B}', \mathcal{I}')$ is a GDD with an α-resolution $\mathcal{B}'_1, \ldots, \mathcal{B}'_b$ with $|\mathcal{B}'_i| = \overline{m}'_1$ for $i = 1, \ldots, b$. Furthermore, suppose that the parameters of $\mathcal{G}'$ are $v', \overline{m}'_1 b, \overline{m}'_1 r, k';\ m_1, m'_2;\ \overline{m}'_1\lambda_1, \overline{m}'_1\lambda_2$ and that the groups of $\mathcal{G}'$ are $\mathcal{P}'_i$, $i = 1, \ldots, m'_2$. We also suppose $\mathcal{P} \cap \mathcal{P}' = \emptyset$. We label the blocks of $\mathcal{G}$ by B_i, $i = 1, \ldots, b$, and the blocks in $\mathcal{B}'_i$ by B'_{ij}, $j = 1, \ldots, \overline{m}'_1$, and define sets B^*_{ij}, $i = 1, \ldots, b$, $j = 1, \ldots, \overline{m}'_1$ by

$$B^*_{ij} = (B'_{ij}) \cup (B_i).$$

(Here (B) stands for the set of points incident with block B.) Further, let $\mathcal{B}^*_i = \{B^*_{ij} : j = 1, \ldots, \overline{m}'_1\}$.

PROPOSITION 1. *The incidence structure* $\mathcal{G}^* = (\mathcal{P}^*, \mathcal{B}^*, \mathcal{I}^*)$, *where* $\mathcal{P}^* = \mathcal{P} \cup \mathcal{P}'$, $\mathcal{B}^* = \bigcup_{i=1}^{b} \mathcal{B}^*_i$ *and* $\mathcal{I}^*$ *is given by set-theoretical inclusion, is a GDD with parameters* $v + v'$, $\overline{m}'_1 b$, $\overline{m}'_1 r$, $k + k'$; $m_1, m_2 + m'_2$; $\overline{m}'_1\lambda_1, \overline{m}'_1\lambda_2$ *and groups* $\mathcal{P}_1, \ldots, \mathcal{P}_{m_2}$, $\mathcal{P}'_1, \ldots, \mathcal{P}'_{m'_2}$.

PROOF: We establish the "balance" properties for $\mathcal{G}^*$ and leave the rest to the reader.

The substructure of $\mathcal{G}^*$ defined by $\mathcal{P}$ and $\mathcal{B}^*$ is an $\overline{m}'_1$-multiple of a GDD isomorphic to $\mathcal{G}$ and with groups $\mathcal{P}_1, \ldots, \mathcal{P}_{m_2}$, and the substructure of $\mathcal{G}^*$ defined by $\mathcal{P}'$ and $\mathcal{B}^*$ is isomorphic to $\mathcal{G}'$ and has groups $\mathcal{P}'_1, \ldots, \mathcal{P}'_{m'_2}$. So two points in the same $\mathcal{P}_i$, or in the same $\mathcal{P}'_i$, are on $\overline{m}'_1\lambda_1$ blocks of $\mathcal{G}^*$. Also, two points, one from $\mathcal{P}_i$ and one from $\mathcal{P}_j$, where $i \neq j$, are on $\overline{m}'_1\lambda_2$ blocks of $\mathcal{G}^*$ and similarly for a pair of points, one from $\mathcal{P}'_i$ and one from $\mathcal{P}'_j$.

Consider $P \in \mathcal{P}$ and $P' \in \mathcal{P}'$. P is on r blocks of $\mathcal{G}$, say $B_{i_1}, \ldots, B_{i_r}$. So the set of blocks of $\mathcal{G}^*$ which are incident with P is $\bigcup_{l=1}^{r} \mathcal{B}^*_{i_l}$. But P' is on $\alpha = \overline{m}'_1 r/b = \overline{m}'_1 k/v$

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blocks of each $\mathcal{B}'_i$ and hence is on this number of blocks of each $\mathcal{B}^*_i$. Therefore P and P' are common to $\overline{m}'_1 rk/v = \overline{m}'_1\lambda_2$ blocks of $\mathcal{G}^*$. ∎

REMARKS. (a) Since $\mathcal{G}$ is not singular, neither $\mathcal{G}'$ nor $\mathcal{G}^*$ is singular.

