10

Supersymmetry: boson–fermion unification

The previous chapters, and foremost Chapter 8, show that the development of fundamental physics is inherently based on the idea of unification, and in three related but distinct ways [☞ Conclusion 8.1 on p. 300]. However, one aspect remains in which the objects in fundamental physics, as discussed so far, remain separated:

1. The basic building blocks of matter – quarks and leptons – have spin $\frac{1}{2}\hbar$ and so are *fermions*: they are subject to Pauli's exclusion principle (no two fermions can coexist in the same state) and an ensemble of fermions obeys the Fermi–Dirac statistics.
2. Interaction mediators – gauge and Higgs[1] fields – have integral spin and so are *bosons*: not subject to Pauli's exclusion principle, their ensemble obeys the Bose–Einstein statistics; infinitely many bosons in the same state form a Bose *condensate*.

Digression 10.1 The following parallel practically suggests itself:

1. Subject to Pauli's exclusion principle, two fermions cannot be simultaneously in the same quantum state, i.e., "in the same place" in the Hilbert space – just as in classical physics two material objects cannot be simultaneously in the same place in the real space.
2. Not subject to Pauli's exclusion principle, two bosons *can* be simultaneously in the same quantum state, i.e., "in the same place" in the Hilbert space – just as in classical physics two interaction fields can be simultaneously in the same place in the real space.

Also, matter (substance) elementary particles are fermions, and mediating elementary particles of interaction fields are bosons [☞ Table 2.3 on p. 67]. As if, by extending classical physics into quantum, we transported the "events" of physics from spacetime into the Hilbert space.

[1] Recall Conclusion 7.4 on p. 265: Higgs bosons mediate the interaction of other particles with the *true* vacuum.

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This chapter offers a brief review of the *only possible* way to bridge this last divide: the symmetry transformations that change bosons into fermions and back. The so-extended symmetries of spacetime are called supersymmetries.

The mathematical structure of supersymmetry is a kind of *superalgebras*, i.e., of *supergroups*, which are abstract algebraic structures that mathematicians have studied since the 1960s. The special property of supersymmetries among superalgebras is that they contain the Poincaré algebra (i.e., group) in flat spacetime, as well as the corresponding generalization for anti de Sitter spacetime[2] or with so-called conformal symmetry. In 1971, Yuri A. Gol'fand and Evgeny Likhtman discovered that supersymmetry [☞ Section 10.3] helps in dealing with divergences and renormalization computations in field theory. Besides the conceptual importance, the aim of this chapter is then also to show this practical aspect of supersymmetry application. The interested Reader is, besides texts and monographs in physics [189, 387, 562, 560, 129, 76, 344, 308, 556, 516, 8] and mathematics [178, 125, 535, 461], also directed to the on-line sources [144, 351, 356, 60, 19]; finally, Refs. [115, 186] give a detailed review of the effects and application of supersymmetry in quantum mechanics.

— ❦ —

Supersymmetry that will be considered here is a *global*, i.e., *rigid* symmetry: the symmetry transformation parameters [☞ definition (10.62)] are constants over all spacetime. Of course, there also exists a gauge generalization of supersymmetry, where the supersymmetry transformation parameters are free functions over spacetime, in perfect analogy with the procedure in Section 5.1. Such a gauge, i.e., *local* supersymmetry, turns out *necessarily* to include gravitation, as well as interactions that are mediated by spin-$\frac{3}{2}$ *gravitinos*, the superpartners of spin-2 gravitons. The structure of these models is a fascinating unification of gravitation and the gravitons with particles of lower spin – including gauge 4-vectors, Dirac fermions and scalars, but is also technically much more demanding than the material covered so far, so the interested Reader is directed to the abundant literature, and especially to the textbooks [189, 562, 560, 76]. Besides, it turns out that these "supergravity" models are – by themselves – neither renormalizable nor can they include all the delicate details of the Standard Model without extension within superstring theory, which will be reviewed in Chapter 11.

10.1 The linear harmonic oscillator and its extensions

Before delving into a review of concrete applications of supersymmetry in field theory and elementary particle physics, consider the appearance of supersymmetry in one of the simplest and most familiar quantum-mechanical systems, in the supersymmetric extension of the linear harmonic oscillator.

10.1.1 The harmonic oscillator

The linear harmonic oscillator is very well known and studied in full within every quantum mechanics course, so we recall only the basic relations, to set up the notation. With the standard notation

$$[A,B] := AB-BA \quad \text{and} \quad \{A,B\} := AB+BA, \tag{10.1}$$

[2] The generalization of empty spacetime when the cosmological constant is positive (as is the case with the real spacetime in which we live) is called de Sitter geometry, whereas the empty spacetime with a negative cosmological constant is called anti de Sitter geometry [☞ relations (9.81)]. Supersymmetry turns out not to be definable in spacetimes with de Sitter geometry ($\Lambda > 0$). Thus, the value of the cosmological constant is an indirect measure of supersymmetry breaking, if the fundamental description of Nature indeed is supersymmetric.

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in the "excitation representation," we have

$$H_{\rm LHO} := \tfrac{1}{2}\hbar\omega\{a^\dagger, a\} = \hbar\omega(a^\dagger a + \tfrac{1}{2}), \qquad [a, a^\dagger] = 1; \tag{10.2a}$$

$$\mathscr{H}_{\rm LHO} = \Big\{ |n\rangle \,:\, \langle n|n'\rangle = \delta_{n,n'},\ \sum_n |n\rangle\langle n| = \mathbb{1},\ \ n, n' \in 0, 1, 2, \dots \Big\}, \tag{10.2b}$$

$$a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{10.2c}$$

as well as

$$H_{\rm LHO}|n\rangle = E_n|n\rangle, \qquad E_n = \hbar\omega(n + \tfrac{1}{2}). \tag{10.2d}$$

The ground state, $|0\rangle$ is characterized by the fact that

$$|0\rangle \;:\quad a|0\rangle = 0 \quad \text{and} \quad E_0 = \tfrac{1}{2}\hbar\omega \neq 0. \tag{10.3}$$

The Hilbert space (10.2b) is sketched in Figure 10.1(a), on p. 362. Since every observable physical quantity $\widetilde{\mathcal{F}}$ for the linear harmonic oscillator may be expressed as a function of operators $a, a^\dagger$, *[✐ why?]* the relations (10.2a) and (10.2c) suffice to compute every matrix element $\langle n'|\widetilde{\mathcal{F}}|n\rangle$:

$$\widetilde{\mathcal{F}} = \sum_{p,q=0}^{\infty} c_{p,q}(a^\dagger)^p(a)^q, \qquad \langle n'|(a^\dagger)^p(a)^q|n\rangle = \begin{cases} N_{p,q}\,\delta_{n'-p,n-q}, & q \leqslant n \text{ and } p \leqslant n', \\ 0 & \text{otherwise,} \end{cases} \tag{10.4a}$$

$$N_{p,q} = \sqrt{\underbrace{n(n-1)\cdots(n-q+1)}_{q}\,\underbrace{(n-q+1)(n-q+2)\cdots(n-q+p)}_{p}}. \tag{10.4b}$$

The linear harmonic oscillator is said to be completely solved.

10.1.2 The fermionic extension

Now extend the oscillator (10.2) with a degree of freedom represented by the operators $b, b^\dagger$, which obey

$$\{b, b^\dagger\} = 1 \quad \text{and} \quad \{b, b\} = 0 = \{b^\dagger, b^\dagger\} \quad \Rightarrow \quad b^2 = 0 = b^{\dagger 2}, \tag{10.5}$$

$$[a, b] = 0, \quad [a, b^\dagger] = 0, \quad [a^\dagger, b] = 0, \quad [a^\dagger, b^\dagger] = 0, \tag{10.6}$$

and where the Hamiltonian for the extended system is

$$H_{\rm LHO^+} = \tfrac{1}{2}\hbar\omega\{a^\dagger, a\} + \tfrac{1}{2}\hbar\widetilde{\omega}[b^\dagger, b] \;=\; \hbar\big(\omega\, a^\dagger a + \widetilde{\omega}\, b^\dagger b\big) + \tfrac{1}{2}\hbar(\omega - \widetilde{\omega}). \tag{10.7}$$

Just as in the well-known algebraic analysis of the linear harmonic oscillator, suppose that the operator $b^\dagger b$ (as it occurs in the Hamiltonian) has eigenstates

$$b^\dagger b\,|\nu\rangle_f = \nu|\nu\rangle_f. \tag{10.8}$$

Then,

$$b^\dagger b\big(b^\dagger|\nu\rangle_f\big) = b^\dagger(1 - b^\dagger b)|\nu\rangle_f = \begin{cases} b^\dagger(1-\nu)|\nu\rangle_f & = (1-\nu)\big(b^\dagger|\nu\rangle_f\big), \\ b^\dagger|\nu\rangle_f - \underline{\underline{b^{\dagger 2}}}\, b|\nu\rangle_f & = \big(b^\dagger|\nu\rangle_f\big), \quad b^{\dagger 2} \equiv 0, \end{cases} \tag{10.9}$$

computed in two different ways, produces the relation $(1-\nu)b^\dagger|\nu\rangle_f = b^\dagger|\nu\rangle_f$. That is, $\nu\, b^\dagger|\nu\rangle_f = 0$, so that

$$\text{either} \quad b^\dagger|\nu\rangle_f \equiv 0, \qquad \text{or} \quad \nu = 0 \ \text{ and } \ b^\dagger|0\rangle_f \propto |1\rangle_f. \tag{10.10}$$

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Similarly,

$$b^\dagger b\big(b|\nu\rangle_f\big) = \begin{cases} b^\dagger \underline{\underline{b^2}}\, b|n\rangle_f & \equiv 0, \quad b^2 \equiv 0, \\ (1-bb^\dagger)b|\nu\rangle_f & = b(1-b^\dagger b)|\nu\rangle_f = b(1-\nu)|\nu\rangle_f = (1-\nu)\big(b|\nu\rangle_f\big), \end{cases} \tag{10.11}$$

computed in two different ways, produces the relation $(1-\nu)b|\nu\rangle_f = 0$. Thus,

$$\text{either} \quad b|\nu\rangle_f \equiv 0, \qquad \text{or} \quad \nu = 1 \;\; \text{and} \;\; b|1\rangle_f \propto |0\rangle_f. \tag{10.12}$$

Consistently with these results, we have that

$$b|0\rangle_f \equiv 0, \qquad b^\dagger|0\rangle_f = |1\rangle_f, \qquad b|1\rangle_f = |0\rangle_f, \qquad b^\dagger|1\rangle_f \equiv 0. \tag{10.13}$$

We define for the extended system:

$$|n,\nu\rangle := |n\rangle \otimes |\nu\rangle_f, \qquad n = 0,1,2,3,\dots, \quad \nu = 0,1, \tag{10.14a}$$

which defines the $b, b^\dagger$-extended Hilbert space:

$$\mathscr{H}_{\text{LHO}^+} := \Big\{\, |n,\nu\rangle \;:\; \langle n,\nu|m,\mu\rangle = \delta_{n,m}\delta_{\nu,\mu},\; \sum_{n,\nu} |n,\nu\rangle\langle n,\nu| = \mathbb{1} \,\Big\}, \tag{10.14b}$$

where $n, n' = 0,1,2,3\dots$ and $\nu, \nu' = 0,1$, and where the energy levels are given as

$$H_{\text{LHO}^+}|n,\nu\rangle = E_{n,\nu}|n,\nu\rangle, \qquad E_{n,\nu} = \hbar\big[\omega(n+\tfrac{1}{2}) + \widetilde{\omega}(\nu - \tfrac{1}{2})\big]. \tag{10.14c}$$

The energy of the ground state, $|0,0\rangle$, is

$$E_{0,0} = \tfrac{1}{2}\hbar(\omega - \widetilde{\omega}). \tag{10.15}$$

Since $n = 0,1,2,3\dots$, it follows that the $a^\dagger$-excitations of the familiar linear harmonic oscillator are not limited by Pauli's exclusion principle, and so are identified as bosonic excitations/particles. Since $\nu = 0,1$, it follows that the (single possible) $b^\dagger$-excitation does obey Pauli's exclusion principle, and so is identified as a fermionic excitation/particle with which the linear harmonic oscillator is extended.

The Hilbert space of this fermion-extended linear harmonic oscillator is sketched in Figure 10.1(b), where the white nodes represent bosonic states and the black ones are fermionic states. In that figure, $\widetilde{\omega}$ is chosen to be equal to $\frac{4}{5}\omega$, so that the difference in the energies of the ground state and the first fermionic excitation, $|0,1\rangle$, is $\frac{4}{5}$ of the energy gap between the ground state and the first bosonic excitation, $|1,0\rangle$.

Digression 10.2 By the way, there exist two distinct conventions for **Hermitian conjugation**:

1. the physicists' rule [189, 76], where $(XY)^\dagger = Y^\dagger X^\dagger$ regardless whether "X" and "Y" are commuting or anticommuting objects;
2. the mathematicians' rule [178, 124], where $(XY)^\dagger = (-1)^{\pi(X)\pi(Y)} Y^\dagger X^\dagger$ and where $\pi(X) = 0$ for commuting X and $\pi(X) = 1$ for anticommuting X.

These rules coincide except for anticommuting (fermionic) objects, $\chi\psi = -\psi\chi$:

$$\text{physicists' rule:} \quad (\psi\chi)^\dagger = +\chi^\dagger\psi^\dagger, = -\psi^\dagger\chi^\dagger, \tag{10.16a}$$

$$\text{mathematicians' rule:} \quad (\psi\chi)^\dagger = -\chi^\dagger\psi^\dagger, = +\psi^\dagger\chi^\dagger. \tag{10.16b}$$

The product of two real fermions is imaginary by the physicists' rule, but real by the mathematicians' rule. Herein, we adopt the physicists' practice and rule.

10.1.3 The supersymmetric oscillator

With the operators $a, a^\dagger, b$ and $b^\dagger$, we define the bilinear operators $(b^\dagger a)$ and $(a^\dagger b)$, for which we compute

$$[H_{\text{LHO}^+}, b^\dagger a] = \hbar(\widetilde{\omega} - \omega)b^\dagger a, \qquad [H_{\text{LHO}^+}, a^\dagger b] = \hbar(\omega - \widetilde{\omega})a^\dagger b, \tag{10.17}$$

$$\{a^\dagger b, b^\dagger a\} = a^\dagger a + b^\dagger b. \tag{10.18}$$

This shows that the choice $\widetilde{\omega} \to \omega$ gives a special case, where the operators

$$H := \hbar\omega(a^\dagger a + b^\dagger b), \qquad Q := \sqrt{2\hbar\omega}\, a^\dagger b, \qquad Q^\dagger := \sqrt{2\hbar\omega}\, b^\dagger a, \tag{10.19}$$

define the so-called supersymmetry algebra, for which

$$\{Q^\dagger, Q\} = 2H, \qquad [H, Q] = 0 = [H, Q^\dagger] \tag{10.20}$$

are the defining relations. The last two relations show that the operators Q and $Q^\dagger$ generate symmetries of this specially tuned ($\widetilde{\omega} \to \omega$) fermion-extended oscillator. The first relation identifies the operators Q and $Q^\dagger$ as square-roots of this specially tuned fermion-extended Hamiltonian H.

Finally, we compute

$$Q^\dagger|n{+}1, 0\rangle = \sqrt{2\hbar\omega(n{+}1)}|n, 1\rangle, \qquad \text{and} \qquad Q|n, 1\rangle = \sqrt{2\hbar\omega(n{+}1)}|n{+}1, 0\rangle, \tag{10.21}$$

$$\tfrac{1}{2}\{Q^\dagger, Q\}|n, \nu\rangle = H|n, \nu\rangle = \hbar\omega(n{+}\nu)|n, \nu\rangle, \tag{10.22}$$

so that

$$E_{n,\nu} = \hbar\omega(n{+}\nu). \tag{10.23}$$

Thus, for every $n = 0, 1, 2, 3\ldots$, the states $|n{+}1, 0\rangle$ and $|n, 1\rangle$ form a degenerate pair of states that the operators Q and $Q^\dagger$ map one into another, as is shown in Figure 10.1(c).

It is now clear that the ground state, $|0, 0\rangle$, is the only non-degenerate state and that it has a vanishing energy; the spectrum in Figure 10.1(c) fully exhausts the Hilbert space (10.14b) for this specially tuned ($\widetilde{\omega} = \omega$) extended harmonic oscillator. The action of the operators $Q, Q^\dagger$ on the Hilbert space (10.14b) is manifestly a symmetry. With respect to this symmetry, only the ground state $|0, 0\rangle$ is invariant, while for every $n = 1, 2, 3\ldots$, $\big(|n{+}1, 0\rangle; |n, 1\rangle\big)$ is a boson–fermion pair of superpartner states, a so-called *supermultiplet*.

Definition 10.1 *A symmetry is called **supersymmetry** if (1) it maps bosonic states into fermionic ones and vice versa, and (2) it is generated by operators Q and $Q^\dagger$ the anticommutator of which contains the Hamiltonian H of the system.*

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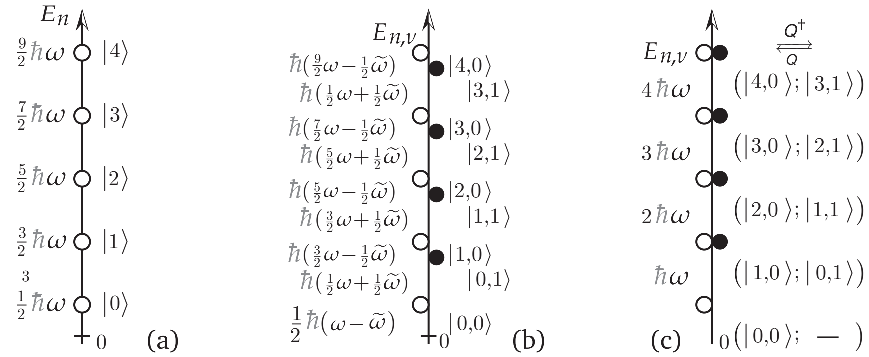

Figure 10.1 A sketch of Hilbert spaces: (a) the linear harmonic oscillator, (b) its fermionic extension with $\widetilde{\omega} \approx \frac{4}{5}\omega$, (c) its supersymmetric fermionic extension.

