SOME REMARKS ON A COMBINATORIAL THEOREM OF ERDÖS AND RADO

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P. Erdös and R. Rado [1] proved that to each pair of positive integers n and k, with $k \geq 3$, there corresponds a least positive integer $\varphi(n, k)$ such that if $\mathcal{F}$ is a family of more than $\varphi(n, k)$ sets, each set with n elements, then some k of the sets have pair-wise the same intersection. They also proved

$$(1) \qquad (k-1)^n \leq \varphi(n, k) \leq n!(k-1)^n \{1 - \sum_{i=0}^{n-1} \frac{i}{(i+1)!(k-1)^i}\}$$

and conjectured that there is a constant c such that

$$(2) \qquad \varphi(n, k) < c^n (k-1)^n$$

It is clear that $\varphi(1, k) = k - 1$ for all k. (This is also a consequence of (1).) The only other value of φ which is known is $\varphi(2, 3) = 6$. That $\varphi(2, 3) \leq 6$ follows from (1), and it is not difficult to see that in the family {(1, 2), (1, 3), (2, 3), (4, 5), (4, 6), (5, 6)} no three sets have pairwise the same intersection.

The main result that we establish in this paper is

$$(3) \qquad \varphi(n, k) \geq \begin{cases} \left((k-1)^2 + [\frac{k-1}{2}]\right)^{\frac{n}{2}} & \text{if n is even,} \\ (k-1)\left((k-1)^2 + [\frac{k-1}{2}]\right)^{\frac{n-1}{2}} & \text{if n is odd.} \end{cases}$$

It is clear that the lower bound for $\varphi(n, k)$ given by (3) is

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better than that given by (1) for all $k \geq 3$ and $n \geq 2$.

In order to prove (3) we shall need some preliminary theorems and results.

THEOREM 1. For all positive integers a, b and k, with $k \geq 3$, we have

$$\varphi(a+b, k) \geq \varphi(a, k)\varphi(b, k). \tag{4}$$

Proof. Let $\{A_1, A_2, \ldots, A_{\varphi(a, k)}\}$ and $\{B_1, B_2, \ldots, B_{\varphi(b, k)}\}$ be families of sets having the desired property, that is, no k of the A's, and no k of the B's, have pairwise the same intersection. As the notation implies, each A has a elements and each B has b elements. We assume also that $A_i \cap B_j = \phi$ for all i and j. Let

$$\mathcal{F} = \{A_i \cup B_j : i = 1, 2, \ldots, \varphi(a, k), \; j = 1, 2, \ldots, \varphi(b, k)\} .$$

The number of sets in $\mathcal{F}$ is $\varphi(a, k)\varphi(b, k)$ and each member of $\mathcal{F}$ has $a+b$ elements. The proof of the theorem will be complete if we show that no k members of $\mathcal{F}$ have pairwise the same intersection.

Suppose there exist distinct sets $F_1, F_2, \ldots, F_k$ in $\mathcal{F}$ and a set $S \subset \bigcup \mathcal{F}$ such that $F_i \cap F_j = S$ for $i, j = 1, 2, \ldots, k, i \neq j$. Partition S into two sets R and T, an element being placed in R if it belongs to $\bigcup A_i$ and in T if it belongs to $\bigcup B_i$. Then if $F_i = A_{m_i} \cup B_{n_i}$, we must have $A_{m_i} \cap A_{m_j} = R$ and $B_{n_i} \cap B_{n_j} = T$ for $i, j = 1, 2, \ldots, k, i \neq j$. If the sets $A_{m_1}, A_{m_2}, \ldots, A_{m_k}$ are all distinct or if the sets $B_{n_1}, B_{n_2}, \ldots, B_{n_k}$ are all distinct we have a contradiction. If this is not the case, then $A_{m_1} = A_{m_2} = \ldots = A_{m_k}$ and $B_{n_1} = B_{n_2} = \ldots = B_{n_k}$ and hence $F_1 = F_2 = \ldots = F_k$. This contradicts the fact that the F's were chosen as distinct members of $\mathcal{F}$. The proof of the theorem is complete.

It follows easily from (4) that

$$(5) \qquad \varphi(n, k) \geq \begin{cases} \varphi(2, k)^{n/2}, & \text{if } n \text{ is even,} \\ (k-1)\varphi(2, k)^{n-1/2}, & \text{if } n \text{ is odd.} \end{cases}$$

We turn our attention now to the derivation of a lower bound for $\varphi(2, k)$.

THEOREM 2.

$$(6) \qquad \varphi(2, k) \geq (k-1)^2 + [\frac{k-1}{2}].$$

Proof. Let $N = \{1, 2, \ldots, 2k-1\}$. Let us take the case where k is odd and let $\ell = \frac{k-1}{2}$. We show how to select $(k-1)^2 + \ell$ subsets of N, each set with two elements, no k of which have pairwaise the same intersection. Let

$$\mathfrak{F}_1 = \{(i, j): i = 1, 2, \ldots, \ell;\ j = k+1, \ldots, 2k-1\}$$
$$\mathfrak{F}_2 = \{(i, j): i = \ell+1, \ldots, k-1;\ j = k+\ell+1, \ldots, 2k-1\}$$
$$\mathfrak{F}_3 = \{(i, j): i = \ell+1, \ldots, k-1;\ j = \ell+2, \ldots, k;\ i < j\}$$
$$\mathfrak{F}_4 = \{(i, j): i = k, \ldots, k+\ell-1;\ j = k+1, \ldots, k+\ell;\ i < j\}.$$