(b) If $\mathcal{G}$ is a (v,b,r,k,λ)-design and $\mathcal{G}'$ is a $(v,\overline{m}'_1 b,\overline{m}'_1 r,k,\overline{m}'_1\lambda)$-design with an α-resolution $\mathcal{B}'_1, \ldots, \mathcal{B}'_b$ with $|\mathcal{B}'_i| = \overline{m}'_1$ for $i = 1, \ldots, b$, then the construction method above yields a (semiregular) GDD with two groups of points (namely $\mathcal{P}$ and $\mathcal{P}'$). In this situation the parameters of $\mathcal{G}^*$ are $2v,\overline{m}'_1 b,\overline{m}'_1 r,2k;v,2;\overline{m}'_1\lambda,(\overline{m}'_1 rk)/v$.

PROPOSITION 2. *$\mathcal{G}^*$ is semiregular if and only if $\mathcal{G}'$ is semiregular.*

PROOF:

$$
\begin{aligned}
G^* \text{ is semiregular } &\iff (v+v')\overline{m}'_1\lambda_2 = \overline{m}'_1 r(k+k') \\
&\iff v'(\overline{m}'_1\lambda_2) = (\overline{m}'_1 r)k' \qquad \text{(since } \mathcal{G} \text{ is semiregular)} \\
&\iff \mathcal{G}' \text{ is semiregular.}
\end{aligned}
$$

∎

The most obvious way to apply our construction method to obtain semiregular GDDs is to let $v = v'$ (whence $m_2 = m'_2$), $k = k'$, $\mathcal{G}$ be a semiregular GDD with an α-resolution with $\overline{m}'_1$ blocks in each α-resolution class and $\mathcal{G}'$ be an $\overline{m}'_1$-multiple of a GDD isomorphic to $\mathcal{G}$. In this situation the α-resolution of $\mathcal{G}$ induces an α-resolution of $\mathcal{G}'$ with $\overline{m}'_1$ blocks in each α-resolution class.

REMARKS. (a) Our construction method generalises the dual form of the construction method considered in Theorem 7(ii) of Mavron [**5**].

(b) Semiregular GDDs with an α-resolution can be constructed from a BIBD and a GDD with a parallelism by using the method of [**9**]. The method of [**8**, Section 4], can also be used to construct semiregular GDDs with an α-resolution.

(c) Suppose we apply our construction method with $\mathcal{G}$ a regular GDD and $m_2 = m'_2$. Then, for the constructed $\mathcal{G}^*$, we have that

(i) two points common to a group of $\mathcal{G}$ or to a group of $\mathcal{G}'$ are on $\overline{m}'_1\lambda_1$ ($\neq \overline{m}'_1\lambda_2$) blocks of $\mathcal{G}^*$,

(ii) two points in different groups of $\mathcal{G}$ or of $\mathcal{G}'$ are on $\overline{m}'_1\lambda_2$ blocks of $\mathcal{G}^*$, and

(iii) $|\mathcal{P}| = |\mathcal{P}'|$ and a point of $\mathcal{P}$ and a point of $\mathcal{P}'$ are on $(\overline{m}'_1 rk)/v$ ($\neq \overline{m}'_1\lambda_2$) blocks of $\mathcal{G}^*$.

Thus, in the terminology of [**7**], $\mathcal{G}^*$ is a "group divisible 3-associate design".

Next, we consider the situation in which the α-resolution of $\mathcal{G}'$ is an affine α-resolution. When this is so $\mathcal{G}'^d$ is a semiregular GDD. Since $\mathcal{G}'$ is not singular we

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have, by Result 2, that $\mathcal{G}'$ is also semiregular, and so by Proposition 2, $\mathcal{G}^*$ is semiregular. Since the construction of semiregular GDDs with dual properties is our aim, the following theorem is our main result.

