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ON SEMIGROUP ALGEBRAS AND SEMISIMPLE SEMILATTICE SUMS OF RINGS

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Let P be a semilattice. In (**5**), a ring T is called a *supplementary semilattice sum of subrings* T_α $(\alpha \in P)$ if the following conditions hold: $T = \sum_{\alpha\in P} T_\alpha$, $T_\alpha T_\beta \subseteq T_{\alpha\beta}$ for all $\alpha, \beta \in P$, and $T_\alpha \cap \left(\sum_{\alpha\neq\beta} T_\beta\right) = 0$ for each $\alpha \in P$. Thus, as an abelian group, T is a direct sum of the additive subgroups T_α $(\alpha \in P)$, and the multiplicative structure of T is strongly influenced by the semilattice P. Properties of these rings have been studied extensively in (**2**), (**3**), (**5**), and (**6**).

Let π be a property of rings. A ring is called a *π-ring* if it has property π. An ideal I of a ring is a *π-ideal* if I is a π-ring. A ring is *π-semisimple* if it has no nonzero π-ideals. Assume that the property π satisfies the following conditions: (a) homomorphic images of π-rings are π-rings, and (b) ideals of π-rings are π-rings. For example, the properties of being nil, nilpotent, left quasi-regular, or von Neumann regular are such properties.

It is known (see (**5**) and (**6**)) that if each ring T_α $(\alpha \in P)$ is π-semisimple, then the supplementary semilattice sum T of subrings T_α $(\alpha \in P)$ is also π-semisimple. J. Weissglass has posed the following converse problem (**6**, Question 1, p. 477): find a condition on the semilattice P such that if T is any supplementary semilattice sum of subrings T_α $(\alpha \in P)$, then each T_α $(\alpha \in P)$ must be π-semisimple whenever T is π-semisimple. By proving a theorem on semigroup rings, we obtain the answer to Weissglass' problem: P must be trivial. (Facts about semigroup rings can be found in (**1**), (**5**), and (**6**).) In particular, if π is the property of being nil, nilpotent, or left quasi-regular, Weissglass' problem is answered by setting $T = RS$ and $T_\alpha = RS_\alpha$ $(\alpha \in P)$ in the following result.

Theorem. *Let P be any semilattice with at least 2 elements. If P has a zero (minimal) element, denote it by μ. For any field R, there exist semigroups S_α $(\alpha \in P)$ such that*

(1) $S = \bigcup_{\alpha\in P} S_\alpha$ *is a semilattice P of semigroups S_α,*

(2) *the semigroup ring RS_α has a nonzero nilpotent ideal whenever $\alpha \neq \mu$, and*

(3) *the semigroup ring RS is Jacobson semisimple.*

Proof. For each $\alpha \in P$, let F_α be the free semigroup without identity on the symbols

$$\{x_{1\gamma}, x_{2\gamma}, x_{3\beta} \mid \gamma \geqq \alpha, \beta > \alpha\}.$$

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Let G_α be the semigroup obtained by adjoining a zero element z to F_α. Let D_α be the semigroup obtained by imposing the following relations on G_α:

(i) $x_{1\gamma}^2 = x_{1\gamma}x_{2\gamma} = x_{2\gamma}x_{1\gamma} = x_{2\gamma}^2$ for all $\gamma \geqq \alpha$, and

(ii) $x_{i\gamma}x_{j\beta} = z = x_{j\beta}x_{i\gamma}$ for $i, j \in \{1, 2, 3\}$ and $\beta \neq \gamma$.

Define $S_\alpha = D_\alpha - \{x_{1\gamma}, x_{2\gamma} \mid \gamma > \alpha\}$ for $\alpha \neq \mu$, and define $S_\mu = D_\mu - \{x_{1\gamma}, x_{2\gamma} \mid \gamma \geqq \mu\}$ whenever μ exists. Thus D_α is an ideal extension of S_α for each $\alpha \in P$. Let $\varphi_{\beta\gamma}: S_\beta \to D_\gamma$ be the inclusion map whenever $\beta, \gamma \in P$ with $\beta \geqq \gamma$. Then $S = \bigcup_{\alpha \in P} S_\alpha$ is a semilattice P of semigroups S_α via the homomorphisms $\varphi_{\beta\gamma}$; that is, for $x \in S_\sigma$ and $y \in S_\tau$, $x \cdot y = (x\varphi_{\sigma,\sigma\tau})(y\varphi_{\tau,\sigma\tau}) \in S_{\sigma\tau}$. (See (**4**, Theorem III.7.2) for details.)