Digression 10.3 The dimensions (units) of the quantum-mechanical supersymmetry generator follow directly from relations (10.20), and are given as $[Q] = \frac{\sqrt{ML}}{T}$.

The system described by the creation and annihilation operators, $a^\dagger, b^\dagger$ and a, b respectively, for which the (anti)commutation relations (10.2a) and (10.5)–(10.6) hold and the Hamiltonian is specified by the first of equations (10.19), is the supersymmetric harmonic oscillator. In the general case the states are represented by wave-functions, which are functions of time and of the general form:

$$\begin{aligned} \phi(t) &:= \textstyle\sum_n \phi_n(t)|n,0\rangle, && \text{and} & \psi(t) &:= \textstyle\sum_n \psi_n(t)|n{-}1,1\rangle, \\ &= \textstyle\sum_n \phi_n(t)\frac{(a^\dagger)^n}{\sqrt{n!}}|0,0\rangle, && & &= \textstyle\sum_n \psi_n(t)\frac{(a^\dagger)^{n-1}b^\dagger}{\sqrt{(n-1)!}}|0,0\rangle, \end{aligned} \tag{10.24}$$

where $\phi(t)$ is a bosonic state and $\psi(t)$ a fermionic one. Let $\mathscr{B}$ and $\mathscr{F}$ be the vector spaces spanned by bosonic and fermionic wave-functions, respectively. Then the operators Q and $Q^\dagger$ map

$$Q \oplus Q^\dagger \ : \quad \mathscr{B} := \Big\{\sum_n \phi_n(t)|n,0\rangle\Big\} \ \leftrightarrows \ \mathscr{F} := \Big\{\sum_n \psi_n(t)|n{-}1,1\rangle\Big\}, \tag{10.25}$$

except for the ground state, $|0,0\rangle$, which both Q and $Q^\dagger$ annihilate. The ground state thus forms the *kernel* of the supersymmetry mapping (10.25) [☞ the lexicon entry for "kernel," in Appendix B.1]. Since the mapping $Q \oplus Q^\dagger$ acts both ways, the kernel could – in general – have both a bosonic and a fermionic component, so the precise statement is that

$$\big\{\phi_0(t)|0,0\rangle\big\} = \ker(Q \oplus Q^\dagger) \cap \mathscr{B}. \tag{10.26}$$

The function $\phi_0(t)|0,0\rangle$, as a special mode in the expansion (10.24), is often referred to as the "zero mode."

In the general supersymmetric case[3] it is possible that the mapping (10.25) has both bosonic and fermionic components in the kernel, i.e., it is possible that there exist n_B bosonic and n_F fermionic states that are annihilated by both Q and $Q^\dagger$. With such a generalization in mind, we have:

[3] A quantum-mechanical system with the general Hamiltonian for which there exist adequately general operators Q and $Q^\dagger$ so that the relations (10.20) hold is supersymmetric [☞ Refs. [115, 186] for a classification and examples].

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Definition 10.2 (the Witten index) *For a quantum-mechanical system with a Hamiltonian H and a Hermitian-conjugate pair of operators $(Q, Q^\dagger)$ that satisfy the relations (10.20), define*

$$\iota_W := n_B - n_F \qquad \text{(the Witten index)}, \tag{10.27}$$

$$n_B = \dim\Big(\ker(Q \oplus Q^\dagger) \cap \mathscr{B}\Big), \qquad n_F = \dim\Big(\ker(Q \oplus Q^\dagger) \cap \mathscr{F}\Big), \tag{10.28}$$

where $\mathscr{B}$ and $\mathscr{F}$ are the vector spaces of bosonic and fermionic states, so that the Hilbert space of the system is $\mathscr{H} = \mathscr{B} \oplus \mathscr{F}$, and $(Q \oplus Q^\dagger) : \mathscr{B} \leftrightarrows \mathscr{F}$.

In 1981, Edward Witten showed that this *index* – by definition integral – can change only with radical changes in the Hamiltonian, such as the radical change in the potential from the harmonic $\frac{1}{2}m\phi^2$ to the anharmonic $\frac{1}{4}\lambda\phi^4$. For example, if the potential is given as

$$V(\phi) = \tfrac{1}{2}\big(m\phi + \lambda\phi^2\big)^2, \quad \text{with } |m|, |\lambda| < \infty, \tag{10.29}$$

the Witten index continues to have the constant value ($\iota_W = 2$) for arbitrary finite values of the parameter m while $\lambda \neq 0$. The value of the index changes discontinuously (into $\iota_W = 1$) in the parameter subspace where $\lambda = 0$. The Witten index is similarly constant with almost all continuous changes in parameters such as the parameters in the Lagrangian density (7.9). Using this stability, Witten proved the theorem within field theory [573]:

Theorem 10.1 (Witten) *Supersymmetry may be broken spontaneously only if $\iota_W = 0$. Conversely, supersymmetry must remain an exact symmetry while $\iota_W \neq 0$.*

This theorem then automatically also holds within quantum mechanics (adequate for this section), and within statistical physics.

That is, the Witten index ι_W is an *obstruction* for supersymmetry breaking. By definition integral, ι_W cannot change continuously with continuous changes in parameters and so can change only abruptly. This property makes the Witten index one of the first examples of *quasi-topological* invariants in physics, after Dirac's quantization of the magnetic monopole (5.98) charges. However, the relationship between the Witten index and (super)symmetry breaking is definitely the first example where such an invariant plays the role of an obstruction for a physical process such as the breaking of a symmetry and the accompanying phase transition.

Digression 10.4 It proves useful to list the parameters of a model, then designate the subspaces of this parameter space according to the values of the Witten index; this produces the first, rough sketch of the phase diagram for the system.

If the parameter space has at least two subspaces (two phases), each labeled by "its" value of the Witten index, then a change of the parameters that moves from one into the other subspace describes a phase transition. In a phase transition, the Hilbert space of the model changes radically: if we treat the potential (10.29) quantum mechanically, so $\phi = \phi(t)$, the radical change is seen from the fact that:

1. For $\lambda \neq 0$, the Hilbert space $\mathscr{H}_{\lambda\neq 0}$ consists of wave-functions that must decay asymptotically as $\exp\{-\alpha|\phi|^3\}$, for $\phi \to \pm\infty$ and a suitable $\alpha > 0$.
2. For $\lambda = 0$, the Hilbert space $\mathscr{H}_{\lambda=0}$ consists of wave-functions that must decay asymptotically as $\exp\{-\beta|\phi|^2\}$, for $\phi \to \pm\infty$ and a suitable $\beta > 0$.

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Since $\exp\{-\alpha|\phi|^3\}$ decays faster than $\exp\{-\beta|\phi|^2\}$, then $\mathscr{H}_{\lambda\neq 0} \subsetneq \mathscr{H}_{\lambda=0}$, and the Hilbert space over the generic part of the parameter space (where $\lambda \neq 0$) is thus *more limited* than the Hilbert space over the special subspace where $\lambda = 0$, and where the Hilbert space is strictly larger.

10.1.4 Exercises for Section 10.1

✎ ***10.1.1*** *Compute the results (10.4).*

✎ ***10.1.2*** *Find an alternative to equation (10.13), or prove that this is the only possibility.*

✎ ***10.1.3*** *Using the definitions (10.19), compute equations (10.20).*

✎ ***10.1.4*** *Compute equation (10.23).*

✎ ***10.1.5*** *Verify (or disprove) the claims made in Digression 10.4.*

10.2 Supersymmetry in descriptions of Nature

The previous section introduced and defined supersymmetry as a symmetry of a very simple model, which may perhaps appear to be an artificial toy, an abstract example that is not applicable in the "real world." However, the early history of the discovery and application of supersymmetry is a meandering and branching story that indicates both a wide applicability, as well as the fact that many ideas in physics are conceived of in one area, but are then applied more successfully and notably in another area. Something like that was already seen in the telegraphic review of the discovery of spontaneous symmetry breaking, on p. 252.

10.2.1 Applications of supersymmetry

While supersymmetry in fundamental physics is still awaiting experimental confirmation [182], this fermion–boson symmetry has found rather successful applications elsewhere. In fact, novel applications of supersymmetry are still being discovered, so that this review is, at best, a starting point for the interested Reader.

Supersymmetry and hadrons Already in 1966–8, Hironari Miyazawa had discovered the (approximate) boson–fermion symmetry as a formal mapping between mesons (bosons) and baryons (fermions). Miyazawa's approach required the use of the $\mathfrak{su}(6|21)$ *superalgebra*, which was a very unfamiliar structure at the time, and this phenomenological approach did not gain much acceptance. Recall Pauli's denigrating stance towards group theory and its methods [☞ p. 150], which remained well-entrenched until Gell-Mann and Ne'emann used $SU(3)_f$ in hadron classification – seven or eight years after Miyazawa! Much later, it turned out [☞ e.g., Ref. [100]] that Miyzawa's approach together with the quark model (which was accepted only several years after Miyazawa's work) yields quite good results, and is useful in hadron phenomenology.

Supersymmetry and strings In 1971, Jean-Loup Gervais and Bunji Sakita [549] discovered the boson–fermion symmetry in fermionic string theories, which is actually a superconformal symmetry – a combination of supersymmetry and conformal symmetry. At the time, string theory competed with the quark model in attempting to describe hadrons and strong interactions. As the quark model soon (1973–4) proved to be superior in describing hadrons and strong interactions,

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this application of supersymmetry also fell by the wayside until 1984, when (super)string theory was revived as a theory of fundamental physics, and not of hadronic bound states [☞ Chapter 11].

Supersymmetry and field theory In the same year, 1971, Yuri A. Gol'fand and Evgeny Likhtman discovered that the use of supersymmetry in field theory removes a large number of divergent results and markedly simplifies (and sometimes even trivializes) the problem of renormalization [☞ Sections 5.3.3 and 6.2.4]. Similar conclusions were soon – and independently – published by Dmytro V. Volkov and V. P. Akulov, in 1972, as well as Julius Wess and Bruno Zumino in 1974. Also in 1974, Abdus Salam and John A. Strathdee introduced the notion of *superspace* as a supersymmetric extension of spacetime, and *superfield* as fields defined over superspace, and which contain both bosonic and fermionic fields as components. These ideas soon generated significant interest, and in less than ten years, Marcus Grisaru, S. James Gates, Jr., Martin Roček and Warren Siegel had already published the first textbook on supersymmetry, superspace, superfields and supergravity [189]; for more details and topically organized original references, see Ref. [76].

Supersymmetry and nuclear structure On the other, phenomenological side, supersymmetry is used also in the analysis of nuclear structure; see Ref. [364] for experimental confirmation, a recent article [185], the review [399] and references therein. Indeed, atomic nuclei of adjacent isotopes and elements, which differ only in one neutron or proton, may be treated as superpartners: Suppose a particular atomic nucleus ${}^{A}_{Z}X$ has an even atomic number (the number of protons and neutrons together) and so is a boson. Then the nuclei that have one neutron more or less, ${}^{A\pm1}_{Z}X'$, or one proton more or less, ${}^{A\pm1}_{Z\pm1}X''$, are fermions. The formal boson–fermion (supersymmetric) transformations

$$\begin{array}{ccccc} & & {}^{A+1}_{Z}X' & & \\ & & \downarrow\!\uparrow & & \\ {}^{A-1}_{Z-1}X'' & \leftrightarrows & {}^{A}_{Z}X & \leftrightarrows & {}^{A+1}_{Z+1}X'' \\ & & \downarrow\!\uparrow & & \\ & & {}^{A-1}_{Z}X' & & \end{array} \tag{10.30}$$

may all be used to predict the structure and the energy levels of the ${}^{A\pm1}_{Z}X'$ and ${}^{A\pm1}_{Z\pm1}X''$ nuclei, starting with the known properties of the ${}^{A}_{Z}X$ nucleus. This approximate supersymmetry may even be used for estimating information about nuclei that in comparison to a well-known ${}^{A}_{Z}X$ nucleus have both an additional proton and an additional neutron, ${}^{A\pm2}_{Z\pm1}X'''$ [401], which fit in the corners of the diagram (10.30), as well as the so-called *hypernuclei*, which are short-lived nuclei that captured a Λ^0 baryon [400] and which extend the diagram (10.30) in a third dimension. This application of supersymmetry is similar to Gell-Mann's application of $SU(3)$ algebra in classifying hadrons.

Supersymmetry as an approximate, phenomenological symmetry Supersymmetry may be applied in a similar, approximate and phenomenological fashion wherever bosonic states clearly differ from fermionic but have (approximately) the same energy [☞ Theorem 10.3 on p. 369 and Eq. (10.20)], and where the process by which a bosonic state may be transformed into a fermionic one and back is easy to identify. The simplest example in atomic physics would be the simple ionization of any neutral atom. Indeed,

1. a neutral atom has $A+Z$ spin-$\frac{1}{2}$ particles: Z protons, $(A-Z)$ neutrons and Z electrons;
2. simple ionization removes a single electron, leaving the atom with one fewer electrons.

If $A+Z$ is even, the original neutral atom was a boson, and the once-ionized atom is a fermion, and vice versa. In any case, the ionization process turns a bosonic state into a fermionic one or the other way around. The same holds for molecules, and the question is only whether the application of supersymmetry may help to discover anything new about these relatively well studied systems. Leaving this to the interested Reader☝, we return to the supersymmetry in field theory and as a possible fundamental symmetry.

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Supersymmetry in lower-dimensional systems By far the majority of the real physical systems extend through all three dimensions of real space. However, there do exist physical systems that may be regarded, to a good approximation, as 2-dimensional (such as the monolayer systems in *solid state physics*: crystals and materials that consists of mostly a single layer of atoms, molecules or ions) or even just 1-dimensional (such as the enormously long molecules of DNA in *biophysics*).

Supersymmetry may, of course, also be discovered in such systems, as is the case with the monolayer system of graphene, where supersymmetry and the Witten index successfully describe the appearance of the unconventional quantum Hall effect; see, e.g., the articles [402, 153, 5, 408, 347] as well as the references cited therein.

Three levels of fundamental physics Even in fundamental physics, supersymmetry [☞ Definition 10.1 on p. 361] may occur in either of the three very different (albeit closely related) levels; see also Section 11.2 for a slightly different layering of the (super)string theoretical system, and so also the layered appearance and application of supersymmetry. These are:

1. The description of the physical system itself – whether in the classical Hamiltonian formalism, or in the formalism of quantum mechanics or field theory – in the real $(3+1)$-dimensional spacetime. If supersymmetric, the list of supersymmetry generators contain the Hamiltonian density for the given physical system, and also the linear momentum densities. The algebra of operators that are assigned to these physical quantities is then given by relations that contain the algebra (10.20), but are typically rather more complicated (10.63).
2. In analyzing any physical system, the dynamics and the evolution in time are important, and the so-called dimensional reduction to the worldline offers a frequently used approach to analysis. In this approach, for every physical quantity:[4]
   (a) First neglect the dependence on spatial coordinates, and treat the result as a (relativistic or non-relativistic, as needed) quantum-mechanical system.
   (b) All symmetries of the higher-dimensional theory remain to be symmetries of the dimensionally reduced quantum-mechanical "shadow," but the dynamics of the 1-dimensional system – and of the supersymmetry algebra (10.20) or (10.31) too – is simpler to analyze.
   (c) A dynamical solution to the 1-dimensional system and its symmetries (which contain the "shadows" of the Lorentz symmetries of the original higher-dimensional system) are used to reconstruct a corresponding dynamical solution to the original higher-dimensional system.
3. In the Schrödinger picture, every quantum description of any model has a Hilbert space of state functions (or state operators), upon which the Hamiltonian of the system has an induced action. If the system has a supersymmetry, it then manifests as an (induced) supersymmetry in the Hilbert space. Owing to the separate role of time in the Schrödinger picture, this supersymmetry always has the 1-dimensional algebra (10.31).

Conclusion 10.1 *Every supersymmetric model always contains an inherently 1-dimensional (induced) supersymmetry (10.31) in the Hilbert space, which is physically distinct from the dimensional reduction in the second item of the above list, even if they turn out to be mathematically isomorphic. See also Digression 10.1 on p. 357.*

[4] Although many researchers intuitively use this conceptual approach, to the best of my knowledge, the first formal description of this conceptual approach to the research program appeared in Ref. [197].

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10.2.2 Additional (super)symmetry

The introductory form of supersymmetry, given in relations (10.20), and Definition 10.1 on p. 361, suggests some simple generalizations.

On one hand, it is evidently possible to find systems with several pairs of supersymmetry generators, e.g., proton and neutron ones in supersymmetric models of nuclear structure; see the diagram (10.30). Denote such replicas by $Q_i, Q^{\dagger j}$, so that the defining relations (10.20) become

$$\{ Q^{\dagger i}, Q_j \} = 2\delta^i{}_j H, \qquad [H, Q_i] = 0 = [H, Q^{\dagger j}], \qquad i,j = 1,2,\dots,N. \tag{10.31}$$

Equivalently, it is possible to introduce a real basis

$$\mathcal{Q}_j := Q_j + Q^{\dagger j} \qquad \text{and} \qquad \mathcal{Q}_{N+j} := i(Q^{\dagger j} - Q_j), \qquad j = 1,2,\dots,N, \tag{10.32a}$$

$$\{ \mathcal{Q}_I, \mathcal{Q}_J \} = 2\delta_{IJ} H, \qquad [H, \mathcal{Q}_I] = 0, \qquad I,J = 1,2,\dots,2N, \tag{10.32b}$$

and then generalize to a supersymmetric algebra (10.32b) with an *odd* number of real generators $\mathcal{Q}_I$. In this real (Hermitian) basis, $\mathcal{Q}_I^2 = H$ holds, and $\mathcal{Q}_I$ may literally be treated as square-roots of the Hamiltonian. On the other hand, the supersymmetry algebra may be defined *starting* with the relations (10.32b), including the case of an odd number of real operators $\mathcal{Q}_I$. The supersymmetry (10.31)–(10.32) is referred to as "$2N$-extended supersymmetry."

