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A note on modules over regular rings

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It is shown that a von Neumann regular ring R is left self-injective if and only if every finitely generated torsion-free left R-module is projective. It is further shown that a countable self-injective strongly regular ring is Artin semi-simple.

In this note we show that a (von Neumann) regular ring R is left (right) self-injective if and only if every finitely generated torsion-free left (right) R-module is projective. This answers a question of R.S. Pierce [3]. Using sheaf-theoretical techniques, Pierce proved the above theorem for a commutative R. In passing, we show that if R is self-injective regular and has no non-zero nilpotent elements, then a maximal ideal of R is projective as an R-module if and only if it is a direct summand of R. A consequence is that a countable self-injective regular ring is Artin semi-simple, provided it has no non-zero nilpotent elements.

All rings that we consider possess an identity and all modules are unitary left modules. A module M over a ring R is called a *torsion module* if $\mathrm{Hom}_R(S, R) = 0$, for every submodule S of M. Every R-module M possesses a (unique) largest torsion submodule $t(M)$. If $t(M) = 0$, we call M *torsion-free*. A ring R is called (von Neumann) *regular* if each element a in R satisfies the equation $a = axa$, for some element x in R. R is called left (right) *self-injective* if R is injective as a left (right) R-module. A regular ring R is called *left continuous* [5] if its principal left ideals form a complete lattice

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L under set inclusion and for any directed set $\{A_i\}$, $i \in I$, of elements of L, $A \wedge \left(\bigvee_{i \in I} A_i\right) = \bigvee_{i \in I} (A \wedge A_i)$, for any A in L. We shall need the following characterisation (see [5]): A regular ring R is left continuous if and only if every left ideal of R is essential in a direct summand of the left R-module R. A regular left self-injective ring is always left continuous; but the converse is not always true. An R-module M is called *quasi-projective* in case $\mathrm{Hom}_R(M, -)$ is right exact on all short exact sequences of the form $0 \to K \to M \to L \to 0$. We shall make use of the following property of quasi-projective modules (see [4]): If M is a projective module and $M \oplus N$ is quasi-projective, where N is a quotient module of M, then N is projective.

PROPOSITION 1. *A regular ring R is left continuous if and only if every cyclic torsion-free left R-module is projective.*

IF: Let I be a left ideal of R and T be a left ideal containing I such that T/I is the torsion part $t(R/I)$ of R/I. Then R/T is cyclic torsion-free and hence is projective, by hypothesis. Thus T is a summand of R. If $a \in T$ with $Ra \cap I = 0$, then T/I would contain an isomorphic copy of the torsion-free module Ra, a contradiction. Thus I is essential in the summand T and, by the theorem quoted above, R is left continuous.

ONLY IF: Let R be left continuous and $S = R/I$ be a torsion-free cyclic left R-module. By hypothesis, I is essential in a summand E of R. Suppose $I \neq E$. Let $f : F/I \to R$ be a non-zero morphism, where F/I is a cyclic submodule of E/I (f exists since E/I is torsion-free). If $K/I = \ker f$, then $K \neq F$ and $F/K \cong \mathrm{Im} f$ is projective, since $\mathrm{Im} f$ is a principal left ideal of the regular ring R. Thus K is a summand of F and $I \subset K \subset F$, a contradiction to the fact that I is essential in F. Hence $I = E$ is a summand and $S = R/I$ is projective.

PROPOSITION 2. *Let R be a regular ring. Then the following properties are equivalent:*

(i) R is left self-injective;

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(ii) *every finitely generated torsion-free left* R*-module is quasi-projective;*

(iii) *every finitely generated torsion-free left* R*-module is projective.*

Assume *(ii)*. Let A be a finitely generated torsion-free R-module and B a finitely generated free R-module having A as a quotient module. By hypothesis, $A \oplus B$ is quasi-projective. Then by Lemma 3.2 of [4], A is projective. This implies *(iii)* and thus *(ii)* and *(iii)* are equivalent.

Assume *(iii)*. Let R' be the injective hull of the left R-module R. Suppose $a \in R'$ and $a \notin R$. Then $S = R + Ra$ is a finitely generated torsion-free R-module and hence is projective. Since R is regular, by Kaplansky [2], the cyclic submodule R is a summand of S. This contradicts the fact that R is essential in S. Thus $R = R'$ is self-injective. This proves *(i)*.

Assume *(i)*. Since a left self-injective ring is left continuous, by Proposition 1, every cyclic torsion-free left R-module is isomorphic to a summand of R and hence is projective and injective. By finite induction, any finitely generated torsion-free left R-module is projective (and injective). This proves *(iii)*.

REMARK. Professor R.S. Pierce writes that the implication *(iii)* $\Rightarrow$ *(i)* has also been obtained independently by E.R. Gentile.

R.S. Pierce works out a sheaf-theoretical representation of modules over commutative regular rings and uses this representation to consider problems about such modules. With easy modifications one can show that almost all the results of Pierce in [3] carry over to the case when R is a regular ring without non-zero nilpotent elements. Such rings are just the *strongly regular rings*, that is, those rings R in which to each element a there exists an x in R such that $a = a^2x$ $(= xa^2)$. The extension is made possible since the idempotents in a strongly regular ring are central.

