THE NUMBER OF IRREDUCIBLE TOURNAMENTS

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An *n-tournament* is a set of n labelled points, each pair A, B of which is joined either by the oriented line AB or by the oriented line BA. There are $N = n(n-1)/2$ such pairs and so F_n different n-tournaments, where $F_n = 2^N$. A tournament is *reducible* if the points can be separated into two non-empty subsets $\mathscr{A}$ and $\mathscr{B}$, such that every line joining a point in $\mathscr{A}$ to a point in $\mathscr{B}$ is directed towards the point in $\mathscr{B}$. Rado [3] showed that an irreducible tournament is *strongly connected*; i.e. for every ordered pair of points A, B, there is a sequence of correctly oriented lines $AC_1, C_1C_2, \ldots, C_hB$ in the tournament, and conversely that a strongly connected tournament is irreducible.

We write f_n for the number of different irreducible n-tournaments and $P(n) = f_n/F_n$ for the probability that an n-tournament is irreducible.

Every n-tournament $\mathscr{C}$ contains a unique maximal irreducible sub-tournament $\mathscr{B}$ such that every edge joining a point in $\mathscr{C} - \mathscr{B}$ to a point in $\mathscr{B}$ is directed towards the point in $\mathscr{B}$. There are just $\binom{n}{s} f_s F_{n-s}$ different n-tournaments in which $\mathscr{B}$ has s nodes. Since s may have any value from 1 to n, we have

$$F_n = f_n + \sum_{s=1}^{n-1} \binom{n}{s} f_s F_{n-s}. \tag{1}$$

This result, in the equivalent form,

$$P(n) = 1 - \sum_{s=1}^{n-1} \binom{n}{s} P(s) 2^{-s(n-s)}$$

is due to Moon and Moser [1]; they deduce that

$$|P(n) - 1 + n2^{1-n}| < 2^{1-n}$$

and Moon [2] improves this by replacing 2^{1-n} by $n^2 2^{3-2n}$ on the right-hand side. Part of my object here is to improve this further by deducing from (1) a complete asymptotic expansion for $P(n)$ or, what is equivalent, for f_n when n is large.

We write T_n for the number of non-isomorphic n-tournaments (or, if we change our phraseology a little, for the number of different n-tournaments on n unlabelled points), and t_n for the number of non-isomorphic irreducible n-tournaments. By similar reasoning to the above, we find that

$$T_n = t_n + \sum_{s=1}^{n-1} t_s T_{n-s}. \tag{2}$$

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We use this to find an asymptotic expansion of t_n in terms of $T_n, T_{n-1}, \ldots$. This is useful, since Moon [2] gives a result from which we can calculate the asymptotic expansion of T_n. In particular, he shows that $T_n \sim 2^N/n!$.

If we put

$$F_n = n!\,G_n, \quad f_n = n!\,g_n\,(n \geqq 1), \quad G_0 = 1, \quad g_0 = -1$$

in (1), we obtain

$$\sum_{s=0}^{n} g_s G_{n-s} = 0 \qquad (n \geqq 1). \tag{3}$$

If we put $T_0 = 1$, $t_0 = -1$ in (2), we have

$$\sum_{s=0}^{n} t_s T_{n-s} = 0 \qquad (n \geqq 1). \tag{4}$$

In what follows we use C for a positive number, not always the same at each occurrence, but always independent of n. The constant implied in the $O(\ \ )$ notation is a C. The similarity of (3) and (4) makes it convenient to prove the following result.

LEMMA. *If* $H_0 = 1$, $h_0 = -1$, $0 \leqq h_n < H_n\ (n \geqq 1)$,

$$\sum_{s=0}^{n} h_s H_{n-s} = 0 \qquad (n \geqq 1) \tag{5}$$

and

$$C < n!\,2^{-N} H_n < C,$$

then, for any fixed positive integer r, *we have*

$$h_n = H_n + \sum_{s=1}^{r-1} \beta_s H_{n-s} + O(H_{n-r}), \tag{6}$$

where

$$\beta_s = \sum_{t=0}^{s} h_t h_{s-t} = -2h_s + \sum_{t=1}^{s-1} h_t h_{s-t} \qquad (s \geqq 1). \tag{7}$$

By (5),

$$\sum_{s=0}^{r-1} (h_s H_{n-s} + H_s h_{n-s}) = -\sum_{s=r}^{n-r} h_s H_{n-s} = -E_r,$$

say. Now

$$0 \leqq E_r = \sum_{j=r}^{n-r} h_j H_{n-j} \leqq \sum_{j=r}^{n-r} H_j H_{n-j} \leqq 2 \sum_{j=r}^{[n/2]} H_j H_{n-j} \leqq C H_r H_{n-r} \sum_{j=r}^{[n/2]} A_j B_j,$$

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where

$$A_j = \frac{r!(n-r)!}{j!(n-j)!} \leqq n^{j-r}$$

and

$$2\log B_j/\log 2 = j^2+(n-j)^2-r^2-(n-r)^2 = 2(j-r)(j+r-n) \leqq -(j-r)(n-2r).$$