Superalgebras (10.31) and (10.32) may be further extended by adding bosonic operators (with various possible actions upon the considered physical system), as well as by adding commutation relations among these additional bosonic operators and the operators given by (10.31), i.e., (10.32). For example, to the relations (10.32b) we may add a matrix of operators Z_{IJ}, so that the relations (10.32b) are replaced with

$$\{ \mathcal{Q}_I, \mathcal{Q}_J \} = 2\delta_{IJ} H + Z_{IJ}, \qquad [H, \mathcal{Q}_I] = 0, \qquad I,J = 1,2,\dots,2N, \tag{10.33}$$

where

$$\delta^{IJ} Z_{IJ} = 0, \qquad [\mathcal{Q}_I, Z_{JK}] = 0 = [H, Z_{IJ}], \qquad [Z_{IJ}, Z_{KL}] = 0, \tag{10.34}$$

which represents a *central* extension of the superalgebra (10.32b). On the other hand, the last group of commutation relations, $[Z_{IJ}, Z_{KL}] = 0$, may also be replaced by

$$[Z_{IJ}, Z_{KL}] = f_{IJKL}{}^{MN} Z_{MN}, \tag{10.35}$$

so that the operators Z_{IJ} generate some nontrivial Lie algebra [☞ Appendix A]. The physical meaning of some of the operators Z_{IJ} may well be spacetime (such as translations, rotations and Lorentz boosts), in which case at least some of the commutators $[H, Z_{IJ}]$ become non-vanishing. The remaining Z_{IJ}'s may generate "internal" symmetries such as the gauge symmetries corresponding to changes in the phases of complex wave-functions, weak isospin and color in the Standard Model. In the equations (10.33)–(10.35), it was assumed that $[Z_{IJ}] = [H] = \frac{ML^2}{T^2}$, so that the coefficients $f_{IJKL}{}^{MN}$ must have these same dimensions (units) – or be scaled by an appropriate constant of such dimensions. In a concrete application, this may well need to be modified by introducing appropriate constants ($\hbar, c$, etc.) in these equations.

Extending this analysis to include *fermionic* (super)symmetry operators, and correspondingly to superalgebras where the binary operation is the *supercommutator*:

$$[X, Y\} := XY - (-1)^{|X||Y|} YX, \qquad |X| = \begin{cases} 0 & \text{if} \quad X \quad \text{is a boson,} \\ 1 & \text{if} \quad X \quad \text{is a fermion.} \end{cases} \tag{10.36}$$

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The general theory of algebraic structure imposes only the requirement that the various (anti)commutation relations (10.20)–(10.35) be self-consistent, for which the verification of the generalization of the Jacobi identities is necessary and sufficient:

$$0 \equiv \big[B_1,[B_2,B_3]\big] + \big[B_2,[B_3,B_1]\big] + \big[B_3,[B_1,B_2]\big], \tag{10.37a}$$
$$0 \equiv \big[B_1,[B_2,F_3]\big] + \big[B_2,[F_3,B_1]\big] + \big[F_3,[B_1,B_2]\big], \tag{10.37b}$$
$$0 \equiv \big\{F_1,[F_2,B_3]\big\} + \big\{F_2,[F_1,B_3]\big\} + \big[B_3,\{F_1,F_2\}\big], \tag{10.37c}$$
$$0 \equiv \big[F_1,\{F_2,F_3\}\big] + \big[F_2,\{F_3,F_1\}\big] + \big[F_3,\{F_1,F_2\}\big], \tag{10.37d}$$

where B_1, B_2, B_3 are any three bosonic operators and F_1, F_2, F_3 are any three fermionic operators from the considered superalgebra.

Digression 10.5 A superalgebra $\mathfrak{S}$ is the generalization of the algebraic structure of *algebra*, the elements of which are either *even* (bosonic) $B_1, B_2, \ldots \in \mathfrak{S}^0$, or *odd* (fermionic) $F_1, F_2, \ldots \in \mathfrak{S}^1$. The binary "multiplication" operation is called the "supercommutator," denoted $[\ ,\ \}$, such that:

$$[B_1,B_2\} := [B_1,B_2] \in \mathfrak{S}^0, \quad [B_1,F_1\} := [B_1,F_2] \in \mathfrak{S}^1, \quad [F_1,F_2\} := \{F_1,F_2\} \in \mathfrak{S}^0. \tag{10.38a}$$

The supersymmetry algebra is then specified by the defining relations

$$[X_a, X_b\} = i f_{ab}{}^c X_c, \tag{10.39}$$

which define the Killing–Cartan metric tensor:

$$g_{ab} := f_{ac}{}^d f_{bd}{}^c. \tag{10.40}$$

For example, in the supersymmetry algebra (10.32b), define $X_0 = H$ and $X_I = \mathcal{Q}_I$ where $I = 1, 2, \ldots, 2N$. Then

$$f_{00}{}^0 = 0 = f_{00}{}^I, \qquad f_{IJ}{}^0 = -i\delta_{IJ}, \qquad f_{0I}{}^J = 0 = f_{IJ}{}^K, \tag{10.41}$$

so the complete Killing–Cartan metric tensor vanishes identically:

$$g_{00} = f_{00}{}^0 f_{00}{}^0 + f_{0K}{}^0 f_{00}{}^K + f_{00}{}^L f_{0L}{}^0 + f_{0K}{}^L f_{0L}{}^K = 0, \tag{10.42a}$$
$$g_{0J} = f_{00}{}^0 f_{J0}{}^0 + f_{0K}{}^0 f_{J0}{}^K + f_{00}{}^L \underline{f_{JL}{}^0} + f_{0K}{}^L f_{JL}{}^K = 0, \tag{10.42b}$$
$$g_{IJ} = f_{I0}{}^0 f_{J0}{}^0 + \underline{f_{IK}{}^0} f_{J0}{}^K + f_{I0}{}^L \underline{f_{JL}{}^0} + f_{IK}{}^L f_{JL}{}^K = 0, \tag{10.42c}$$

where the only nonzero factors are underlined. This high level of degeneracy prevents an effective application of standard (Lie-algebraic) methods of classification and study.

Also, representations of supersymmetry algebras are vector spaces of the form $\mathscr{B} \oplus \mathscr{F}$, where $\mathscr{B}$ denotes the vector space of bosonic wave-functions and $\mathscr{F}$ is the vector space of fermionic wave-functions, which the supersymmetry transformations map into each other, generalizing the relation (10.25). Note that $\mathscr{H} = \mathscr{B} \oplus \mathscr{F}$ is actually a complete Hilbert space for the considered model, and in supersymmetric theories one automatically and by definition considers the (super)symmetries of this complete Hilbert space. Automatically, we obtain results of the form

$$\langle b|F|b\rangle \equiv 0 \equiv \langle f|F|f\rangle, \quad \langle f|B|b\rangle \equiv 0 \equiv \langle f|B|f\rangle, \qquad \forall |b\rangle \in \mathscr{B}, \quad \forall |f\rangle \in \mathscr{F}, \tag{10.43}$$

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where B is any bosonic operator and F any fermionic operator; they are called super-selection rules and hold in all models with supersymmetry. This result is consistent with the definition of the "fermionic number," which is 1 for all fermions (states, functions, operators, ...) and 0 for all bosons. In products, this number is added, and it is defined modulo 2. Thus, e.g.,

$$\mathrm{F}\big(\langle b|F|b\rangle\big) = \mathrm{F}\big(\langle b|\big) + \mathrm{F}\big(F\big) + \mathrm{F}\big(|b\rangle\big) = 0+1+0 = 1 \neq 0, \tag{10.44a}$$
$$\mathrm{F}\big(\langle f|F|f\rangle\big) = \mathrm{F}\big(\langle f|\big) + \mathrm{F}\big(F\big) + \mathrm{F}\big(|f\rangle\big) = 1+1+1 = 3 \simeq 1 \pmod 2, \quad \neq 0, \tag{10.44b}$$

and so on. It turns out that this "fermionic number" may be defined consistently in spacetimes of all dimensions, and that it differentiates spinorial from tensorial representations of the Lorentz group. Also, the Witten index (10.27) may be formally defined as

$$\iota_W = \operatorname*{Tr}_{\mathscr{H}}\big[(-1)^{\mathrm{F}}\big]. \tag{10.45}$$

10.2.3 Exercises for Section 10.2

✎ ***10.2.1*** *By explicit computation show that the operators $Q_i, Q^{\dagger j}$ and H that satisfy the algebra (10.33) also satisfy the Jacobi identities (10.37).*

✎ ***10.2.2*** *By explicit computation show that the operators $\mathcal{Q}_i, Z_{IJ}$ and H that satisfy the algebra (10.31) also satisfy the Jacobi identities (10.37).*

10.3 Supersymmetric field theory

In the 1960s (before the experimental confirmation and consequent wide acceptance of the quark model!), many elementary particle physics researchers explored how much and what may all be proven and established about the behavior of leptons and hadrons – without a detailed knowledge of their dynamics, i.e., without knowing the "microscopic" theory of these interactions. Also, attempts were made to combine the symmetries of spacetime, such as the rotational (i.e., angular momentum or spin) $SU(2)$ group of symmetries, with the so-called *internal* symmetries of elementary particles, such as isospin and its $SU(3)_f$ generalization by Gell-Mann and Ne'emann. The successful non-relativistic combination $SU(2)\times SU(3)_f \subset SU(6)$ surprisingly turned up the frustration:[5] a fully relativistic generalization could not be found, rousing suspicions of a profound obstruction.

Indeed, in 1965, Lochlainn O'Raifeartaigh published a proof [396, 397] of the theorem that today bears his name, and which may be paraphrased simply as [344]:

> ***Theorem 10.2 (O'Raifeartaigh)*** *The Hilbert space of the states of a particle with finite and non-vanishing mass is invariant with respect to the action of the Lie group of transformations that contains the Poincaré group (Lorentz transformations and spacetime translations).*

A sharper version of one key aspect of this theorem was provided by P. Roman and C. J. Koh the same year [463]:

> ***Theorem 10.3 (Roman–Koh)*** *Distinct particles and states transformed into each other by a Lie group have the same Lorentz-invariant mass $m := \sqrt{\mathrm{p}\cdot\mathrm{p}}$.*

Only two years later, Sidney Coleman and Jeffrey Mandula (in 1967) proved the theorem [111] for all relativistic field theories:

[5] This $SU(6)$ is indeed part of Miyazawa's $\mathfrak{su}(6|21)$ superalgebra framework mentioned in Section 10.2.1.

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Theorem 10.4 (Coleman–Mandula) *In any model of particles with finite and non-vanishing masses and which (directly or indirectly) interact with each other, the only permissible symmetries form the Poincaré group and some Lie group, the elements of which commute with all of the Poincaré group of symmetries.*

It then follows that no (bosonic) symmetry transformation can change the fermionic number of any state or particle upon which the operator acts, i.e., the fermionic number of the wave-function that represents this state or particle.

In 1975, Rudolf Haag, Jan Łopuszanski and Martin Sohnius noticed the "hole" in these results: It was tacitly assumed that the symmetry operators were bosonic, so that the symmetries form a Lie group, and the group generators satisfy a Lie algebra where the operation of multiplication is a commutator. The more general algebraic structures defined by bosonic *as well as fermionic* operators together are *superalgebras*, where the binary operation is a supercommutator (10.36). Within this extension of the Lie algebras, Haag, Łopuszanski and Sohnius proved the theorem [255]:

Theorem 10.5 (Haag–Łopuszanski–Sohnius) *In every model with a* **(1)** *finite number of distinct types of particles,* **(2)** *each of which has a finite and non-vanishing mass, and* **(3)** *with an asymptotically complete S-matrix,*[6] *the only permissible symmetries form a so-called supersymmetric extension of the product of the Poincaré group and some Lie group, the elements of which commute with all of the Poincaré group of symmetries [☞ Definition 10.3].*

Definition 10.3 *The Poincaré algebra, $\mathfrak{po}(1,3) = \mathfrak{spin}(1,3) \niplus \mathfrak{tr}(\mathbb{R}^{1,3})$, [☞ Section A.5.3] is generated by Lorentz transformations (A.110) and spacetime translations (A.109), i.e., the operators $L_{\mu\nu}$ and P_μ, respectively, which satisfy the relations schematically given as [☞ also the definition (10.64)]*

$$[L, L'] = L,'' \qquad [L, P] = P', \qquad [P, P'] = 0. \tag{10.46}$$

The supersymmetric extension of the Poincaré algebra then has the additional spin-$\frac{1}{2}$ generators Q, which satisfy the relations schematically given as

$$\{Q, Q'\} = P \oplus Z, \qquad [L, Q] = \tfrac{1}{2} Q', \qquad [P, Q] = 0, \tag{10.47a}$$

$$[Z, Z'] = Z,'' \qquad [L, Z] = 0, \qquad [Z, P] = 0, \qquad [Z, Q] = 0. \tag{10.47b}$$

The generators Q are called **supercharges***, and Z are* **central charges***.*

Comment 10.1 *Theorem 10.5 also guarantees that in all relativistic field theories only the supersymmetric generators, and exclusively with spin $\frac{1}{2}$, may change the spin (and also the fermionic number) of the particles upon which they act, and to extend the symmetries into supersymmetry. Also, it is known that the inclusion of massless particles does not change the conclusion of the theorem if those particles are Yang–Mills gauge bosons and their superpartners (***gauginos***).*

Digression 10.6 Without inserting any dimensionful constants such as $\hbar$ or c in the equations (10.46)–(10.47), the implied dimensions of these field theory supersymmetry generators Q and central charges Z are $[Q] = \sqrt{\frac{ML}{T}}$ and $[Z] = \frac{ML}{T}$, which differ from their quantum-mechanical counterparts because of $P_0 = -H/c$; see Digressions 10.3 on p. 362 and 10.7 on p. 378.

[6] The S-matrix by definition maps all possible incoming states of the system into all possible outgoing states.

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10.3.1 Supersymmetry stabilizes the vacuum

Upon review – with the benefit of a century's worth of hindsight – the need to include ***quantumness*** in the description of Nature may be understood as the only ***universal property*** that stabilizes the atoms and so also all the tangible matter [☞ Digression 2.3 on p. 45]. Besides, the quantumness of physics unifies the concepts (our idealized mnemonic imagery) of particles and waves [☞ Section 8.1.1 on p. 297].

On the other hand, the need to include (general) ***relativity*** in the description of Nature was seen in Chapters 5, 6 and 9 to be part of a ***universal gauge principle*** that connects the existence of unmeasurable degrees of freedom in the description of Nature with *local* symmetries, then interactions and curvature of spacetime in which the physical particles move and fields extend. Besides, the special theory of relativity unifies space and time into spacetime, energy and momentum into 4-momentum, rotations and boosts into the Lorentz group, etc. The general theory of relativity unifies the notion of gravitation and acceleration, and provides the inherent relation between the curvature of spacetime and the presence of matter.

On the third hand, already the classical and certainly the quantum field theory indicate that the precise definition of observable quantities is not infrequently a very delicate task – the naive expressions even for the energy of empty spacetime not infrequently diverge [☞ Digression 3.13 on p. 123, and Sections 5.3.3 and 6.2.4]. Besides, in interactive field theories that include gravity even the ground state of a system is not guaranteed to have a non-negative energy, nor in fact is it guaranteed to have a globally defined energy bounded from below.

Regarding this last issue, supersymmetry helps (which is stated here with no detailed and mathematically strict justification and proof):

> ***Conclusion 10.2*** ***Supersymmetry*** *offers (as best as known) the only* ***universal mechanism*** *for stabilizing the vacuum: in* **every** *system without gravity [☞ Ref. [73] for energy positivity conditions without supersymmetry],*
>
> 1. *the minimum of energy is zero if and only if the system is supersymmetric;*
> 2. *the minimum of energy is positive if the system has a spontaneously broken supersymmetry.*

> ***Comment 10.2*** *If the description of the system includes the general theory of relativity (to describe gravity), the energy is not a globally well-defined quantity, and statements of non-negativity of energy do not have an invariant meaning.*

Besides, supersymmetry is the only property that may unify bosons and fermions [☞ Theorems 10.2 on p. 369, and 10.4 on p. 370]. For a compact and comprehensive summary of these properties, see Table P.1 on p. xiii, i.e., Table 11.1 on p. 409.

Technical advantages of supersymmetry

Even when spontaneously broken, supersymmetry also has two technically very advantageous consequences:

1. it significantly lessens (or even eliminates) the need for renormalization of parameters in field theory;
2. it prevents the "mixing" of characteristic energies.

That is, in any model (in field theory without supersymmetry) where in the classical version there exist two distinct characteristic energies (such as the energy of electro-weak unification, $m_W c^2 \sim 10^2$ GeV and the energy of grand unification $m_X \sim 10^{15}$ GeV), quantum effects "spoil" results such as masses of the order $10^2\text{GeV}/c^2$ via renormalization "corrections" of order $10^{15}\text{GeV}/c^2$.