The following proposition investigates the projective maximal ideals in a self-injective regular ring. The authors are greatly indebted to Professor R.S. Pierce for communicating an example which motivated the proof.

PROPOSITION 3. *Let* R *be (left) self-injective and strongly regular. Then a maximal (left) ideal of* R *is projective if and only if it is a direct summand of* R .

First note that R (being strongly regular) is a duo ring, that is, every one-sided ideal is two sided; and that any maximal ideal A of R is completely prime, that is, $ab \in A$ implies that either $a \in A$ or $b \in A$. Suppose that M is a maximal ideal of R such that M is projective but not a summand and hence not finitely generated. By the regularity of R , $M = \bigoplus_{i \in I} Re_i$, $e_i^2 = e_i$, where I is infinite (see [2]). Since the idempotents in R are central, the e_i's $(i \in I)$ are orthogonal. Let $\underline{\underline{X}}$ denote the space of all the maximal ideals of R under the hull kernel topology and, for each i in I , let $\underline{\underline{N}}(e_i) = \{A \in X \mid e_i \notin A\}$. Then the $\underline{\underline{N}}(e_i)$ are open and closed [3] and $\underline{\underline{X}} \setminus \{M\} = \cup\, \underline{\underline{N}}(e_i)$. Observe that $\underline{\underline{N}}(e_i) \cap \underline{\underline{N}}(e_j) = \emptyset$, for $i \neq j$. Let J and K be two infinite subsets of I satisfying $J \cap K = \emptyset$ and $J \cup K = I$. Let $\underline{\underline{B}} = \bigcup_{i \in J} \underline{\underline{N}}(e_i)$ and $\underline{\underline{C}} = \bigcup_{i \in K} \underline{\underline{N}}(e_i)$. Then $\underline{\underline{B}} \cap \underline{\underline{C}} = \emptyset$ and $\underline{\underline{B}} \cup \underline{\underline{C}} = \underline{\underline{X}} \setminus \{M\}$. Since $\underline{\underline{X}}$ is compact and J, K are infinite, $\underline{\underline{B}}$ and $\underline{\underline{C}}$ are not closed in $\underline{\underline{X}}$. Since $\underline{\underline{X}}$ is extremely disconnected [3], the closures $\overline{\underline{\underline{B}}}$ and $\overline{\underline{\underline{C}}}$ are open. This implies that $\overline{\underline{\underline{B}}} \cap \overline{\underline{\underline{C}}} = \emptyset$. But $\underline{\underline{B}} \neq \overline{\underline{\underline{B}}}$ and $\underline{\underline{C}} \neq \overline{\underline{\underline{C}}}$, so that $|(\overline{\underline{\underline{B}}} \cup \overline{\underline{\underline{C}}}) \setminus (\underline{\underline{B}} \cup \underline{\underline{C}})| \geq 2$ and this contradicts the fact that $\underline{\underline{B}} \cup \underline{\underline{C}} = \underline{\underline{X}} \setminus \{M\}$. Hence the assertion.

REMARK. Proposition 3 does not hold if R is not self-injective. To see this, let P be the set of all positive prime integers, $Z(p)$ the field of integers modulo p where $p \in P$, $\prod Z(p)$ the direct product and $\oplus Z(p)$ the direct sum of the $Z(p)$'s . Let R be the subring of $\prod Z(p)$ consisting of all sequences $x = \langle \ldots, x_p, \ldots \rangle$, $x_p \in Z(p)$ with the property that to each x in R there corresponds a rational number s/t such that $(s, t) = 1$ and $tx_p \equiv s \pmod p$ for almost all primes p in P . It is easy to verify that R contains $\oplus Z(p)$ and $R/(\oplus Z(p)) \cong Q$, the field of rational numbers. R is commutative regular

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and is not self-injective, since it is an essential R-submodule of $\prod Z(p)$. The ideal $\oplus Z(p)$ is maximal in R and is a projective R-module. But $\oplus Z(p)$ is not a summand, since it is essential in R .

COROLLARY. *A countable self-injective strongly regular ring is Artin semi-simple.*

Now a countably generated left ideal of a regular ring is always projective. Hence, by Proposition 3, every maximal (left) ideal of the given ring R is a summand and we conclude that R is Artin semi-simple.

REMARK. Note that the above corollary immediately yields the well known result: *A complete boolean algebra can not be countably infinite.*

PROPOSITION 4. *Let R be a regular ring. Then every torsion-free left R-module is projective if and only if R is Artin semi-simple.*

We prove only the necessity: Since the projective R-modules are always torsion-free, it is clear that an arbitrary direct product of projective left R-modules is again projective and hence, by Chase [1], R is left perfect. Already the Jacobson radical of R is zero and hence R is Artin semi-simple.

REMARK. Proposition 4 remains true if the word 'projective' is replaced by 'quasi-projective'.

Note added in proof, 2 November 1970. Dr M.L. Teply has kindly pointed out that the implication *(i)* $\Longleftrightarrow$ *(iii)* of Proposition 2 has been obtained in a stronger form by V. Cateforis, *Pacific J. Math.* 30 (1969), 39-45.

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