In view of (ii), each element of D_α ($\alpha \in P$) may be written as z or a monomial that is homogeneous in $\gamma \in P$ (i.e., a monomial in $x_{1\beta}, x_{2\beta}, x_{3\beta}$ for some $\beta > \alpha$ or a monomial in $x_{1\alpha}$ and $x_{2\alpha}$). *We will assume that all elements of S_α and D_α ($\alpha \in P$) are written in this form.* As usual, the *support* of an element $t = \sum r_k s_k \in RS$, denoted by supp t, is $\{s_k \in S \mid r_k \neq 0\}$.

For any ring Q, let $J(Q)$ denote the Jacobson radical of Q.

We now establish (2) and (3) by proving a sequence of lemmas.

Lemma 1. *For any $\alpha \in P$, the support of any element of $J(RS_\alpha)$ cannot contain a monomial involving an $x_{i\beta}$ for any $\beta > \alpha$ and $i \in \{1, 2, 3\}$.*

Proof. Fix $\beta > \alpha$, and let $B = \{x_{i_1\beta}x_{i_2\beta} \ldots x_{i_m\beta} \mid m \geqq 1,\ i_j = 1, 2 \text{ or } 3 \text{ for all } j\} \subseteq D_\alpha$; that is, B is the set of all monomials in D_α involving an $x_{i\beta}$ entry by (ii). To obtain a contradiction, we assume that $t \in J(RS_\alpha)$ and $(\operatorname{supp} t) \cap B \neq \varnothing$. By (i) and (ii), $(\operatorname{supp} x_{3\beta}tx_{3\beta}) \cap B \neq \varnothing$, $0 \neq x_{3\beta}tx_{3\beta} \in J(RS_\alpha)$, and the only monomials in supp $x_{3\beta}tx_{3\beta}$ that have degree >1 start and end with $x_{3\beta}$. Choose $t' \in RS_\alpha$ such that

$$x_{3\beta}tx_{3\beta} + t' + x_{3\beta}tx_{3\beta}t' = 0.$$

If $(\operatorname{supp} t') \cap B = \varnothing$, then $(\operatorname{supp} x_{3\beta}tx_{3\beta}t') \cap B = \varnothing$ by (ii). But then $B \cap \operatorname{supp}(x_{3\beta}tx_{3\beta} + t' + x_{3\beta}tx_{3\beta}t') \neq \varnothing$, which contradicts the fact that $x_{3\beta}tx_{3\beta} + t' + x_{3\beta}tx_{3\beta}t' = 0$. Hence $(\operatorname{supp} t') \cap B \neq \varnothing$.

By (i) we assume that each member of B is written with as many $x_{1\beta}$ entries as possible (and hence as few $x_{2\beta}$ entries as possible). Then we can order B as follows:

$$x_{j_1\beta}x_{j_2\beta} \ldots x_{j_m\beta} > x_{k_1\beta}x_{k_2\beta} \ldots x_{k_n\beta}$$

if either (a) $m > n$ or else (b) $m = n$, $j_1 = k_1$, $j_2 = k_2, \ldots, j_{q-1} = k_{q-1}$, $j_q > k_q$ for some $q \leqq n$. Thus, if $e, f, g, h \in B$ such that $e > f$, $g > h$, and both e and f end in $x_{3\beta}$, then $eg > fg$ and $eg > eh$. Let e and g be maximal in $(\operatorname{supp} x_{3\beta}tx_{3\beta}) \cap B$ and $(\operatorname{supp} t') \cap B$, respectively. Then $eg \in \operatorname{supp} x_{3\beta}tx_{3\beta}t'$, and $eg \notin (\operatorname{supp} x_{3\beta}tx_{3\beta}) \cup (\operatorname{supp} t')$. This contradicts the assumption that $x_{3\beta}tx_{3\beta} + t' + x_{3\beta}tx_{3\beta}t' = 0$.

Lemma 2. *For any $\alpha \in P$, $rz \in J(RS_\alpha)$ implies $r = 0$.*

Proof. The ideal Rz of RS_α is (ring) isomorphic to the field R; so $J(RS_\alpha) \cap Rz = J(Rz) = 0$.

Lemma 3. *$J(RS_\mu)=0$ if μ exists, and $J(RS_\alpha)=\{rx_{2\alpha}-rx_{1\alpha}\mid r\in R\}$ is a nilpotent ideal of RS_α for $\alpha\neq\mu$.*

Proof. Let $\alpha\in P$, and let H be the ideal of RD_α generated by the set

$$\{rz, rx_{1\alpha}, rx_{2\alpha}\mid r\in R\}.$$

By Lemma 1, $J(RS_\alpha)\subseteq H$. Hence $J(RS_\alpha)=J(RS_\alpha)\cap(H\cap RS_\alpha)=J(H\cap RS_\alpha)=(H\cap RS_\alpha)\cap J(H)\subseteq J(H)$. The mapping $\theta\colon H\to R[x]$ given by

$$\left(az+bx_{2\alpha}+\sum_{i=1}^{n}c_ix_{1\alpha}^i\right)\theta=bx+\sum_{i=1}^{n}c_ix^i$$

is a ring homomorphism of H onto the polynomial ring $R[x]$. (Note that if $\alpha=\mu$, then $b=c_1=0$.) Thus $(J(H))\theta\subseteq J(R[x])=0$; so

$$J(RS_\alpha)\subseteq J(H)\subseteq\ker\theta=\{az+bx_{2\alpha}-bx_{1\alpha}\mid a,b\in R\}.$$

If $\alpha=\mu$, then $b=0$, and hence $J(RS_\mu)=0$ by Lemma 2.