Hence

$$E_r \leqq CH_r H_{n-r} \sum_{j=r}^{[n/2]} n^{j-r} 2^{-(j-r)(n-2r)/2} \leqq CH_{n-r}.$$

It follows that

$$\sum_{s=0}^{r-1} (h_s H_{n-s} + H_s h_{n-s}) = O(H_{n-r}).$$

If we replace n, r by $n-t$, $r-t$, where $0 \leqq t < r$ we have

$$\sum_{s=0}^{r-t-1} (h_s H_{n-t-s} + H_s h_{n-t-s}) = O(H_{n-r}).$$

We multiply the left-hand side by h_t and sum from $t=0$ to $t=r-1$. The sum is

$$\sum_{t=0}^{r-1} \sum_{s=0}^{r-t-1} (h_t h_s H_{n-t-s} + h_t H_s h_{n-t-s}) = \sum_{v=0}^{r-1} H_{n-v} \sum_{t=0}^{v} h_t h_{v-t} + \sum_{v=0}^{r-1} h_{n-v} \sum_{t=0}^{v} h_t H_{v-t}$$

$$= \sum_{v=0}^{r-1} \beta_v H_{n-v} - h_n,$$

by (5) and (7), where $\beta_0 = 1$. But the corresponding sum on the right-hand side is $O(H_{n-r})$ and so we have (6).

We remark that it may be readily deduced from (5) and (7) that

$$H_n + \sum_{s=1}^{n} \beta_s H_{n-s} = -h_n,$$

a result which compares somewhat oddly with (6).

To use our lemma we consider first the case in which $H_n = G_n$. Here we have

$$G_1 = G_2 = 1, \quad G_3 = \tfrac{4}{3}, \quad G_4 = \tfrac{8}{3}; \quad g_1 = 1, \quad g_2 = 0, \quad g_3 = \tfrac{1}{3}, \quad g_4 = 1;$$

and so

$$\beta_1 = -2, \quad \beta_2 = 1, \quad \beta_3 = -\tfrac{2}{3}, \quad \beta_4 = -\tfrac{4}{3}.$$

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Substituting in (6), we obtain

$$f_n = F_n - 2nF_{n-1} + n(n-1)F_{n-2} - \tfrac{2}{3}n(n-1)(n-2)F_{n-3}$$
$$-\tfrac{4}{3}n(n-1)(n-2)(n-3)F_{n-4} + O(n^5 F_{n-5})$$
$$= 2^N\left\{1 - \binom{n}{1}2^{2-n} + \binom{n}{2}2^{4-2n} - \binom{n}{3}2^{8-3n}\right.$$
$$\left. - \binom{n}{4}2^{15-4n} + O(n^5 2^{-5n})\right\}.$$

Since $P_n = f_n/F_n = f_n 2^{-N}$, this also gives us the asymptotic expansion of P_n.

Next we have

$$T_1 = T_2 = 1, \quad T_3 = 2, \quad T_4 = 4, \quad T_5 = 12,$$

by [2]. Hence, by (4),

$$t_0 = -1, \quad t_1 = 1, \quad t_2 = 0, \quad t_3 = t_4 = 1, \quad t_5 = 6$$

and so in this case

$$\beta_1 = -2, \quad \beta_2 = 1, \quad \beta_3 = -2, \quad \beta_4 = 0, \quad \beta_5 = -10.$$

Hence

$$t_n = T_n - 2T_{n-1} + T_{n-2} - 2T_{n-3} - 10T_{n-5} + O(T_{n-6}). \tag{8}$$

This is of little value until we have an asymptotic expansion for T_n. Moon [2] gives an exact expression for T_n from which it is easy to deduce an asymptotic expansion with a little calculation. We find that

$$T_n = G_n + \tfrac{2}{3}(n-2)G_{n-2} + \tfrac{4}{45}(n-4)(10n-41)G_{n-4} + O(n^3 G_{n-6})$$

and so, by (8),

$$t_n = G_n - 2G_{n-1} + \tfrac{1}{3}(2n-1)G_{n-2} - \tfrac{2}{3}(2n-3)G_{n-3} + \tfrac{2}{45}(n-4)(20n-67)G_{n-4} + O(n^2 G_{n-5}).$$

Hence

$$T_n = \frac{2^N}{n!}\left\{1 + \binom{n}{3}2^{5-2n} + \frac{1}{3}\binom{n}{5}(10n-41)2^{15-4n} + O(n^9 2^{-6n})\right\}$$

and

$$t_n = \frac{2^N}{n!}\left\{1 - 2\binom{n}{1}2^{1-n} + \frac{2}{3}\binom{n}{2}(2n-1)2^{3-2n} - 4\binom{n}{3}(2n-3)2^{6-3n}\right.$$
$$\left. + \frac{1}{3}\binom{n}{5}(20n-67)2^{14-4n} + O(n^7 2^{-5n})\right\}.$$

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