THEOREM 3. *Suppose the α-resolution of $\mathcal{G}'$ is an affine α-resolution of $\mathcal{G}'$ with each pair of blocks in the same α-resolution class meeting in ρ_1' points and each pair of blocks in different α-resolution classes meeting in ρ_2' points.*

(a) *If $\mathcal{G}^d$ is an affine (k/m_2)-resolvable (b, v, k, r, ρ)-design, then $\mathcal{G}^{*d}$ is an affine $((k+k')/(m_2+m_2'))$-resolvable $(\overline{m}_1'b, v+v', k+k', \overline{m}_1'r, \rho+\rho_2')$-design.*

(b) *If $\mathcal{G}^d$ is a semiregular GDD with parameters $b, v, k, r; \overline{m}_1, \overline{m}_2; \rho_1, \rho_2$ and groups $\mathcal{B}_\gamma = \{B_{(\gamma-1)\overline{m}_1+\beta} : \beta = 1, \ldots, \overline{m}_1\}, \gamma = 1, \ldots, \overline{m}_2$, then $\mathcal{G}^{*d}$ is a semiregular GDD with parameters $\overline{m}_1'b, v+v', k+k', \overline{m}_1'r; \overline{m}_1'\overline{m}_1, \overline{m}_2; \rho_1+\rho_2', \rho_2+\rho_2'$ and groups $\bigcup_{f=(i-1)\overline{m}_1+1}^{i\overline{m}_1} \mathcal{B}_f^*, i = 1, \ldots, \overline{m}_2$.*

PROOF: We shall prove only Part (b). Part (a) can be established in a similar manner. (At the point in the proof of Part (b) where (7) is applied to $\mathcal{G}$ one applies (8).)

(b) For convenience we denote $\mathcal{B}_f^*$ by $\mathcal{B}^*(i,j)$, where i and j are the unique integers such that $f = (i-1)\overline{m}_1 + j$, $1 \leqslant i \leqslant \overline{m}_2$ and $1 \leqslant j \leqslant \overline{m}_1$.

Let $B^* \in \mathcal{B}^*(i,j)$ and $B^{**} \in \mathcal{B}^*(h,l)$. We consider three cases.

CASE 1: $i = h$, $j = l$

B^* and B^{**} meet in k points of $\mathcal{P}$ and ρ_1' points of $\mathcal{P}'$.

CASE 2: $i = h$, $j \neq l$

B^* and B^{**} meet in ρ_1 points of $\mathcal{P}$ and ρ_2' points of $\mathcal{P}'$.

CASE 3: $i \neq h$

B^* and B^{**} meet in ρ_2 points of $\mathcal{P}$ and ρ_2' points of $\mathcal{P}'$.

Now

$$\begin{aligned}
\rho_2' - \rho_1' &= \frac{k' - \rho_1'}{\overline{m}_1'} && \text{((6) applied to } \mathcal{G}') \\
&= \frac{\overline{m}_1' r - \overline{m}_1' \lambda_1}{\overline{m}_1'} && \text{((7) applied to } \mathcal{G}') \\
&= r - \lambda_1 \\
&= k - \rho_1 && \text{((7) applied to } \mathcal{G}).
\end{aligned}$$

Clearly we have $k + \rho_1' = \rho_1 + \rho_2'$ and so $\mathcal{G}^{*d}$ is a GDD with groups $\bigcup_{j=1}^{\overline{m}_1} \mathcal{B}^*(i,j) =$

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$\bigcup_{f=(i-1)\overline{m}_1+1}^{i\overline{m}_1} \mathcal{B}_f^*$. But then $\overline{m}_1' b(\rho_2+\rho_2') = \overline{m}_1' r(k+k')$ is easily established, since $\mathcal{G}^d$ and $\mathcal{G}'^d$ are semiregular. Hence, $\mathcal{G}^{*d}$ is semiregular. □

REMARKS. (a) The construction of Theorem 3(a) generalises the dual form of that of Theorem 7(iii) of Mavron [5]. Note that we have used a different method of proof to Mavron in generalising his results.