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Since masses are by definition invariant with respect to the action of Lorentz symmetries as well as all gauge symmetries, no fundamental symmetry principle – except supersymmetry – can "protect" them from such catastrophic quantum corrections.[7] Thus, in models without supersymmetry we may expect only one (the largest) effective characteristic energy, which in theories with gravity must be the Planck mass, $M_P \sim 10^{19}\text{GeV}/c^2$. All other masses then would have to be a multiple of this big mass and there is no reason for the existence of minuscule dimensionless coefficients such as [☞ result (7.132b), and Tables 4.1 on p. 152, and C.2 on p. 526]

$$\frac{m_{\nu_e}}{M_P} \lesssim 10^{-28}, \qquad \frac{m_e}{M_P} \sim 10^{-23}, \qquad \frac{m_u}{M_P} \sim 10^{-22}, \tag{10.48}$$

for them to remain stable with respect to quantum corrections, and much smaller than $O(1)$ numbers. It follows that in models without supersymmetry there is neither a fundamental reason for the masses of the elementary particles to be so many orders of magnitude smaller than the Planck mass, nor a mechanism that would "protect" such minuscule masses (were we to choose them so "by hand") from quantum corrections.

The presence of supersymmetry in any theoretical model (and so too in the Standard Model), has an important effect on the appearance (and stability with respect to quantum corrections) of experimentally established minuscule parameters such as (10.48) [189, 562, 560, 76]:

Theorem 10.6 *In any supersymmetric model, quantum effects do not change the part of the Lagrangian density that stems from the so-called* ***superpotential*** *[☞ Section 10.3.2].*

Corollary 10.1 *Although – all by itself – supersymmetry cannot* ***cause*** *minuscule parameters such as (10.48), supersymmetry does "protect" them if they enter via Lagrangian terms that stem from the superpotential, and in particular owing to the shift in the Higgs field in the process of spontaneous symmetry breaking. In practice, that includes all masses.*

This property of supersymmetry is exceptionally advantageous in the *technical* sense, because of the fact that most field theory models are analyzed and used in practice within perturbative computational frameworks described in Procedure 5.1. Renormalization is inherently a feature of iterative additions of ever higher contributions within a perturbative computational framework. Therefore, the appearance of divergences, the need for renormalization as well as the property of softening and limiting this need via supersymmetry is – by definition – a technical and not a conceptual property. This characterization holds even if some of the "non-perturbative" results and properties of a particular model are known [☞ Section 6.3], and they are:

1. statements about the existence of alternative vacua which cannot be computed by perturbative methods defined about the usual vacuum, but where the results are again obtained by some kind of perturbative computation about some such alternative vacuum,
2. general statements about the whole Hilbert space.

— ❦ —

In all Yang–Mills type gauge field theories [☞ Chapters 5 and 6], the divergences can be removed from precisely defined expressions for measurable physical quantities [☞ Section 3.3.4, especially the closing part and the discussion about Digression 3.11 on p. 122, to begin with]. In as much as the renormalization procedure has not satisfied the intuition and conceptual insight of some of the most influential twentieth-century physicists, the number of *live* physicists who do not accept

[7] The notable exceptions to this reasoning are the "pseudo-Goldstone modes" mentioned in Section 7.2.3.

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renormalization pragmatically as a "procedure that works" is ever smaller [☞ paraphrasing Max Planck, on p. 124 and Digression 3.11 on p. 122]. However, the renormalization *procedure* is, indubitably, a technical detail of the current understanding of Nature, and not a fundamental principle of this (not even the current) understanding.

It should then be clear that the original motivation for supersymmetry stemmed from the very practical fact, of which Gol'fand and Likhtman discovered the first glimpses in 1971, that this peculiar type of symmetry automatically removes many of the divergences that occur in field theory. A detailed analysis of this general procedure is far outside the scope of this book, although some of the simplest aspects will nevertheless be made visible.

Sections 5.3.3 and 6.2.4 showed concrete (albeit the simplest) Feynman computations with diagrams where the need for renormalization appears. In the remainder of this section we will consider one (the simplest) conceptual problem in field theory, and then also the mechanism by which supersymmetry completely removes this problem.

Vacuum energy

Consider, for example, a scalar field with the Lagrangian density (7.9), where we set for simplicity $\lambda \to 0$:

$$\mathscr{L}_{\text{KG}} = \tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \tfrac{1}{2}\big(\tfrac{mc}{\hbar}\big)^2\phi^2 \;=\; \tfrac{1}{2c^2}\dot{\phi}^2 - \tfrac{1}{2}\big[\vec{\nabla}^2 + \big(\tfrac{mc}{\hbar}\big)^2\big]\phi^2. \tag{10.49}$$

The Euler–Lagrange equation of motion derived from this Lagrangian density is

$$\big[\tfrac{1}{c^2}\partial_t^2 - \vec{\nabla}^2 + \big(\tfrac{mc}{\hbar}\big)^2\big]\phi(\mathrm{x}) = 0, \tag{10.50}$$

the so-called Klein–Gordon equation. If we expand $\phi(\mathrm{x})$ in plane waves,

$$\phi(\mathrm{x}) = \tfrac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3\vec{k}\;\phi_{\vec{k}}(\mathrm{x}), \qquad \phi_{\vec{k}}(\mathrm{x}) := f_{\vec{k}}(t)\,e^{i\vec{k}\cdot\vec{r}}, \tag{10.51}$$

the $\vec{\nabla}^2$-term produces the eigenvalue $-\vec{k}^2$, and the equation of motion becomes

$$\big[\partial_t^2 + \big(\vec{k}^2c^2 + \tfrac{m^2c^4}{\hbar^2}\big)\big]f_{\vec{k}}(t) = 0. \tag{10.52}$$

The wave-modes $\phi_{\vec{k}}(\mathrm{x})$ are linearly independent, so every plane wave $\phi_{\vec{k}}(\mathrm{x})$ behaves as an independent degree of freedom, "counted" by the vectors $\vec{k}$, and with the dynamics of the harmonic oscillator with the frequency $c\sqrt{\vec{k}^2 + \frac{m^2c^2}{\hbar^2}}$. The presence of interactions (as would be produced by the $\frac{1}{4}\lambda\phi^4$ term in the Lagrangian density (7.9) and which we have omitted for simplicity) couples these independent oscillators but does not reduce their number nor does it destroy their linear independence. Every such *quantum* oscillator has its stationary states with energies [☞ relation (3.37)]

$$E_{n,\vec{k}} = E_{\vec{k}}(n + \tfrac{1}{2}), \qquad E_{\vec{k}} := \hbar c\sqrt{\vec{k}^2 + \tfrac{m^2c^2}{\hbar^2}} = \sqrt{(\hbar\vec{k})^2c^2 + m^2c^4}, \tag{10.53}$$

and the energy of the entire field (summed over all oscillators, of course) in the ground state is

$$E_{\text{vacuum}} = \tfrac{1}{2}\int \mathrm{d}^3\vec{k}\;E_{\vec{k}} = 2\pi\int_0^\infty k^2\mathrm{d}k\;\sqrt{\hbar^2k^2c^2 + m^2c^4}. \tag{10.54}$$

This evidently diverges $\sim k^4$ as $k \to \infty$: there are (continuously) infinitely many vectors $\vec{k}$ and all except $\vec{k} = \vec{0}$ have a positive magnitude $\vec{k}^2 > 0$.

For the free electromagnetic field, the result is virtually identical, only with the ultra-relativistic expression $E_{\vec{k}} = |\hbar\vec{k}|c$, since $m_\gamma \equiv 0$, so the result for E_{vacuum} diverges again.

However, modeling after the supersymmetric harmonic oscillator in Section 10.1.3, we may construct a supersymmetric model beginning with:

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1. a pair of fields (10.51) combined into a complex scalar field $\phi(\mathrm{x})$;
2. the Lagrangian density of the type (7.19), but with $\lambda \to 0$;
3. adding a complex Weyl fermion $\Psi_+(\mathrm{x})$ of left chirality [☞ Section 5.2.1 on p. 172] and an auxiliary complex field $F(\mathrm{x})$;
4. adding Lagrangian (counter)terms that are specially tuned so that:
   (a) the Hamilton action for the whole system $(\phi; \Psi_+; F)$ is invariant with respect to the linear action of supersymmetry;
   (b) the Euler–Lagrange equations of motion form a system of:
      i. one differential equation of the second order for the complex field $\phi(\mathrm{x})$,
      ii. one pair of differential equations of the first order for the two components of the complex Weyl fermion $\Psi_+(\mathrm{x})$ – which also means that one linear combination of these components is the canonical coordinate while another is the canonically conjugate momentum,
      iii. one *non-differential* equation or the auxiliary complex field $F(\mathrm{x})$.

The non-differential equation obtained in step 4(b)iii holds point-by-point in all of spacetime *separately*, and so can be used – at least in principle – to substitute its solution back into the Lagrangian density, whereupon the differential equations in steps 4(b)i and 4(b)ii need to be re-derived from the so-substituted Lagrangian density. These *differential* equations of motion, however, express the values of the fields $\phi(\mathrm{x})$ and $\Psi_+(\mathrm{x})$ at any one point in spacetime in terms of the values of those fields at infinitesimally nearby points, and so describe dynamical (continually propagating) fields. A detailed analysis of the physical degrees of freedom then shows that all states in the Hilbert space (except for the ground states, with a vanishing energy) occur in boson–fermion pairs, generalizing the situation shown in Figure 10.1(c), on p. 362.

In so-constructed models the result (10.23) guarantees that the equivalent computation for the vacuum energy gives $E_{\text{vacuum}} = 0$. This, in fact, is a direct (and so universal) consequence of the algebra (10.31), and up to a factor c^{-1} also of the algebra (10.47) [☞ Digression 10.6 on p. 370], where

$$\sum_i \{Q^{\dagger i}, Q_i\} = 2NH, \qquad \text{since} \quad \mathrm{Tr}[\delta^i_j] = N, \tag{10.55}$$

and where it is easy to show that the left-hand side is non-negative. The algebraic details of all consistent generalizations of supersymmetry – as long as the trace of the coefficient in front of the Hamiltonian (δ^i_j) on the right-hand side of equation (10.31) is positive – guarantee the non-negativity of the Hamiltonian spectrum, so that $\langle H \rangle \geqslant 0$ is a universal result in all (rigidly) supersymmetric theories.

Supersymmetric states, supersymmetry breaking and details

The states $|\Omega\rangle$ with vanishing energy, for which $\langle\Omega|H|\Omega\rangle = 0$, must in turn satisfy

$$\begin{aligned}0 = \langle\Omega|H|\Omega\rangle &= \Big\langle\Omega\Big|\tfrac{1}{N}\sum_i \{Q^{\dagger i}, Q_i\}\Big|\Omega\Big\rangle = \tfrac{1}{N}\sum_i \Big\{\langle\Omega|Q^{\dagger i}\,Q_i|\Omega\rangle + \langle\Omega|Q_i\,Q^{\dagger i}|\Omega\rangle\Big\}\\ &= \tfrac{1}{N}\sum_i \Big\{\big|Q_i|\Omega\rangle\big|^2 + \big|Q^{\dagger i}|\Omega\rangle\big|^2\Big\},\end{aligned} \tag{10.56}$$

which is a sum of non-negative contributions, so each must vanish separately, whereupon

$$\text{both} \quad Q_i|\Omega\rangle = 0 \quad \text{and} \quad Q^{\dagger i}|\Omega\rangle = 0 \quad \text{for all} \;\; i. \tag{10.57}$$

From there, it follows that

$$U_{\epsilon,\bar\epsilon}|\Omega\rangle = |\Omega\rangle, \qquad U_{\epsilon,\bar\epsilon} := \exp\big\{-i(\epsilon\cdot Q + \epsilon^\dagger\cdot Q^\dagger)\big\}, \tag{10.58}$$

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whereby the states $|\Omega\rangle$ are supersymmetric, i.e., unchanged under the supersymmetry transformation.

There may exist several supersymmetric states, and even continuously many. In the general case, when the bosonic (fermionic) states form a space $\mathscr{V}_B$ (i.e., $\mathscr{V}_F$), the Witten index is given by the relation

$$\iota_W := \chi_E(\mathscr{V}_B) - \chi_E(\mathscr{V}_F), \tag{10.59}$$

where $\chi_E(\mathscr{X})$ is the Euler characteristic of the space $\mathscr{X}$, which reduces to the previous definition (10.27) since the Euler characteristic of a point equals $\chi_E(\cdot) = 1$ – and which also holds for any space that contracts (continuously) to a point, such as $\mathbb{R}^n$ and $\mathbb{C}^n$.

From this analysis it follows that the Hilbert space of every supersymmetric model can only consist of:

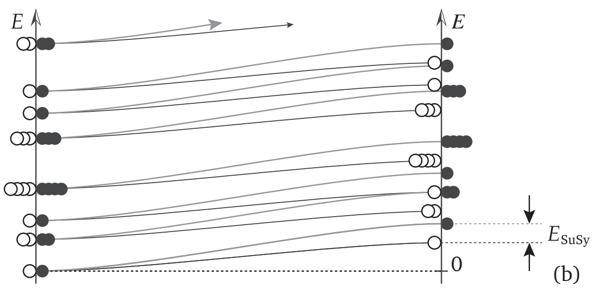
1. supersymmetric states (of zero energy, so these are the ground states of the system),
2. supersymmetric boson–fermion pairs of states with positive energy.

In supersymmetric models, every $E > 0$ energy level must be evenly degenerate. That is, for each bosonic state, $|b_a\rangle$ with $E_a \neq 0$, we construct the fermionic state $|f_{a,I}\rangle := \mathcal{Q}_I|b_a\rangle$ and vice versa:

$$H|b_a\rangle = E_a|b_a\rangle \quad \Rightarrow \quad |b_a\rangle = \frac{H}{E_a}|b_a\rangle = \frac{\mathcal{Q}_I\mathcal{Q}_I}{E_a}|b_a\rangle = \frac{1}{E_a}\big(\mathcal{Q}_I|f_{a,I}\rangle\big), \tag{10.60}$$

which is evidently possible if and only if $E_a \neq 0$. (It is possible to prove further also that the total number of bosonic and fermionic states with a given energy $E_a \neq 0$ must be the same [560].) Thus, only the degeneracy of the ground state(s) (where $E = 0$) is not determined and only the ground state(s) may be non-degenerate, and only if the Witten index is nonzero, $\iota_W \neq 0$.

In supersymmetric models, the Hilbert space is of the form of a direct sum of so-called "sectors," of which every one consists of one ground state (with $E = 0$) and an infinite ladder of boson–fermion pairs of states (with $E > 0$), formally obtained by acting with operators of creation on the given ground state, just as is the case in Figure 10.2(a), and which generalizes the situation shown in Figure 10.1(c), p. 362.

(a)


(b)

Figure 10.2 A sketch of a sector in the Hilbert space of a supersymmetric system, before (a) and after (b) spontaneous supersymmetry breaking; $E_{\rm SuSy}$ is the supersymmetry-breaking parameter. For supersymmetry to be broken, the Witten index must vanish, which means that the ground states must occur in boson–fermion degenerate pairs.

If there are no supersymmetric (ground) states with $E = 0$ energy, supersymmetry is broken: The states in the Hilbert space are formally obtained as a direct sum of sectors, each of which is obtained by choosing a state with lowest, albeit positive, energy and upon which one acts with creation operators. These sectors of the Hilbert space in the general case form semi-infinite ladders

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of bosonic and (independently) fermionic states, and those states are not guaranteed to be ordered in pairs as shown in Figure 10.1 on p. 362, and in Figure 10.2.

Spontaneous supersymmetry breaking If supersymmetry is broken because the system of Euler–Lagrange equations of motion in the list on p. 374 does not have a solution for which the potential energy minimum[8] equals zero, and Hamilton's action functional continues to be supersymmetric, supersymmetry is said to be *spontaneously broken*. In such cases, every sector of the Hilbert space looks like a semi-infinite ladder of states, as shown in Figure 10.2(b), and where the difference between the masses of adjacent bosonic and fermionic states, E_{SuSy}, is the supersymmetry-breaking parameter; in the limiting case $E_{\text{SuSy}} \to 0$, this sector returns from the shape in Figure 10.2(b) into the shape in Figure 10.2(a). In practice, this case is confirmed by analyzing the subsystem of non-differential equations in step 4(b)iii on p. 374, and the simplest model (named after the physicist who discovered this possibility, Lochlainn O'Raifeartaigh) where supersymmetry is spontaneously broken requires at least three distinct super-multiplets $(\phi^a; \Psi^a_+; F^a)$ $a = 1, 2, 3$ [189, 560]; see Digression 10.11 on p. 385.

Explicit supersymmetry breaking If supersymmetry of the Hamilton action is breaking because of the occurrence (or addition "by hand") of a term in the Lagrangian density, supersymmetry is said to be *explicitly broken* by that term. The effect of the explicit supersymmetry breaking on the Hilbert space of course depends on the concrete Lagrangian term that breaks supersymmetry.

— ❦ —

A detailed analysis of the mechanism whereby supersymmetry removes the need for renormalizing parameters in the Lagrangian density that stem from the superpotential [☞ Section 10.3.2] is far beyond the scope of this book. However, at least intuitively, the source of this property is seen from the fact listed in Rule 7 for Feynman calculus with the diagrams in quantum electrodynamics [☞ Procedure 5.2 on p. 193]. That is, each fermionic loop (closed path) in a given Feynman diagram requires an additional factor of (-1) as compared to an otherwise identical diagram where that same loop is bosonic. If then the Feynman diagrams Γ and Γ' differ only by:

1. the loop $\mathscr{C} \subset \Gamma$ is a closed path of particle X in the diagram Γ,
2. the loop $\mathscr{C} \subset \Gamma'$ is a closed path of the superpartner of particle X in the diagram Γ',

the contributions to the amplitude of probability cancel $\mathfrak{M}(\Gamma) + \mathfrak{M}(\Gamma') = 0$.