It is not difficult to check that the families of $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$ and $\mathfrak{F}_4$ are pairwise disjoint and that

$$|\mathfrak{F}_1| = \ell(k-1),$$

$$|\mathfrak{F}_2| = (k-\ell)(k-\ell-1)^2,$$

and

$$|\mathfrak{F}_3| = |\mathfrak{F}_4| = \frac{\ell(\ell+1)}{2}.$$

Let $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3 \cup \mathfrak{F}_4$. Then

$$\begin{aligned} |\mathfrak{F}| &= |\mathfrak{F}_1| + |\mathfrak{F}_2| + |\mathfrak{F}_3| + |\mathfrak{F}_4| \\ &= \ell(k-1) + (k-\ell)(k-\ell-1)^2 + \ell(\ell+1) \\ &= (k-1)^2 + \ell \\ &= (k-1)^2 + [\frac{k-1}{2}]. \end{aligned}$$

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One can readily show that each of $1, 2, \ldots, 2k-1$ appears in exactly $k-1$ members of $\mathfrak{F}$. Thus if k members of $\mathfrak{F}$ are to have pairwise the same intersection, they must be pairwise disjoint. But this contradicts the fact that $|\bigcup \mathfrak{F}| = 2k-1$.

The case where k is even can be disposed of in a very similar fashion and we shall not present the details here. It follows from (5) and (6) that (3) holds.

We mention briefly what is perhaps the most interesting special case of this problem, namely the case where $k = 3$. It is not difficult to verify that among the following sets no three have pairwise the same intersection:

$$\begin{array}{llll} (1,2,7), & (4,6,8), & (2,3,9), & (7,8,9), \\ (1,3,7), & (5,6,8), & (4,5,10), & (7,8,10), \\ (2,3,7), & (1,2,9), & (4,6,10), & (7,9,10), \\ (4,5,8), & (1,3,9), & (5,6,10), & (8,9,10). \end{array}$$

Thus,

$$\varphi(3, 3) \geq 16.$$

From (4) it now follows easily that

$$\varphi(3m, 3) \geq 16^m$$
$$\varphi(3m+1, 3) \geq 2(16)^m$$
$$\varphi(3m+2, 3) \geq 6(16)^m .$$

This lower bound for $\varphi(n, 3)$ is better than the one afforded by (3).

The determination of $\varphi(n, k)$ is closely related to the following extremal problem in number theory: What is the largest positive integer $f(n, k)$ $(k \geq 3)$ for which there exists a sequence of integers $a_1, a_2, \ldots, a_{f(n,k)}$ satisfying

(i) $1 \leq a_1 < a_2 < \ldots < a_{f(n,k)} \leq n$

(ii) No k of the a's have pairwise the same greatest common divisor?

Erdös [2] proved that there is a constant c_1 such that

$$f(n, k) \geq f(n, 3) > c_1^{\frac{\log n}{\log \log n}}$$

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and pointed out that if one could prove (2), it would follow that

$$f(n, k) < c_2^{\frac{\log n}{\log \log n}}$$

for some constant c_2.

The following result appears to be new: For every $\epsilon > 0$ and every fixed m and k,

$$(7) \qquad f(n, k) > \varphi(m, k)^{\frac{\log n}{(1+\epsilon) m \log \log n}},$$

provided $n \geq n_0(m, k, \epsilon)$.

To prove (7), let $\{A_1, A_2, \ldots, A_{\varphi(m, k)}\}$ be a family of sets each with m elements, and with no k of the sets having pairwise the same intersection. Let $\bigcup A_i = \{a_1, a_2, \ldots, a_\ell\}$. Let r be a positive integer and consider the first ℓr primes and arrange these in an array

$$A = \begin{bmatrix} P_1^1 & P_1^2 & P_1^3 & \ldots & P_1^r \\ P_2^1 & P_2^2 & P_2^3 & \ldots & P_2^r \\ P_3^1 & P_3^2 & P_3^3 & \ldots & P_3^r \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ P_\ell^1 & P_\ell^2 & P_\ell^3 & \ldots & P_\ell^r \end{bmatrix}.$$

From the primes in the j^{th} column of A form the $\varphi(m, k)$ numbers

$$N_t^j = \prod_{a_i \in A_t} P_i^j \qquad t = 1, 2, \ldots, \varphi(m, k).$$

It is clear that no k of the N's have pairwise the same greatest common divisor. Now form the set S of the $\varphi(m, k)^r$

numbers

$$N^1_{i_1} N^2_{i_2} \cdots N^r_{i_r}$$

where $i_1, i_2, \ldots, i_r$ take on the values $1, 2, \ldots, \varphi(m, k)$. An argument similar to that used to prove Theorem 1 can be used to show that no k of the numbers in S have pairwise the same greatest common divisor.

Each number in S is the product of rm primes, the largest of which is at most $P_{r\ell}$. (P_s denotes the s^{th} prime.) Thus the largest number in S is at most

$$\prod_{P_{r\ell - rm} < P \leq P_{r\ell}} P .$$

Let $\epsilon > 0$ be given and choose

$$r = \left[\frac{\log n}{(1+\epsilon) m \log \log n} \right] .$$

Then the prime number theorem and some straight forward calculations show that, if n is sufficiently large,

$$\prod_{P_{r\ell - rm} < P \leq P_{r\ell}} P < n .$$

It follows that (7) holds.

REFERENCES

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