(b) The only construction methods for affine α-resolvable BIBDs with $\alpha > 1$ known to the author are

(i) that of Shrikhande and Raghavarao [9],
(ii) a construction method in [8, see p. 170], and
(iii) that of Theorem 3(a) above.

4. ILLUSTRATIONS

In this section we give some constructions of semiregular GDDs with dual properties that illustrate the method developed in Section 3. Throughout this section a self-dual GDD with parameters

$$q^{n+2},\ q^{n+2},\ q^{n+1}\sigma,\ q^{n+1}\sigma;\ q,\ q^{n+1};\ q^{n+1}\frac{\sigma(\sigma-1)}{q-1},\ q^n\sigma^2$$

will be referred to as an $\mathcal{S}(q, n, \sigma)$. $\mathcal{S}(q, n, 1)$'s are transversal designs and are known to exist for q any prime power and all $n \geqslant 0$. Given an $\mathcal{S}(q, n, 1)$ and a $(q, \sigma, (\sigma(\sigma-1))/(q-1))$-design we can apply the method of [8, Section 4], to construct an $\mathcal{S}(q, n, \sigma)$.

Consider an affine σ-resolvable $(q^{n+1}, (q(q^{n+1}-1))/(q-1), (\sigma(q^{n+1}-1))/(q-1), q^n\sigma, (\sigma(q^n\sigma-1))/(q-1))$-design $\mathcal{G}^d$. The dual $\mathcal{G}$ of $\mathcal{G}^d$ is a semiregular GDD with parameters

$$\frac{q(q^{n+1}-1)}{q-1},\ q^{n+1},\ q^n\sigma,\ \frac{\sigma(q^{n+1}-1)}{q-1};\ q,\ \frac{q^{n+1}-1}{q-1};\ q^n\frac{\sigma(\sigma-1)}{q-1},\ q^{n-1}\sigma^2.$$

(Such affine σ-resolvable designs can be constructed using the method of [9].) With $\mathcal{G}'$ an $\mathcal{S}(q, n, \sigma)$, Theorem 3(a) yields that $\mathcal{G}^{*d}$ is an affine σ-resolvable design with parameters

$$\left(q^{n+2},\ \frac{q(q^{n+2}-1)}{q-1},\ \frac{\sigma(q^{n+2}-1)}{q-1},\ q^{n+1}\sigma,\ \frac{\sigma(q^{n+1}\sigma-1)}{q-1}\right).$$

Next, given an $\mathcal{S}(q, n, 1)$ and a self-dual semiregular GDD $\overline{\mathcal{G}}$ with parameters $v, v, k, k; q, m_2; \lambda_1, \lambda_2$, it is possible to construct a self-dual semiregular GDD $\mathcal{G}$ with parameters $q^{n+1}v,\ q^{n+1}v,\ q^{n+1}k,\ q^{n+1}k;\ q,\ q^{n+1}m_2;\ q^{n+1}\lambda_1,\ q^{n+1}\lambda_2$ (see [8, Section 4]).

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Using an $\mathcal{S}(q, n+1, 1)$ with $\overline{\mathcal{G}}$ in this way yields a self-dual semiregular GDD $\mathcal{G}'$ with parameters $q^{n+2}v$, $q^{n+2}v$, , $q^{n+2}k$, $q^{n+2}k$; q, $q^{n+2}m_2$; $q^{n+2}\lambda_1$, $q^{n+2}\lambda_2$. Propositions 1 and 2 and Theorem 3(b) allow us to infer that $\mathcal{G}^*$ is a semiregular GDD whose dual $\mathcal{G}^{*d}$ is also a semiregular GDD. In this situation $\mathcal{G}^*$ is not symmetric. If $\overline{\mathcal{G}}$ is a transversal design, then $\mathcal{G}^*$ is a transversal design whose dual is a semiregular GDD, but not a transversal design. Examples for $\overline{\mathcal{G}}$ which are not transversal designs are known with $v = \overline{q}^h\left((\overline{q}^h-1)/(\overline{q}-1)\right)$, $k = \overline{q}^{h-1}\overline{\sigma}\left((\overline{q}^h-1)/(\overline{q}-1)\right)$, $m_2 = (\overline{q}^h-1)/(\overline{q}-1)$, $\lambda_1 = \overline{q}^{h-1}\overline{\sigma}\left((\overline{q}^{h-1}\overline{\sigma}-1)/(\overline{q}-1)\right)$ and $\lambda_2 = \overline{q}^{h-2}\overline{\sigma}^2\left((\overline{q}^h-1)/(\overline{q}-1)\right)$, where $\overline{q}$ is any prime power, $h \geqslant 2$ and there is a $(\overline{q}, \overline{\sigma}, (\overline{\sigma}(\overline{\sigma}-1))/\overline{q}-1)$-design (possibly a $(\overline{q}, 1, 0)$-"design"); see [**8**, pp. 170-1]. (Examples with these parameters with $\overline{\sigma} = 1$ are also given in [4].)