It remains of course to precisely determine when and in precisely which perturbative calculations do contributions always occur in such canceling pairs. For the details and a precise formulation of this theorem on non-renormalization in supersymmetric models, the interested Reader is directed to the standard textbooks [189, 562, 560, 76].

10.3.2 Supersymmetry in 1+3-dimensional spacetime

In spacetime, the Hamiltonian is the time component of the 4-momentum operator, so the relation (10.33) may be adapted by replacing the operator Z_{IJ} with the operators of linear momentum. Also, since the supersymmetry generators Q_I, according to the Haag–Łopuszanski–Sohnius theorem, transform as spin-$\frac{1}{2}$ representations of the Lorentz group, the index I must count the components of the corresponding spinor, or several copies of it.

For the purposes of this introductory text, we restrict to *simple* (unextended) supersymmetry, the generators of which form a single Dirac spinor. Generalizations are described in the literature;

[8] By potential energy we mean the value of the total energy, i.e., Hamiltonian where all derivatives of all fields are set to zero.

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see e.g., Refs. [189, 562, 560, 129, 76, 308, 178, 535, 461, 144, 351, 356, 60, 19, 115, 186] for starters.

Superalgebra and supersymmetry

We are interested in the Lorentz group in 1+3-dimensional spacetime, with the very convenient fact (A.5.2) that the algebra of the Lorentz group $Spin(1,3) \cong SL(2;\mathbb{C})$ is isomorphic with the direct sum of two copies of the algebra $\mathfrak{spin}(3) = \mathfrak{su}(2)$, which is in turn very familiar both from classical as well as from quantum mechanics as the group of rotations, i.e., spin. It is then convenient to use the notation that expresses this mathematical structure [☞ Section A.5 to begin with, and the textbooks [189, 560, 76] for more precise and abundant details].

The 4-component Dirac spinor may be decomposed, in a Lorentz-invariant way, into two 2-component Weyl spinors [☞ Section 5.2.1 on p. 172, Appendix A.6 and especially A.6.2], which in Weyl's (chiral) basis of the Dirac matrices (A.132) is

$$\Psi \equiv \Psi_+ + \Psi_- = (\boldsymbol{\gamma}_+\Psi) + (\boldsymbol{\gamma}_-\Psi), \qquad \Psi_+ \mapsto \begin{bmatrix}\psi_\alpha\\ 0\end{bmatrix} \quad \text{and} \quad \Psi_- \mapsto \begin{bmatrix}0\\ \bar{\chi}_{\dot\alpha}\end{bmatrix}, \quad \alpha,\dot\alpha = 1,2. \tag{10.61}$$

In simple supersymmetry, the total number of generators, Q_α and $\overline{Q}_{\dot\alpha}$, is minimal and itself forms a Dirac spinor. The supersymmetry transformation operator, following the definitions (5.20) and (6.2), then is

$$U_{\epsilon,\bar\epsilon} := e^{\delta_Q(\epsilon)} = \mathbb{1} + \delta_Q(\epsilon) + \cdots, \qquad \delta_Q(\epsilon) := -i\big(\epsilon\cdot Q + \bar\epsilon\cdot\overline{Q}\big). \tag{10.62}$$

The defining relations of supersymmetry without any central extension are[9]

$$\big\{\,Q_\alpha\,,\,\overline{Q}_{\dot\alpha}\,\big\} = -2\,\sigma^\mu_{\alpha\dot\alpha}\,P_\mu, \qquad \big[\,L_{\mu\nu}\,,\,Q_\alpha\,\big] = i\hbar(\boldsymbol{\sigma}_{\mu\nu})_\alpha{}^\beta Q_\beta, \tag{10.63a}$$

$$\big[\,L_{\mu\nu}\,,\,P_\rho\,\big] = i\hbar\big(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\big), \qquad \big[\,L_{\mu\nu}\,,\,\overline{Q}_{\dot\alpha}\,\big] = i\hbar(\overline{\boldsymbol{\sigma}}_{\mu\nu})_{\dot\alpha}{}^{\dot\beta}\overline{Q}_{\dot\beta}, \tag{10.63b}$$

$$\big[\,L_{\mu\nu}\,,\,L_{\rho\sigma}\,\big] = i\hbar\big(\eta_{\mu\rho}L_{\nu\sigma} - \eta_{\mu\sigma}L_{\nu\rho} + \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\nu\rho}L_{\mu\sigma}\big), \tag{10.63c}$$

with all other (anti)commutators vanishing, and where the matrices $\boldsymbol{\sigma}_{\mu\nu}$ are defined in relations (A.158) [☞ Appendix A.6.2 in more detail]. The generators P_μ and $L_{\mu\nu}$ have the well-known differential operator representation over spacetime [☞ also relations (A.111)],

$$P_\mu = \tfrac{\hbar}{i}\partial_\mu \quad \text{and} \quad L_{\mu\nu} := \tfrac{\hbar}{i}(\eta_{\mu\rho}x^\rho\partial_\nu - \eta_{\nu\rho}x^\rho\partial_\mu), \tag{10.64}$$

while Q_α and $\overline{Q}_{\dot\alpha}$ are at this point abstract operators. For them to acquire a differential operator representation, spacetime itself must be extended, and we now turn to this.

Superspace

In 1974, Abdus Salam and John Strathdee postulated superspace, as an extension of spacetime and in which spacetime is contained as a subspace. Since then, supersymmetry researchers mostly form two schools: those who fully rely on superspace and the methods of super-functional analysis, and those who regard superspace as an irrelevant crutch. However, it has been proven recently [282] that the canonical relation[10] $[H,t] = i\hbar$ and self-consistency of the supersymmetry algebra (10.31) via Jacobi identities (10.37) implies the existence of superspace. Although the very existence of

[9] The negative sign in the first of relations (10.63) follows from that in relations (3.35) and (3.38).

[10] In spacetime, this is $[p_\mu, x^\nu] = -i\hbar\delta^\nu_\mu$, where the negative sign in the right-hand side stems from the definition $(p_\mu) = (-E/c, \vec{p})$ [☞ the derivation of equation (3.35)], as well as the identification of $E \to i\hbar\frac{\partial}{\partial t}$ and $\vec{p} \to \frac{\hbar}{i}\vec{\nabla}$ in the coordinate representation. Keep in mind that time t in quantum mechanics and spacetime coordinates x^μ in quantum field theory are not eigenvalues of any Hermitian operators but parameters [☞ Ref. [29] for a detailed discussion in quantum mechanics].

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superspace does not *force* us to use it, this will be convenient for the purposes of this book, since it explicitly represents and effectively uses the unification of bosons and fermions.

The standard superspace is the extension of spacetime that in addition to the four bosonic coordinates has another four fermionic, *anticommuting* coordinates:

$$\mathrm{x} \mapsto (x^\mu;\theta^\alpha,\bar\theta^{\dot\alpha}) = (ct,x^1,x^2,x^3;\theta^1,\theta^2,\bar\theta^{\dot 1},\bar\theta^{\dot 2}), \qquad \{\theta^\alpha,\theta^\beta\} = 0 = \{\theta^\alpha,\bar\theta^{\dot\alpha}\}. \tag{10.65}$$

The corresponding derivatives also anticommute:

$$\partial_\alpha := \frac{\partial}{\partial\theta^\alpha}, \quad \bar\partial_{\dot\alpha} := \frac{\partial}{\partial\bar\theta^{\dot\alpha}}, \qquad \{\partial_\alpha,\partial_\beta\} = \{\partial_\alpha,\bar\partial_{\dot\beta}\} = \{\bar\partial_{\dot\alpha},\bar\partial_{\dot\beta}\} = 0. \tag{10.66}$$

It is not hard to verify that the combined operators

$$Q_\alpha := i\partial_\alpha + \hbar\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \qquad \text{and} \qquad \bar Q_{\dot\alpha} := i\bar\partial_{\dot\alpha} + \hbar\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\partial_\mu \tag{10.67}$$

satisfy the relations (10.63) and so, together with the definitions (10.64), give a differential representation of the abstract operators in the algebra (10.63). Newcomers in this field usually find it surprising that there exists a second pair of operators

$$D_\alpha := \partial_\alpha + i\hbar\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \qquad \text{and} \qquad \bar D_{\dot\alpha} := \bar\partial_{\dot\alpha} + i\hbar\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\partial_\mu \tag{10.68}$$

that satisfy

$$\{\, D_\alpha \,,\, \bar D_{\dot\alpha} \,\} = -2\sigma^\mu_{\alpha\dot\alpha}P_\mu = 2i\hbar\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \tag{10.69}$$

as well as the other relations (10.63) upon substituting $Q \to D$ and $\bar Q \to \bar D$, and finally that

$$\left.\begin{array}{l} \{\, D_\alpha \,,\, Q_\beta \,\} = 0 = \{\, D_\alpha \,,\, \bar Q_{\dot\beta} \,\} \\ \{\, \bar D_{\dot\alpha} \,,\, Q_\beta \,\} = 0 = \{\, \bar D_{\dot\alpha} \,,\, \bar Q_{\dot\beta} \,\} \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{l} U^{-1}_{\epsilon,\bar\epsilon} D_\alpha U_{\epsilon,\bar\epsilon} = D_\alpha, \\ U^{-1}_{\epsilon,\bar\epsilon} \bar D_{\dot\alpha} U_{\epsilon,\bar\epsilon} = \bar D_{\dot\alpha}. \end{array}\right. \tag{10.70}$$

Recalling the property (10.70), D_α and $\bar D_{\dot\alpha}$ are usually called *super-covariant derivatives*, although they are in fact *invariant* with respect to supersymmetry transformations; for brevity and to avoid this imprecision, we use “super-derivative” instead. Note that

$$-iQ_\alpha = D_\alpha - 2i\hbar\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \qquad \text{and} \qquad -i\bar Q_{\dot\alpha} = \bar D_\alpha - 2i\hbar\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\partial_\mu. \tag{10.71}$$

Digression 10.7 The definitions (10.64) and relations (10.63) and (10.69) imply that the physical dimensions (units) of the operators in the supersymmetry algebra are

$$[P_\mu] = \frac{M\,L}{T}, \quad [L_{\mu\nu}] = \frac{M\,L^2}{T}, \quad [Q_\alpha] = [\bar Q_{\dot\alpha}] = \sqrt{\frac{M\,L}{T}} = [D_\alpha] = [\bar D_{\dot\alpha}]. \tag{10.72a}$$

Also,

$$[\theta^\alpha] = [\bar\theta^{\dot\alpha}] = \sqrt{\frac{T}{M\,L}}, \quad \text{so} \quad [\hbar\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}] = [x^\mu]. \tag{10.72b}$$

In turn, using the high energy particle physics convention where powers of $\hbar$ and c are implied and unwritten, the dimensions of these field theory operators are expressed by specifying the appropriate *power* of energy:

$$\begin{array}{c} [P_\mu] = 1 = [H], \quad [L_{\mu\nu}] = 0, \quad [Q_\alpha] = [\bar Q_{\dot\alpha}] = \tfrac{1}{2} = [D_\alpha] = [\bar D_{\dot\alpha}], \\ [\theta^\alpha] = [\bar\theta^{\dot\alpha}] = -\tfrac{1}{2}, \quad [x^\mu] = -1, \end{array} \tag{10.72c}$$

implying, e.g., that $\sqrt{\frac{\mathrm{MeV}}{c}}$ are units for Q_α and $\frac{\hbar c}{\mathrm{MeV}}$ for x^μ; see Digressions 10.3 on p. 362 and 10.6 on p. 370, as well as Table C.5 on p. 528.

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Superfields

Since all the operators in the algebra (10.63) are now realized as differential operators (10.64), (10.67) and (10.68) with respect to the coordinates of superspace (10.65), it is natural to introduce functions over superspace, so-called "superfields," $\mathbb{F}(\mathrm{x};\theta,\bar{\theta})$. The very definition of coordinates (10.65) implies that they are nilpotent:

$$\{\theta^\alpha,\theta^\beta\}=0 \quad \stackrel{\alpha=\beta}{\Longrightarrow} \quad 0=\{\theta^\alpha,\theta^\alpha\}=2(\theta^\alpha)^2 \quad\Rightarrow\quad (\theta^\alpha)^2=0,\ \ \alpha=1,2; \tag{10.73a}$$

$$\{\bar{\theta}^{\dot\alpha},\bar{\theta}^{\dot\beta}\}=0 \quad \stackrel{\dot\alpha=\dot\beta}{\Longrightarrow} \quad 0=\{\bar{\theta}^{\dot\alpha},\bar{\theta}^{\dot\alpha}\}=2(\bar{\theta}^{\dot\alpha})^2 \quad\Rightarrow\quad (\bar{\theta}^{\dot\alpha})^2=0,\ \ \dot\alpha=1,2. \tag{10.73b}$$

Therefore, every function of the variables $\theta^\alpha,\bar{\theta}^{\dot\alpha}$ has a formal Taylor expansion that terminates and gives a finite polynomial:

$$\mathbb{F}(\mathrm{x};\theta,\bar{\theta})=\phi(\mathrm{x})+\theta^\alpha\psi_\alpha(\mathrm{x})+\bar{\theta}^{\dot\alpha}\bar{\chi}_{\dot\alpha}(\mathrm{x})+\cdots+\theta^2\bar{\theta}^2\mathcal{F}(\mathrm{x}), \tag{10.74}$$

where the coefficients in the expansion are ordinary functions over ordinary spacetime and where $\theta^2:=\frac{1}{2}\varepsilon_{\alpha\beta}\theta^\alpha\theta^\beta$ and $\bar{\theta}^2:=\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\bar{\theta}^{\dot\alpha}\bar{\theta}^{\dot\beta}$ [☞ Appendix A.6.2 and especially Comment A.3 on p. 490, for notation]. If $\mathbb{F}(\mathrm{x};\theta,\bar{\theta})$ is given as a commuting, scalar superfield and since the $\theta,\bar{\theta}$ coordinates anticommute, the coefficient functions – called *component fields* – alternate between being commuting and anticommuting:

0. $\phi(\mathrm{x})$ is a commuting function and a scalar,
1. $\psi_\alpha(\mathrm{x})$ and $\bar{\chi}_{\dot\alpha}(\mathrm{x})$ are anticommuting functions and spin-$\frac{1}{2}$ spinors,
   ⋮
4. $\mathcal{F}(\mathrm{x})$ is a commuting function and a scalar.

Alternatively, the component fields may be defined as the coefficients in the Taylor expansion over $(\theta^\alpha,\bar{\theta}^{\dot\alpha})$, using the super-derivatives projected to the spacetime subspace of superspace:

$$\phi(\mathrm{x}):=\mathbb{F}(\mathrm{x};\theta,\bar{\theta})\big|; \tag{10.75a}$$

$$\psi_\alpha(\mathrm{x}):=\big[D_\alpha\mathbb{F}(\mathrm{x};\theta,\bar{\theta})\big]\big|; \tag{10.75b}$$

$$\bar{\chi}_{\dot\alpha}(\mathrm{x}):=\big[\bar{D}_{\dot\alpha}\mathbb{F}(\mathrm{x};\theta,\bar{\theta})\big]\big|; \tag{10.75c}$$

$$F(\mathrm{x}):=-\tfrac{1}{4}\big[D^2\mathbb{F}(\mathrm{x};\theta,\bar{\theta})\big]\big|; \tag{10.75d}$$

$$V_{\alpha\dot\alpha}(\mathrm{x}):=-\tfrac{1}{2}\big[[D_\alpha,\bar{D}_{\dot\alpha}]\mathbb{F}(\mathrm{x};\theta,\bar{\theta})\big]\big|,\qquad V_\mu:=\tfrac{1}{2}\bar{\sigma}_\mu^{\dot\alpha\alpha}V_{\alpha\dot\alpha}; \tag{10.75e}$$

$$G(\mathrm{x}):=-\tfrac{1}{4}\big[\bar{D}^2\mathbb{F}(\mathrm{x};\theta,\bar{\theta})\big]\big|; \tag{10.75f}$$

$$\lambda_\alpha(\mathrm{x}):=-\tfrac{1}{4}\big[\bar{D}^2D_\alpha\mathbb{F}(\mathrm{x};\theta,\bar{\theta})\big]\big|; \tag{10.75g}$$

$$\bar{\kappa}_{\dot\alpha}(\mathrm{x}):=-\tfrac{1}{4}\big[D^2\bar{D}_{\dot\alpha}\mathbb{F}(\mathrm{x};\theta,\bar{\theta})\big]\big|; \tag{10.75h}$$

$$\mathcal{F}(\mathrm{x}):=\tfrac{1}{32}\big[(D^2\bar{D}^2+\bar{D}^2D^2)\mathbb{F}(\mathrm{x};\theta,\bar{\theta})\big]\big| \tag{10.75i}$$

where the vertical right-delimiter denotes the projection to the "ordinary" spacetime:

$$(\,X\,)\big|\ :=\ \lim_{\theta,\bar{\theta}\to 0}(\,X\,). \tag{10.76}$$

These definitions use super-derivatives instead of ordinary partial derivatives:

1. Since all definitions (10.75) contain a projection $\theta,\bar{\theta}\to 0$ that annihilates contributions that include $i\hbar\boldsymbol{\sigma}^\mu\cdot\bar{\theta}\partial_\mu$ and $i\hbar\theta\cdot\boldsymbol{\sigma}^\mu\partial_\mu$, the end result is the same as if $D_\alpha\to\partial_\alpha$ and

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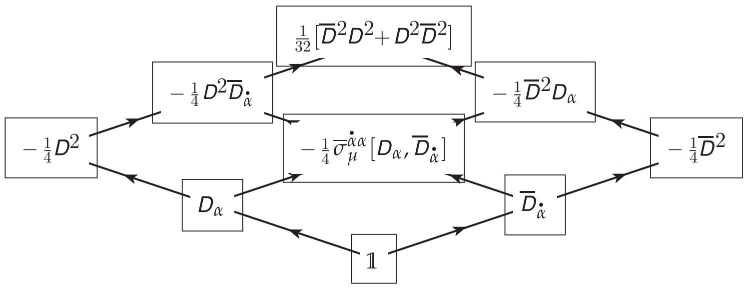
$\overline{D}_{\dot{\alpha}} \to \overline{\partial}_{\dot{\alpha}}$ were used – up to spacetime derivatives of component fields of “lower” physical dimensions (units), obtained upon multiple application of super-derivatives, where one of these derivatives acts on the $\theta, \bar{\theta}$ coordinates in the other.