Assume now that $\alpha\neq\mu$, and let $N_\alpha=\{rx_{2\alpha}-rx_{1\alpha}\mid r\in R\}$. If $az+bx_{2\alpha}-bx_{1\alpha}\in J(RS_\alpha)$, then $az=(az+bx_{2\alpha}-bx_{1\alpha})x_{1\alpha}\in J(RS_\alpha)$; so $a=0$ by Lemma 2. Hence $J(RS_\alpha)\subseteq N_\alpha$. But a straightforward computation using (i) and (ii) shows that $(RS_\alpha)N_\alpha=0=N_\alpha(RS_\alpha)$. Thus N_α is a nonzero nilpotent ideal of RS_α, and hence $N_\alpha\subseteq J(RS_\alpha)$.

For $t=\sum_{\alpha\in P}t_\alpha\in RS$ with $t_\alpha\in RS_\alpha$, let

$$P\text{-supp}\ t=\{\alpha\in P\mid t_\alpha\neq 0\},$$

and

$$\max P\text{-supp}\ t=\{\alpha\in P\text{-supp}\ t\mid \beta\in P\text{-supp}\ t\text{ and }\beta\geqq\alpha\text{ imply }\beta=\alpha\}.$$

Lemma 4. $J(RS)=0$.

Proof. To obtain a contradiction, assume that $0\neq t=\sum_{\alpha\in P}t_\alpha\in J(RS)$ with $t_\alpha\in RS_\alpha$. Let $\beta\in\max P$-supp t. As in the proof of (**5**, Theorem 1), $0\neq t_\beta\in J(RS_\beta)$. By Lemma 3, $\beta\neq\mu$ and $t_\beta=rx_{2\beta}-rx_{1\beta}\in RS_\beta$ for some nonzero $r\in R$. Let $\gamma<\beta$; for $x_{3\beta}\in RS_\gamma$, write $x_{3\beta}\cdot t=\sum_{\alpha\in P}t'_\alpha$. Then $t'_\gamma=rx_{3\beta}x_{2\beta}-rx_{3\beta}x_{1\beta}+az+$terms whose support consists of monomials of degree at least two in either $x_{1\beta}$ or $x_{3\beta}$. (The terms after the first two may be 0.) But $\{\gamma\}=\max P\text{-supp}\ x_{3\beta}t$ for $x_{3\beta}\in RS_\gamma$. Again, as in the proof of (**5**, Theorem 1), $t'_\gamma\in J(RS_\gamma)$. Hence our computed form of t'_γ contradicts Lemma 3. This completes the proof of the Theorem.

Remark. The Theorem of this paper shows that conditions on a non-trivial semilattice P alone are not sufficient for the π-semisimplicity of the semilattice sum T of subrings T_α $(\alpha\in P)$ to force each T_α to be π-semisimple. In particular, additional restrictions must be placed on T to ensure the transfer of π-semisimplicity to each T_α. We have seen that requiring T to be a semigroup ring RS, where S is a semilattice of semigroups S_α, is also not sufficient; the problem arises because the images of the homomorphisms $\{\varphi_{\alpha,\beta}\mid\alpha,\beta\in P,\alpha\geqq\beta\}$ are not in S_β. In case the images of the defining homomorphisms $\varphi_{\alpha\beta}$ are always in S_β ($\beta \in P$), then S is called a *strong semilattice P of semigroups* S_α ($\alpha \in P$) (**4**); for this case conditions on P have been found to ensure the transfer of π-semisimplicity from RS to each RS_α (see (**5**, Theorem 2)). In particular, the Theorem of this paper shows that the "strong" hypothesis on S cannot be dropped in (**5**, Theorem 2).

Another way to ensure the transfer of π-semisimplicity from the semilattice sum T to each T_α ($\alpha \in P$) is to place additional restrictions on the property π. As a consequence (**2**, Theorem 1), π-semisimplicity transfers from T to each T_α ($\alpha \in P$) when either of the following conditions holds: (a) π is a strict, hereditary radical property and P is finite, or (b) π is an A-radical property. (See (**2**) for a discussion of the strong conditions on π in (a) and (b).) It is not known if the condition that P is finite can be removed from (a).

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