Finally, consider a semiregular GDD $\mathcal{G}$ such that $\mathcal{G}^d$ is a singular GDD and a semiregular GDD $\mathcal{G}'$ whose dual is a semiregular GDD. Suppose that the parameters of $\mathcal{G}$ and $\mathcal{G}'$ are such that the construction method of Section 3 can be applied to produce a semiregular GDD $\mathcal{G}^*$. An argument of a similar nature to that which established Theorem 3(b) shows that $\mathcal{G}^{*d}$ is a group divisible 3-associate design.

Let $\mathcal{D}$ be an $ARD(q^{H-2}, q)$, where q is a prime power and $H \geqslant 2$. Also, let $\mathcal{G}$ be a q^{n+1-H}-multiple of $\mathcal{D}^d$, where $n \geqslant H$. Then $\mathcal{G}$ is a semiregular GDD with parameters

$$\frac{q(q^H-1)}{q-1},\ q^{n+1},\ q^n,\ \frac{q^H-1}{q-1};q,\ \frac{q^H-1}{q-1};0,q^{n-1}$$

and $\mathcal{G}^d$ is a singular GDD with parameters

$$q^{n+1},\ \frac{q(q^H-1)}{q-1},\ \frac{q^H-1}{q-1},\ q^n;q^{n+1-H},\ q^H;\frac{q^H-1}{q-1},\ \frac{q^{H-1}-1}{q-1}.$$

(These results concerning the parameters of $\mathcal{G}$ and $\mathcal{G}^d$ follow from the expressions for the parameters of an $ARD(\mu, m)$ as given on p.73 of [3].) Let $\mathcal{G}'$ be a semiregular GDD with parameters

$$q^{n+2}\left(\frac{q^h-1}{q-1}\right),\ q^{h+n+1},\ q^{h+n},\ q^{n+1}\left(\frac{q^h-1}{q-1}\right);q,\ q^{n+1}\left(\frac{q^h-1}{q-1}\right);0,\ q^{h+n-1}$$

whose dual is a semiregular *GDD* with parameters

$$q^{h+n+1},q^{n+2}\left(\frac{q^h-1}{q-1}\right),q^{n+1}\left(\frac{q^h-1}{q-1}\right),q^{h+n};q^h,q^{n+1};q^{n+1}\left(\frac{q^{h-1}-1}{q-1}\right),\ q^n\left(\frac{q^h-1}{q-1}\right).$$

Such a $\mathcal{G}'$ exists for each prime power q, $h \geqslant 2$ and $n \geqslant 0$ (see [**8**, p.172]). The parameters of $\mathcal{G}$ and $\mathcal{G}'$ are such that we can use the construction of this paper to produce a transversal design $\mathcal{G}^*$ whose dual is a group divisible 3-associate design. The intersection numbers for blocks of $\mathcal{G}^*$ are

$$\frac{q^H-1}{q-1}+q^{n+1}\left(\frac{q^{h-1}-1}{q-1}\right),\ \frac{q^H-1}{q-1}+q^n\left(\frac{q^h-1}{q-1}\right)\ \text{and}\ \frac{q^{H-1}-1}{q-1}+q^n\left(\frac{q^h-1}{q-1}\right).$$

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