2. The advantage of using super-derivatives in the definitions (10.75) follows from the relations (10.70): Q and $\overline{Q}$ may be freely anticommuted with the super-derivatives $D, \overline{D}$, which is not true of ordinary partial derivatives $\partial_\alpha, \overline{\partial}_{\dot{\alpha}}$.
3. The super-covariant derivatives may be used for imposing superdifferential constraints, which are then evidently covariant with respect to supersymmetry transformations, implemented by the operator (10.62) [☞ Section 10.3.3].

Note: when acting upon superfields and superdifferential expressions of superfields, whereas the super-derivatives $D_\alpha, \overline{D}_{\dot{\alpha}}$ act as usual, from the left, the supersymmetry generators $Q_\alpha, \overline{Q}_{\dot{\alpha}}$ act *from the right* [189, 76]. Thus,

$$Q_\alpha\big(\mathbb{F}\big) = \mathbb{F}\overleftarrow{Q_\alpha} = +(Q_\alpha\mathbb{F}), \qquad \text{but} \qquad Q_\alpha\big(D_\beta\mathbb{F}\big) = (D_\beta\mathbb{F})\overleftarrow{Q_\alpha} \overset{(10.70)}{=} -\big(Q_\alpha \circ D_\beta\mathbb{F}\big), \tag{10.77}$$

where in the final expressions, both $+Q_\alpha\mathbb{F}$ and $-Q_\alpha D_\beta\mathbb{F}$ act as usual, from the left. It is useful to note that the operators used in the definitions (10.75) form a hierarchy of super-derivatives:



(10.78)

This structure is partially ordered by the physical dimension [☞ Digression 10.7 on p. 378], which grows upward in the diagram (10.78), and by successive application of D_α and $\overline{D}_{\dot{\alpha}}$ denoted by arrows in the diagram (10.78).

Example 10.1 The infinitesimal supersymmetry transformations of any component field may be obtained by computing the projection

$$\begin{aligned}\mathscr{D}[\delta_Q(\epsilon)\mathbb{F}]\big| &\overset{(10.70)}{=} \delta_Q(\epsilon)(\mathscr{D}\mathbb{F})\big| = \big[-i(\epsilon\cdot Q + \bar{\epsilon}\cdot\overline{Q})(\mathscr{D}\mathbb{F})\big]\big| \\ &= \big(\epsilon\cdot(D - 2i\hbar\boldsymbol{\sigma}^\mu\cdot\bar{\theta}\,\partial_\mu) + \bar{\epsilon}\cdot(\overline{D} - 2i\hbar\theta\cdot\boldsymbol{\sigma}^\mu\partial_\mu)\big)(\mathscr{D}\mathbb{F})\big| = \big(\epsilon\cdot D + \bar{\epsilon}\cdot\overline{D}\big)(\mathscr{D}\mathbb{F})\big|,\end{aligned} \tag{10.79}$$

where $\mathscr{D}$ is the specific D-operator from the basis (10.78) that projects on the desired component field within the superfield $\mathbb{F}$. For example,

$$\delta_Q(\epsilon)\phi = \big(\epsilon^\alpha D_\alpha + \bar{\epsilon}^{\dot{\alpha}}\overline{D}_{\dot{\alpha}}\big)\mathbb{F}\big| = \epsilon^\alpha\psi_\alpha + \bar{\epsilon}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}; \tag{10.80}$$

$$\begin{aligned}\delta_Q(\epsilon)\psi_\alpha &= (\epsilon\cdot D + \bar{\epsilon}\cdot\overline{D})(D_\alpha\mathbb{F})\big| = \tfrac{1}{2}\epsilon^\beta\varepsilon_{\beta\alpha}D^2\mathbb{F}\big| + \bar{\epsilon}^{\dot{\alpha}}\big(\tfrac{1}{2}\{D_\alpha, \overline{D}_{\dot{\alpha}}\} - \tfrac{1}{2}[D_\alpha, \overline{D}_{\dot{\alpha}}]\big)\mathbb{F}\big| \\ &= \tfrac{1}{2}\epsilon^\beta\varepsilon_{\beta\alpha}(-4F) + i\hbar\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}(\partial_\mu\phi) - \tfrac{1}{4}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\big(\bar{\sigma}_\mu^{\dot{\beta}\beta}[D_\beta, \overline{D}_{\dot{\beta}}]\mathbb{F}\big)\big| \\ &= 2\varepsilon_{\alpha\beta}\epsilon^\beta F + i\hbar\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}(\partial_\mu\phi) + \sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}F_\mu; \qquad \text{etc.}\end{aligned} \tag{10.81}$$

---

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Another key property is that every function of the form

$$[D^2\overline{D}^2 f(\mathbb{F}_1,\mathbb{F}_2,\dots)]\big| \tag{10.82}$$

is automatically an invariant under the supersymmetry transformation. More precisely, we have the standard result:

Theorem 10.7 *For every analytic functional expression $f(\mathbb{F}_1,\mathbb{F}_2,\dots)$ constructed from superfields $\mathbb{F}_1$, $\mathbb{F}_2$, etc., the Hamilton action of the form $\int \mathrm{d}x\,[D^2\overline{D}^2 f]\big|$ is supersymmetric:*

$$\delta_Q(\epsilon)\int \mathrm{d}^4x\,[D^2\overline{D}^2 f(\mathbb{F}_1,\mathbb{F}_2,\dots)]\big| = \int \mathrm{d}^4x\,\partial_\mu\,\mathcal{K}^\mu = 0, \tag{10.83}$$

where the functional expression and the component fields of the superfields $\mathbb{F}_i$ satisfy the restrictions that are usual in field theory, and which guarantee that the spacetime integrals (10.83) are well defined and convergent.

Comment 10.3 *The concrete choice of the functional expression $f(\mathbb{F}_1,\mathbb{F}_2,\dots)$ of course depends on which concrete terms one desires in the Lagrangian density:*

$$\mathscr{L} := [D^2\overline{D}^2 f(\mathbb{F}_1,\mathbb{F}_2,\dots)]\big|. \tag{10.84}$$

By definition, the Lagrangian density is said to be supersymmetric if it defines a supersymmetric Hamilton action, which means that $\delta_Q(\epsilon)\mathscr{L} = \partial_\mu\mathcal{K}^\mu$ suffices.

Proof The result (10.83) follows from direct computation with the two terms:

$$\begin{aligned}
\delta_Q(\epsilon)\int \mathrm{d}^4x\,[D^2\overline{D}^2 f(\mathbb{F}_1,\mathbb{F}_2,\dots)]\big| &\overset{(10.71)}{=} \int \mathrm{d}x\Big\{(\epsilon{\cdot}D_\alpha + \bar{\epsilon}{\cdot}\overline{D} + \dots)D^2\overline{D}^2 f(\mathbb{F})\Big\}\Big|\\
&\overset{\theta,\bar{\theta}\to 0}{=} \int \mathrm{d}^4x\Big\{\epsilon^\alpha \underbrace{D_\alpha D^2}_{\equiv 0}\overline{D}^2 f(\mathbb{F}) + \bar{\epsilon}^{\dot{\alpha}}\overline{D}_{\dot{\alpha}} D^2\overline{D}^2 f(\mathbb{F})\Big\}\Big|\\
&\overset{(A.164)}{=} \int \mathrm{d}^4x\Big\{\bar{\epsilon}^{\dot{\alpha}}\big[-4i\hbar\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\varepsilon^{\alpha\beta}D_\beta + D^2\overline{D}_{\dot{\alpha}}\big]\overline{D}^2 f(\mathbb{F})\Big\}\Big|\\
&= \int \mathrm{d}^4x\Big\{\partial_\mu \underbrace{\big[-4i\hbar\bar{\epsilon}^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\varepsilon^{\alpha\beta}D_\beta\overline{D}^2 f(\mathbb{F})\big]\big|}_{:=\mathcal{K}^\mu} + \bar{\epsilon}^{\dot{\alpha}} D^2 \underbrace{\overline{D}_{\dot{\alpha}}\overline{D}^2}_{\equiv 0} f(\mathbb{F})\big|\Big\}\\
&= \int_{\text{spacetime}} \mathrm{d}^4x\;\partial_\mu\mathcal{K}^\mu = \oint_{\partial(\text{spacetime})} (\mathrm{d}^3x)_\mu\,\mathcal{K}^\mu,
\end{aligned} \tag{10.85}$$

where the last integral vanishes, since the "boundary" of spacetime is an infinitely distant 3-sphere, where the fields, and also the integral of $\mathcal{K}^\mu$, are routinely required to vanish. ☑

Comment 10.4 *As it is only necessary for the entire integral $\oint_{\partial(\text{spacetime})} (\mathrm{d}^3x)_\mu\,\mathcal{K}^\mu$ to vanish, it would suffice for the expression $\mathcal{K}^\mu$ to vary "at infinity" so that the sum over the infinitely distant (spacetime) 3-sphere should evaluate to zero.*

Digression 10.8 Given an anticommuting (Grassmann) variable θ, integration over θ that is invariant with respect to constant translations $\theta\to\theta+\epsilon$ must in fact be functionally identical to the partial θ-derivative: $\int \mathrm{d}\theta\, f(\theta) \equiv \frac{\partial}{\partial\theta} f(\theta)$; this is called Berezin integration,

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after Felix Alexandrovich Berezin. Since θ must be nilpotent, the result of such integration must be θ-independent, and no loss is incurred by appending the $\theta \to 0$ projection. However, delimited by this trailing projection, the action of a super-derivative such as $D := \frac{\partial}{\partial\theta} + i\hbar\theta\frac{\partial}{\partial x}$ (where x is a commuting variable) is identical to the action of a partial θ-derivative, which in turn is identical to Berezin integration:

$$[D\,f(\theta)]\big| = \big[\frac{\partial}{\partial\theta}f(\theta)\big]\Big| = \int \mathrm{d}\theta\, f(\theta). \tag{10.86}$$

The $\theta,\bar\theta \to 0$ projection (10.76) of the 4-fold super-derivative (10.84) of any superfield function is thus equal to its $\mathrm{d}^2\theta\mathrm{d}^2\bar\theta$-integral. In turn, this re-interprets the integral–super-derivative combination such as in (10.83) as a $\mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta$-integration over the whole superspace and so provides a completely uniform and geometrical treatment. In practice, however, one evaluates these integrals by means of projections of super-derivatives, which is why they are so indicated throughout this chapter.

10.3.3 *The chiral and the gauge superfield*

The superfield $\mathbb{F}(\mathrm{x};\theta,\bar\theta)$ may also be regarded a partially ordered set of component fields, which may be partially ordered by growing physical dimensions (units) akin to the diagram (10.78). Preserving this structure, it is possible to impose constraints on some of the component fields, which is most effectively achieved using super-derivatives.

Super-constraints and the chiral superfield

Using the superfields and super-derivatives D_α and $\bar D_{\dot\alpha}$, it is possible to specify superdifferential equations that – because of the relations (10.70) – transform covariantly with respect to the action of supersymmetry transformations $U_{\epsilon,\bar\epsilon}$, given by equation (10.62).

One of the simplest such superdifferential equations defines the so-called chiral (and the conjugate, anti-chiral) superfield:[11]

$$\textbf{chiral}\quad \bar D_{\dot\alpha}\,\Phi = 0 \qquad \text{and} \qquad \textbf{anti-chiral}\quad D_\alpha\,\bar\Phi = 0. \tag{10.87}$$

It is then not hard to show that

$$\phi := [\Phi]\big|, \qquad \psi_\alpha := [D_\alpha\Phi]\big|, \qquad F := -\tfrac{1}{4}[D^2\Phi]\big| \tag{10.88}$$

are the only non-trivial component fields: two complex scalar fields ϕ and F, and one 2-component complex spin-$\frac{1}{2}$ field ψ_α; the physical dimensions (units) of these two scalar fields, however, are not equal: $[F] = [\phi]\cdot\frac{ML}{T}$. The remaining components either vanish or do not include new fields; for example,

$$[\bar D_{\dot\alpha}\Phi]\big| \overset{(10.87)}{=} 0, \qquad [\bar D^2\Phi]\big| \overset{(10.87)}{=} 0, \tag{10.89}$$

$$[\bar D_{\dot\alpha} D_\alpha\Phi]\big| \overset{(10.69)}{=} [(2\sigma^\mu_{\alpha\dot\alpha}P_\mu - D_\alpha\bar D_{\dot\alpha})\Phi]\big| \overset{(10.87)}{=} [(-2i\sigma^\mu_{\alpha\dot\alpha}\hbar\partial_\mu)\Phi]\big| = -2i\hbar\sigma^\mu_{\alpha\dot\alpha}(\partial_\mu\phi). \tag{10.90}$$

Supersymmetric transformations are easily derived following Example 10.1 on p. 380:

$$\delta_Q(\epsilon)\phi = (\epsilon\cdot D + \bar\epsilon\cdot\bar D)\Phi\big| = \epsilon\cdot\psi; \tag{10.91a}$$

[11] The analogy with complex-analytic functions is fully justified and valuable.

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$$\delta_Q(\epsilon)\psi_\alpha = (\epsilon\cdot D + \bar{\epsilon}\cdot\overline{D})D_\alpha\Phi\big| = \big(\tfrac{1}{2}\epsilon^\beta\varepsilon_{\beta\alpha}D^2 + 2i\hbar\bar{\epsilon}^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\big)\Phi\big|,$$
$$= 2\varepsilon_{\alpha\beta}\epsilon^\beta F + 2i\hbar\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}(\partial_\mu\phi); \tag{10.91b}$$
$$\delta_Q(\epsilon)F = (\epsilon\cdot D + \bar{\epsilon}\cdot\overline{D})(-\tfrac{1}{4}D^2\Phi)\big| = -\tfrac{1}{4}\bar{\epsilon}^{\dot{\alpha}}\big(4i\hbar\sigma^\mu_{\alpha\dot{\alpha}}\varepsilon^{\alpha\beta}\partial_\mu D_\beta\big)\Phi\big|,$$
$$= -i\hbar\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\varepsilon^{\alpha\beta}(\partial_\mu\psi_\beta). \tag{10.91c}$$

Digression 10.9 Iterating the result (10.91), one can show that

$$[\delta_Q(\epsilon_{(1)}),\delta_Q(\epsilon_{(2)})](\phi;\psi_\alpha;F) = 2i\hbar\big(\epsilon_{(2)}\cdot\boldsymbol{\sigma}^\mu\cdot\epsilon_{(1)} - \epsilon_{(1)}\cdot\boldsymbol{\sigma}^\mu\cdot\epsilon_{(2)}\big)\partial_\mu(\phi;\psi_\alpha;F). \tag{10.92a}$$

That is, the commutator of two supersymmetry transformations formally equals a translation in spacetime. However, notice that this translation parameter,

$$\epsilon^\mu_{(1,2)} := \big(\epsilon_{(2)}\cdot\boldsymbol{\sigma}^\mu\cdot\bar{\epsilon}_{(1)} - \epsilon_{(1)}\cdot\boldsymbol{\sigma}^\mu\cdot\bar{\epsilon}_{(2)}\big), \tag{10.92b}$$

is not an ordinary spacetime vector! The supersymmetry transformation parameters, $\epsilon^\alpha_{(1)}, \epsilon^\alpha_{(2)}$ anticommute, and so are nilpotent [☞ relations (10.73)]; the vector (10.92b) is therefore itself (degree-4) *nilpotent*: $(\epsilon^\mu_{(1,2)})^4 \equiv 0$ for any $\mu = 0,1,2,3$. Similarly, $\epsilon\cdot\psi$ is only formally a “shift” in the scalar field ϕ, according to the transformation relation (10.91a), since the expression $\epsilon\cdot\psi(\mathrm{x})$ is in every spacetime point (degree-4) nilpotent and the function $\phi(\mathrm{x})$ in every spacetime point has values that are *ordinary*, i.e., non-nilpotent commuting complex numbers.

Conclusion 10.3 *Although the (symmetrized) iterative application of the supersymmetry generators Q_α and $\overline{Q}_{\dot{\alpha}}$ is equivalent to the application of the spacetime translation generator P_μ, supersymmetry transformations (10.62) do not produce transformations in “real” spacetime.*

The fact that supersymmetry transformations map the fields $\phi(\mathrm{x}) \leftrightarrow \psi_\alpha(\mathrm{x}) \leftrightarrow F(\mathrm{x})$ (and their derivatives) in every spacetime point, however, remains.

Notice that chiral superfields (at the same spacetime point) form the “ring” algebraic structure [☞ the lexicon entry, in Appendix B.1]:

Conclusion 10.4 *The product of two chiral superfields is again a chiral superfield:*

$$\overline{D}_{\dot{\alpha}}\Phi_1 = 0 = \overline{D}_{\dot{\alpha}}\Phi_2, \quad \Rightarrow \quad \overline{D}_{\dot{\alpha}}(\Phi_1\Phi_2) = 0, \tag{10.93}$$

*with the usual rules of distribution between multiplication and addition. It follows that chiral fields (at the same spacetime point) form the “**chiral ring**.” Moreover, it follows that an arbitrary analytic function of chiral superfields (defined by its Taylor expansion) is also a chiral superfield.*

Theorem 10.8 *The most general supersymmetric Lagrangian density for a chiral superfield Φ must be of the form*

$$\mathscr{L}[\Phi] = [D^2\overline{D}^2\,K(\Phi^\dagger,\Phi)]\big| \;+\; [D^2\,W(\Phi)]\big| + [\overline{D}^2\,\overline{W}(\Phi^\dagger)]\big|. \tag{10.94}$$

Proof Since the Lagrangian density must be real, for the first term of the general form (10.84) one selects a real function $K(\Phi^\dagger,\Phi)$, and adds the second term and its Hermitian conjugate where

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$W(\Phi)$ is an arbitrary analytic function. Indeed, this second term is also (and independently!) supersymmetric:

$$
\begin{aligned}
&(\epsilon\cdot Q+\bar\epsilon\cdot\bar Q)\,D^2\,W(\Phi)\big|\\
&\quad\overset{(10.71)}{=}\ \epsilon^\alpha(iD_\alpha-2\hbar\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu)\,D^2\,W(\Phi)\big|+\bar\epsilon^{\dot\alpha}(i\bar D_{\dot\alpha}-2\hbar\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\partial_\mu)\,D^2\,W(\Phi)\big|\\
&\quad\overset{\theta,\bar\theta\to0}{=}\ i\epsilon^\alpha\underbrace{D_\alpha D^2}_{\equiv0}\,W(\Phi)\big|+i\bar\epsilon^{\dot\alpha}(-2i\hbar\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\varepsilon^{\alpha\beta}D_\beta+D^2\bar D_{\dot\alpha})W(\Phi)\big|\\
&\quad=\partial_\mu\underbrace{\big[2\hbar\bar\epsilon^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\varepsilon^{\alpha\beta}D_\beta W(\Phi)\big|\big]}_{\mathcal{K}^\mu}+i\bar\epsilon^{\dot\alpha}D^2\underbrace{\bar D_{\dot\alpha}W(\Phi)}_{=0}\big|=\partial_\mu\mathcal{K}^\mu,
\end{aligned}
\tag{10.95}
$$

so the $\int \mathrm{d}^4x$-integral vanishes again, owing to the usual restrictions on the fields. Listing all possible Lorentz-invariant terms, one shows that the expression (10.94) is the most general form of a supersymmetric Lagrangian density. ☑

The standard choice $K(\Phi^\dagger,\Phi) = \Phi^\dagger\Phi$ gives (after some "D-gymnastics" [☞ relations (A.162)–(A.165)]) the standard Lagrangian density for a scalar ϕ and a fermion ψ_α, and the total resulting Lagrangian density is – up to integration by parts for symmetrization of the expression,

$$
\begin{aligned}
\mathscr{L}[\Phi]=&-(\partial_\mu\phi^*)\eta^{\mu\nu}(\partial_\nu\phi)-\tfrac{i}{2}\bar\sigma^{\mu\dot\alpha\alpha}\big[\bar\psi_{\dot\alpha}(\partial_\mu\psi_\alpha)-(\partial_\mu\bar\psi_{\dot\alpha})\psi_\alpha\big]+F^*F\\
&+F\,W'(\phi)+\tfrac12\varepsilon^{\alpha\beta}\psi_\alpha\psi_\beta\,W''(\phi)\ +\ F^*\,\overline{W}'(\phi^*)+\tfrac12\varepsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\alpha}\bar\psi_{\dot\beta}\,\overline{W}''(\phi^*).
\end{aligned}
\tag{10.96}
$$

Since the Euler–Lagrange equations of motion for the component fields F and F^*,

$$
F^*=-W'(\phi)\qquad\text{and}\qquad F=-\overline{W}'(\phi^*),
\tag{10.97}
$$

are non-differential equations in F and F^*, they may be used to substitute F and F^*:

$$
\begin{aligned}
\mathscr{L}[\Phi]=&-\tfrac{i}{2}\bar\sigma^{\mu\dot\alpha\alpha}\big[\bar\psi_{\dot\alpha}(\partial_\mu\psi_\alpha)-(\partial_\mu\bar\psi_{\dot\alpha})\psi_\alpha\big]-(\partial_\mu\phi^*)\eta^{\mu\nu}(\partial_\nu\phi)\\
&-|W'(\phi)|^2+\tfrac12\varepsilon^{\alpha\beta}\psi_\alpha\psi_\beta\,W''(\phi)+\tfrac12\varepsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\alpha}\bar\psi_{\dot\beta}\,\overline{W}''(\phi^*).
\end{aligned}
\tag{10.98}
$$

The constant $\hbar$ has been eliminated in the expressions such as (10.94)–(10.98) by redefining the component fields to emphasize the similarity with the Lagrangian density (7.34) [☞ Exercise 10.3.6 on p. 388]. A similar redefinition of the fermion fields $\psi_\alpha,\bar\psi_{\dot\alpha}$ and the use of the basis (A.132) for the Dirac $\boldsymbol{\gamma}$-matrices shows the first term (10.98) to give the standard Lagrangian density (5.68a) for Dirac fermions.

The computation (10.94)–(10.98) clearly shows that the $D^2\bar D^2\Phi^\dagger\Phi|$ term produced the standard "kinetic" part of the Lagrangian density, while the terms $D^2W(\Phi)|+\bar D^2\overline{W}(\Phi^\dagger)|$ produce, after eliminating F and F^* via their equations of motion (10.97), the potential

$$
V(\phi)=|W'(\phi)|^2\ \geqslant\ 0.
\tag{10.99}
$$

Finally, the terms $-\psi^2W''(\phi^*)-\bar\psi^2\overline{W}''(\phi^*)$ provide the supersymmetric completion of the potential $|W'(\phi)|^2$. Owing to the relation (10.99) with the potential, the function $W(\Phi)$ is called the *superpotential*.

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Digression 10.10 On one hand, owing to Theorem 10.7 on p. 381, and the similar result (10.95), the Lagrangian density (10.96) is known to be supersymmetric, i.e., its infinitesimal supersymmetry transformation $\delta_Q(\epsilon) = -i(\epsilon\cdot\boldsymbol{Q} + \bar{\epsilon}\cdot\overline{\boldsymbol{Q}})$ changes the Lagrangian density (10.96) into a total derivative. This may also be directly verified by substituting the supersymmetry transformations of the component fields

$$\delta_Q(\epsilon)\phi = \epsilon^\alpha\psi_\alpha, \quad \delta_Q(\epsilon)\psi_\alpha = 2\varepsilon_{\alpha\beta}\epsilon^\beta F + 2i\hbar\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}(\partial_\mu\phi),$$
$$\delta_Q(\epsilon)F = i\hbar\sigma^\mu_{\alpha\dot{\alpha}}\varepsilon^{\alpha\beta}\bar{\epsilon}^{\dot{\alpha}}(\partial_\mu\psi_\beta) \tag{10.100a}$$

into the Lagrangian density (10.96).

However, the Lagrangian density (10.98) is *not* invariant with respect to the supersymmetry transformations (10.100a)! These transformations represent the original (and linear) supersymmetry action upon the superfield Φ, i.e., upon the component fields – including F. The elimination of F by (10.97) changes this action, so that the transformation rules (10.100a) also change, and the Lagrangian density (10.98) is invariant with respect to the so-changed transformations. As $W'(\phi)$ is nonlinear in the general case, these changed supersymmetry transformation rules are also nonlinear.

Digression 10.11 The simplest model in which supersymmetry is spontaneously broken was found by O'Raifeartaigh, and has the superpotential

$$\boldsymbol{D}^2[\lambda\Phi_0 + m\Phi_1\Phi_2 + g\Phi_0\Phi_1^{\,2}]\big| + h.c., \tag{10.101a}$$

where Φ_0, Φ_1 and Φ_2 are three chiral superfields. The non-differential equations of motion for the auxiliary components F_0, F_1 and F_2 are (10.98)

$$F_0 = -\lambda - g\phi_1^{\,2}, \quad F_1 = -m\phi_1 - 2g\phi_0\phi_1, \quad F_2 = -m\phi_2, \tag{10.101b}$$

which make the potential in this model into (10.99)

$$V = \sum_{k=0}^{2}|F_k|^2 = \big|\lambda + g\phi_1^{\,2}\big|^2 + \big|m\phi_2 + 2g\phi_0\phi_1\big|^2 + \big|m\phi_1\big|^2. \tag{10.101c}$$

This cannot possibly vanish: the last term can vanish only where $\phi_1 = 0$ and where the potential becomes $V = |\lambda|^2 + |m\phi_2|^2 > 0$. Supersymmetry is therefore broken spontaneously as there is no solution to the equations of motion where $V = 0$.

One of the original ideas for the application of supersymmetry was to find superfields where the bosons and fermions of the Standard Model [☞ Table 2.3 on p. 67] would all be component fields of the *same* superfields. However, the component fields of the same superfield differ only by spin, and not by charges (electric, weak isospin or color), so this is not possible: the Standard Model fermions have completely different charges from the bosons.

It follows that identifying the 2-component Weyl fermion $\psi_\alpha(\mathrm{x}) \in \Phi(\mathrm{x};\theta,\bar{\theta})$ with a left-handed chiral half of a Dirac 4-component wave-function of the electron, the complex scalar field $\phi(\mathrm{x}) \in \Phi(\mathrm{x};\theta,\bar{\theta})$ is the electron superpartner, the so-called *selectron*, which must be added to the list in Table 2.3 on p. 67. Bosons with the charges given in Table 7.1 on p. 275 have not been detected experimentally, while supersymmetry implies their existence; one jokes that supersymmetry is already 50% experimentally verified. However, this means that supersymmetry – in Nature – must be broken, and in such a way that the masses of the bosonic superpartners of the elementary

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fermions from Table 2.3 on p. 67 are sufficiently big, bigger than the masses of the elementary fermions by the amount E_{SuSy}/c^2, so that this explains why they have not been detected experimentally so far [☞ examples in Figure 10.2 on p. 375].

The gauge superfield

Consider now the real superfield $\mathbb{A}^\dagger = \mathbb{A}$. The component fields may be found by means of Taylor-esque projections (10.75); specifically, the projection (10.75e) finds the real 4-vector field $A_\mu(\mathrm{x}) \in \mathbb{A}(\mathrm{x};\theta,\bar\theta)$. On the other hand, the same component of the combined superfield $i(\Phi - \Phi^\dagger)$ gives

$$\tfrac{1}{4}\bar\sigma_\mu^{\dot\alpha\alpha}\big[[D_\alpha,\bar D_{\dot\alpha}]i(\Phi-\Phi^\dagger)\big]\big| = 2\hbar\big(\partial_\mu\,\Re e(\phi)\big), \tag{10.102}$$

so that the superfield transformation

$$\mathbb{A} \;\to\; \mathbb{A}' = \mathbb{A} + i(\Phi - \Phi^\dagger) \quad\ni\quad A_\mu \;\to\; A'_\mu = A_\mu + \partial_\mu\big(2\hbar\,\Re e(\phi)\big) \tag{10.103}$$

contains the gauge transformation (5.89) of the vector component field, where $-\frac{\hbar}{c}\,\Re e(\phi)$ plays the role of the gauge local parameter. Of course, if the chiral superfields Φ_i are intended one for each Standard Model elementary fermion and we introduce the real superfield $\mathbb{A} \ni A_\mu$ for the electromagnetic field, to parametrize the gauge transformation of the electromagnetic field we must introduce a separate chiral superfield $\boldsymbol{\Lambda}$, the scalar component of which plays the role of the gauge local parameter.

A detailed analysis [189, 562, 560, 76] of the component fields in the combination $\mathbb{A} + i(\boldsymbol{\Lambda} - \bar{\boldsymbol{\Lambda}})$ shows that a suitable choice of the superfield $\boldsymbol{\Lambda}$ eliminates the component fields in the "lower half" of the superfield $\mathbb{A}$, as per diagram (10.78). However, it is more practical to define the *chiral–anti-chiral* pair of fermionic superfields:

$$\mathbb{A}_\alpha := (\bar D^2 D_\alpha \mathbb{A}) \qquad\text{and}\qquad \bar{\mathbb{A}}_{\dot\alpha} := (D^2 \bar D_{\dot\alpha}\mathbb{A}), \tag{10.104}$$

which satisfy

$$\mathbb{A} = \mathbb{A}^\dagger \quad\Rightarrow\quad \varepsilon^{\alpha\beta}D_\alpha\mathbb{A}_\beta = \varepsilon^{\dot\alpha\dot\beta}\bar D_{\dot\alpha}\bar{\mathbb{A}}_{\dot\beta}, \tag{10.105}$$

and the components of which include

$$\mathbb{A}_\alpha\big| =: \lambda_\alpha, \qquad\qquad \bar{\mathbb{A}}_{\dot\alpha}\big| =: \bar\lambda_{\dot\alpha}, \tag{10.106a}$$

$$D_\alpha\mathbb{A}_\beta\big| =: \varepsilon_{\alpha\beta}\mathrm{D} + i(\boldsymbol{\sigma}^{\mu\nu})_\alpha{}^\gamma\,\varepsilon_{\beta\gamma}F_{\mu\nu}, \qquad \bar D_{\dot\alpha}\bar{\mathbb{A}}_{\dot\beta}\big| =: \varepsilon_{\dot\alpha\dot\beta}\mathrm{D} + i(\bar{\boldsymbol{\sigma}}^{\mu\nu})^{\dot\gamma}{}_{\dot\alpha}\,\varepsilon_{\dot\beta\dot\gamma}F_{\mu\nu}, \tag{10.106b}$$

$$D^2\mathbb{A}_\alpha\big| = -i\hbar\sigma^\mu_{\alpha\dot\alpha}\varepsilon^{\dot\alpha\dot\beta}(\partial_\mu\bar\lambda_{\dot\beta}), \qquad \bar D^2\bar{\mathbb{A}}_{\dot\alpha}\big| = -i\hbar\sigma^\mu_{\alpha\dot\alpha}\varepsilon^{\alpha\beta}(\partial_\mu\lambda_\beta). \tag{10.106c}$$

Here,

$$F_{\mu\nu} := (\partial_\mu A_\nu - \partial_\nu A_\mu), \tag{10.107}$$

and the component fields from the "lower half" of the original superfield $\mathbb{A}$ show up neither in the expressions (10.106) nor in any other projection of the superfields $\mathbb{A}_\alpha$ and $\bar{\mathbb{A}}_{\dot\alpha}$. A supersymmetric Lagrangian density that includes the standard $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ Lagrangian density is then obtained from the expression

$$\begin{aligned}\mathscr{L}[\mathbb{A}] &= -\tfrac{1}{4}[D^2\,\varepsilon^{\alpha\beta}\mathbb{A}_\alpha\mathbb{A}_\beta]\big| - \tfrac{1}{4}[\bar D^2\,\varepsilon^{\dot\alpha\dot\beta}\bar{\mathbb{A}}_{\dot\alpha}\bar{\mathbb{A}}_{\dot\beta}]\big|\\ &= -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{i\hbar}{2}\bar\sigma^{\mu\,\dot\alpha\alpha}\big[\bar\lambda_{\dot\alpha}(\partial_\mu\lambda_\alpha) - (\partial_\mu\bar\lambda_{\dot\alpha})\lambda_\alpha\big] + 2\mathrm{D}^2.\end{aligned} \tag{10.108}$$

The first term is – up to a (re)scaling of the field A_μ – the Lagrangian density that is identical to the density in the expression (5.76) for electromagnetic fields. The equations of motion for the

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spinor fields $(\lambda_\alpha, \bar{\lambda}_{\dot{\alpha}})$ are the Dirac equations, which is typical for spin-$\frac{1}{2}$ fermions, but the mass of these spinors vanishes. These then are the superpartners of the gauge fields and are in general called *gauginos*.[12] Notice that the relation (10.105) equates the component functions that occur in D- and $\overline{D}$-projections (10.106b), but leaves $\bar{\lambda}_{\dot{\alpha}}$ formally independent of λ_α. The condition (10.105), however, guarantees that the Dirac spinor $(\lambda_\alpha, \bar{\lambda}_{\dot{\alpha}})$ has four *real* independent components.

Supersymmetric electrodynamics

The minimal supersymmetric Lagrangian density where the chiral field $\mathbb{F}$ interacts with the gauge superfield $\mathbb{A}$, for the supersymmetric version of electrodynamics for example, is obtained in the form

$$\mathscr{L} = -\tfrac{1}{4}[D^2\,\varepsilon^{\alpha\beta}\mathbb{A}_\alpha\mathbb{A}_\beta]\big| - \tfrac{1}{4}[\overline{D}^2\,\varepsilon^{\dot{\alpha}\dot{\beta}}\overline{\mathbb{A}}_{\dot{\alpha}}\overline{\mathbb{A}}_{\dot{\beta}}]\big| + \big[D^2\overline{D}^2\,\overline{\Phi}\,e^{q_\Phi\mathbb{A}}\,\Phi\big]\big|. \tag{10.109}$$

This Lagrangian density is invariant with respect to the gauge transformations

$$\mathbb{A} \to \mathbb{A} + i(\boldsymbol{\Lambda} - \overline{\boldsymbol{\Lambda}}), \qquad \Phi \to e^{iq_\Phi\boldsymbol{\Lambda}}\Phi, \qquad \overline{\Phi} \to e^{-iq_\Phi\overline{\boldsymbol{\Lambda}}}\overline{\Phi}, \tag{10.110}$$

which coincide with the transformations (5.14a) for the component fields A_μ, and that of the $\psi_\alpha \in \Phi$ with the left-handed chiral projection of the transformation (5.14b). Expanding the expression (10.109) produces the Lagrangian density for supersymmetric electrodynamics, where the additional terms involve the superpartners of both the photon (itself represented by the 4-vector potential A_μ), and the left-handed chiral electron (represented by the fermion field ψ_α).

To extend this minimal model so as to include also the right-handed chiral electron, we must introduce another chiral field, $\Phi^c := C(\Phi)$, which is defined so that ψ^c_α is the left-handed chiral spin-$\frac{1}{2}$ fermionic field with the electric charge opposite to that of the electron. Then, $\overline{\psi}^c_{\dot{\alpha}} \in \overline{\Phi^c}$ is the right-handed chiral spin-$\frac{1}{2}$ fermions field with the electric charge equal to the electron charge. Therefore, the Lagrangian density for electrodynamics with a massive electron must be of the form

$$\begin{aligned}\mathscr{L} = &-\tfrac{1}{4}[D^2\,\varepsilon^{\alpha\beta}\mathbb{A}_\alpha\mathbb{A}_\beta]\big| - \tfrac{1}{4}[\overline{D}^2\,\varepsilon^{\dot{\alpha}\dot{\beta}}\overline{\mathbb{A}}_{\dot{\alpha}}\overline{\mathbb{A}}_{\dot{\beta}}]\big| + \big[D^2\overline{D}^2\,\overline{\Phi}\,e^{q_\Phi\mathbb{A}}\,\Phi\big]\big| + \big[D^2\overline{D}^2\,\overline{\Phi^c}\,e^{-q_\Phi\mathbb{A}}\,\Phi^c\big]\big|\\ &+ m\Big\{[D^2\Phi\Phi^c]\big| + [\overline{D}^2\overline{\Phi}\,\overline{\Phi^c}]\big|\Big\},\end{aligned} \tag{10.111}$$

where we added terms in the second row, which produce the Lagrangian terms $m(\psi\cdot\psi^c + \overline{\psi}\cdot\overline{\psi^c})$ for the electron, as well as (after eliminating the auxiliary scalar fields F and F^c using their non-differential equations of motion) $m^2\phi\phi^c \equiv m^2|\phi|^2$ for the selectron.

The minimal supersymmetric Standard Model

The construction of the complete supersymmetric Standard Model is now seen as a generalization of the procedure that led us to the Lagrangian density (10.111). For the details of this construction, the interested Reader is directed to the abundant literature [☞ textbooks [189, 562, 560, 76] to begin with]. However, note:

1. On one hand, supersymmetry conceptually unites bosons and fermions – and requires that every boson has a fermion superpartner, and *vice versa*.
2. On the other hand, the concrete bosons and fermions of the Standard Model cannot be each others' superpartners, since the (gauge and Higgs) bosons in the Standard Model transform differently from the fundamental fermions in the Standard Model with respect to the action of the gauge group $SU(3)_c \times SU(2)_w \times U(1)_Q$.

[12] Superpartners of bosonic particles are named using the boson's name with an attached *-ino* suffix, such as *photino*, *gluino* and *higgsino*. The superpartners of fermionic particles are named by attaching an *s-* prefix to the fermion's name, such as *selectron*, *sneutrino* and *squark*.

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Also, it turns out that the details of the fermion mass hierarchy require introducing not one but two chiral superfields for each Higgs field in the Standard Model, and it follows:

> ***Conclusion 10.5*** *The so-called Minimally Supersymmetric Standard Model (MSSM) requires a little over twice as many particles as the Standard Model.*

Considering this simple counting of degrees of freedom used to describe Nature, the reason for supersymmetrizing the Standard Model certainly is not economy. However, recall the conceptual and practical (technical) consequences of supersymmetry [☞ Section 10.3.1]:

1. vacuum stabilization,
2. mass hierarchy stabilization, and
3. simplification of the renormalization procedure.

Note also the fact that before the invention of supersymmetry, which successfully solves these problems of the Standard Model, these problems were hardly mentioned. Of course, that owes partly to the approach of describing Nature pragmatically and axiomatically:

> ***Comment 10.5*** *Theoretical models are constructed with the aim to describe, in a logically coherent and consistent theoretical system, the known phenomena without predicting nonexistent phenomena, and while keeping the necessary assumptions as few as possible.*
>
> *These assumptions (axioms) are re-examined only when the resulting theoretical system "paints" the development of the model "into a corner" and when within this theoretical system it is not possible to construct a model that does not err in a concrete aspect of the description of Nature, or when an opportunity emerges to explain it in a conceptually more fundamental or practically simpler system of assumptions.*

Of course, the question remains: In what measure is supersymmetry of help in models of quantum physics that contain the general theory of relativity? Considering that the complete theory of quantum gravity does not exist yet, a final answer to this question then does not exist either☝. However, the next chapter will permit us to say a little more about this.

10.3.4 Exercises for Section 10.3

✎ ***10.3.1*** *Prove that the left-hand side of the relation (10.55) is non-negative.*

✎ ***10.3.2*** *Show that the operators (10.67) and (10.68) satisfy the operatorial relations (10.69) and (10.70).*

✎ ***10.3.3*** *Confirm the relations (10.91).*

✎ ***10.3.4*** *Confirm the result (10.95).*

✎ ***10.3.5*** *By iterative and consistent use of relations (10.69) and definitions (10.88) (a.k.a. "D-gymnastics"), derive equation (10.96) from (10.94).*

✎ ***10.3.6*** *Return the proper factors of* $\hbar$ *and* c *in the expressions (10.96)–(10.98).*

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10.4 Classification of off-shell supermultiplets

Recall Procedure 5.1 on p. 193, which is generally accepted as the only systematic procedure applicable in all known models of quantum field theory [☞ also Procedure 11.1 on p. 416]. For the purposes of this procedure we must define an integral of the general type

$$\int \mathbf{D}[\,\phi\,]\, e^{iS[\phi]/\hbar} \tag{10.112}$$

to be computed over all possible fields, here represented by the symbol ϕ. $S[\phi]$ is the classical Hamilton action, which according to Hamilton's principle has a minimum for the choice of fields ϕ that satisfy the classical (Euler–Lagrange) equations of motion, i.e., for fields ϕ that are *on shell*. For such *classical* fields, $S[\phi]$ is minimal, and the integrand $\exp\{iS[\phi]/\hbar\}$ oscillates minimally. By contrast, a choice of the fields that are "far" from such classical fields then causes the integrand to oscillate very fast, so that the contributions mostly cancel. The naive reasoning then is that the contributions of the classical fields dominate the formal integral (10.112). This is in no way proven rigorously, as the space of the choices of the fields ϕ is infinite-dimensional: although the contribution to the integral (10.112) from any one non-classical field is infinitesimally small, there are continuously many such fields and the sum over them may well even diverge.

However, we certainly know that the fields over which the integration (10.112) is to be performed must a priori be *off-shell*, i.e., not subject to any differential equation, and foremost not the classical (Euler–Lagrange) equations of motion: that would be outright contradictory. *Quantum* supersymmetric models then must be constructed using *off-shell* supermultiplets (collections of particles and their superpartners); in models of supersymmetric quantum field theory, both the known particles and all their superpartners must be represented by *off-shell* fields.

With that in mind it is then surprising that four decades after the introduction of supersymmetry in field theory there is still no complete theory of *off-shell* representation of supersymmetry algebras☝. Recent research in this direction [139, 140, 141, 142] indicates a fantastic and combinatorially complex multitude of possibilities, very different from the well-known theory of the finite-dimensional unitary representations of Lie algebras, and even the supersymmetric *on-shell* representations, which are well known.

The remainder of this section gives a telegraphic description of this research, mostly so as to indicate some open possibilities for research. However, this introduction is restricted to *intact* supermultiplets[13] – those that have not been constrained or gauged in any way. Constrained and gauged (gauge-equivalences of) supermultiplets are indeed very widely used, and the interested Reader is directed to the textbooks [189, 562, 560, 76].

10.4.1 *One-dimensional supersymmetry as the common denominator*

Recall the three levels of theoretical analysis of physical systems [☞ description on p. 366] where supersymmetry may show up, and especially the second and third levels of analysis, where supersymmetry reduces to supersymmetric quantum mechanics, with the algebra

$$\{\,Q_I\,,\,Q_J\,\} = 2\delta_{IJ}H, \qquad [\,H\,,\,Q_I\,] = 0, \qquad I,J = 1,\dots,N, \tag{10.113}$$

where $H = i\hbar\partial_\tau$ is the Hamiltonian and τ the proper time, and where in the general case one does not require N to be even as in the relations (10.32). Note that we revert to the quantum-mechanical normalization, $[Q_I] = \frac{\sqrt{M}L}{T}$.

[13] The adjective "intact" is simply shorter than the detailed "unconstrained and ungauged."

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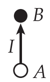
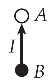
Following the lesson from the conclusion in Digression 10.9 on p. 383, or even just simply the pragmatic application of supersymmetry as a transformation that maps bosons into fermions and back, the goal of classifying representations of supersymmetry is to classify all possible supermultiplets, i.e., collections of bosons and fermions that supersymmetry maps one into the other. To this end, it is convenient to introduce a graphical notation as described in Table 10.1.[14]

Table 10.1 The correspondence between Adinkras and supersymmetry transformations gives: node ↔ component field; white/black node ↔ boson/fermion; Ith color/index edge ↔ Q_I; dashed edge ↔ − sign; edge direction ↔ 1 (∂_τ when following an edge in the opposite direction). In addition, the Adinkras are drawn putting the nodes at levels proportional to their relative units, so the implicit edge directions are upward.

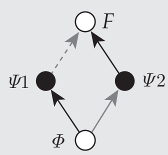
Adinkra	Supersymmetry transf.	Adinkra	Supersymmetry transf.
	$Q_I \begin{bmatrix}\psi_B\\ \phi_A\end{bmatrix} = \begin{bmatrix} i\dot{\phi}_A\\ \psi_B\end{bmatrix}$		$Q_I \begin{bmatrix}\psi_B\\ \phi_A\end{bmatrix} = \begin{bmatrix} -i\dot{\phi}_A\\ -\psi_B\end{bmatrix}$
	$Q_I \begin{bmatrix}\phi_A\\ \psi_B\end{bmatrix} = \begin{bmatrix} \dot{\psi}_B\\ i\phi_A\end{bmatrix}$		$Q_I \begin{bmatrix}\phi_A\\ \psi_B\end{bmatrix} = \begin{bmatrix} -\dot{\psi}_B\\ -i\phi_A\end{bmatrix}$

Edges are here labeled by the index I; for a fixed I, they are drawn in the Ith color.

The next two examples of supermultiplets of $N = 2$ supersymmetric quantum mechanics should clarify the application of the rules in Table 10.1.

Example 10.2 The simplest supermultiplet is of the general form that reflects the basis of the type (10.78):

$$Q_1\,\phi = \psi_1, \qquad Q_2\,\phi = \psi_2, \tag{10.114a}$$
$$Q_1\,\psi_1 = i\dot{\phi}, \qquad Q_2\,\psi_1 = -iF, \tag{10.114b}$$
$$Q_1\,\psi_2 = iF, \qquad Q_2\,\psi_2 = i\dot{\phi}, \tag{10.114c}$$
$$Q_1\,F = \dot{\psi}_2, \qquad Q_2\,F = -\dot{\psi}_1. \tag{10.114d}$$



The black edges depict the action of the Q_1 supercharge and the gray edges the Q_2-action. The fact that in a two-colored quadrangle (10.114) an odd number of edges must be dashed (i.e., the corresponding supercharge action has an additional −1 sign) follows from the fact that

$$\left.\begin{array}{ll} \phi \xrightarrow{Q_1} \psi_1 \xrightarrow{-Q_2} F: & F = -Q_2\big(Q_1(\phi)\big),\\ \phi \xrightarrow{Q_2} \psi_1 \xrightarrow{Q_1} F: & F = Q_1\big(Q_2(\phi)\big), \end{array}\right\} \quad\Rightarrow\quad Q_1 Q_2 = -Q_2 Q_1, \tag{10.115}$$

in agreement with equation (10.31).

[14] A graphical representation of a system of equations offers the evident advantage of heuristic insight and is not at all a new idea [177]; the formalization of such graphs – called *Adinkras* – for the purposes of supersymmetry, however, is of recent origin [139]. They are particularly useful in depicting intact supermultiplets.

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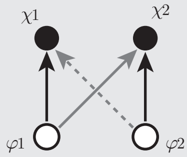
Example 10.3 Another example of an $N = 2$ supermultiplet is

$$\begin{aligned} \mathcal{Q}_1\,\varphi_1 &= \chi_1, & \mathcal{Q}_2\,\varphi_1 &= \chi_2, && (10.116a)\\ \mathcal{Q}_1\,\varphi_2 &= \chi_2, & \mathcal{Q}_2\,\varphi_2 &= -\chi_1, && (10.116b)\\ \mathcal{Q}_1\,\chi_1 &= i\dot{\varphi}_1, & \mathcal{Q}_2\,\chi_1 &= -i\dot{\varphi}_2, && (10.116c)\\ \mathcal{Q}_1\,\chi_2 &= i\dot{\varphi}_2, & \mathcal{Q}_2\,\chi_2 &= i\dot{\varphi}_1. && (10.116d) \end{aligned}$$



Formally, equating $(\phi;\psi_1\psi_2;F) = (\varphi_1;\chi_1,\chi_2;\dot{\varphi}_2)$ identifies the two supermultiplets, but this implies the relation $F = \dot{\varphi}_2$ and so $\varphi_2 = \int \mathrm{d}\tau\, F$, which is evidently non-local. The two supermultiplets, (10.114) and (10.116), thus cannot be considered equivalent off-shell supermultiplets.

Both examples, 10.2 and 10.3, depict supermultiplets that consist of two bosons and two fermions. The difference is indicated by the fact that in Example 10.2 $[F] = [\phi]\cdot\frac{ML^2}{T^2}$, whereas in Example 10.3 $[\varphi_1] = [\varphi_2]$; see Table C.5 on p. 528. It is then evident that the supersymmetric Lagrangian of the form

$$\mathscr{L}_2 := \tfrac{1}{2}\mu\big[(\dot{\varphi}_1)^2 + (\dot{\varphi}_2)^2 + \tfrac{2i}{\hbar}(\chi_1\dot{\chi}_2 - \dot{\chi}_1\chi_2)\big], \tag{10.117}$$

with an appropriate characteristic constant μ, produces the familiar equations of motion: second order in time derivatives for the bosons φ_1, φ_2 and first order for fermions χ_1, χ_2. By contrast, the analogous supersymmetric Lagrangian

$$\mathscr{L}_1 := \tfrac{1}{2}\mu\big[(\dot{\phi})^2 + \tfrac{1}{\hbar^2}F^2 + \tfrac{2i}{\hbar}(\psi_1\dot{\psi}_2 - \dot{\psi}_1\psi_2)\big] \tag{10.118}$$

produces the usual equations of motion for the boson ϕ and the fermions ψ_1, ψ_2, but an algebraic equation for the boson F [☞ step 4(b)iii on p. 374, as well as the equations of motion (10.97)].

This *dynamical* information is thus encoded by the "*height arrangement*" of the nodes in the Adinkra, which defines the relative physical units of the component fields in the depicted supermultiplet.

Digression 10.12 The formal difference between the supermultiplets (10.114) and (10.116) is seen by analyzing the identifications

$$\begin{aligned} (\phi;\psi_1,\psi_2;F) &\ \overset{=}{\longrightarrow}\ (\varphi_1;\chi_1,\chi_2;(\dot{\varphi}_2)),\\ (\phi,(\textstyle\int \mathrm{d}\tau\, F);\psi_1,\psi_2) &\ \overset{=}{\longleftarrow}\ (\varphi_1,\varphi_2;\chi_1,\chi_2). \end{aligned} \tag{10.119}$$

This gives a formal bijection between the two supermultiplets. However, since $\partial_\tau : \varphi_2 \mapsto (\dot{\varphi}_2)$ annihilates the constant term in a power expansion of the function $\varphi_2(\tau)$ and $\partial_\tau^{-1} : F \mapsto (\int \mathrm{d}\tau\, F)$ adds an arbitrary (integration) constant, this formal bijection is not a perfect 1–1 mapping in both ways, and the supermultiplets (10.116) and (10.114) must be considered different.

Digression 10.13 For the *Lagrangians* (not Lagrangian densities!) $\mathscr{L}_1$ and $\mathscr{L}_2$ to have the units of energy and μ to be identifiable as a mass, $[\phi] = [\varphi_i] = L$, $[\psi_i] = [\chi_i] = \frac{\sqrt{ML^2}}{T}$ and $[F] = \frac{ML^3}{T^2}$.

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Supermultiplets that can be depicted with Adinkras (graphs that are constructed based on the rules in Table 10.1 on p. 390 [☞ article [139] for the appropriate theorems and details]) have the property that the supersymmetric mapping from bosonic to fermionic component fields and back may also be represented by a system of superdifferential relations:

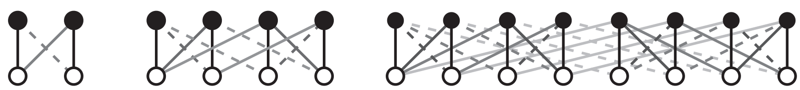
$$D_I\boldsymbol{\Phi}_i = (L_I)_i{}^{\alpha}\boldsymbol{\Psi}_\alpha, \qquad D_I\boldsymbol{\Psi}_\alpha = i\hbar(R_I)_\alpha{}^{i}\dot{\boldsymbol{\Phi}}_i, \tag{10.120}$$

where the index I counts supercharges, i the bosonic superfields $\boldsymbol{\Phi}_i$, α the fermionic superfields $\boldsymbol{\Psi}_\alpha$, chosen so that:

1. component fields $\phi_i = \boldsymbol{\Phi}_i|$ and $\psi_\alpha = \boldsymbol{\Psi}_\alpha|$ (up to a ∂_τ- or ∂_τ^{-1}-prefactor as needed) are the complete system of component fields for the desired supermultiplet, and
2. in every row and every column, the numerical matrices $(L_I)_i{}^{\alpha}$ and $(R_I)_\alpha{}^{i}$ have precisely one nonzero entry, which equals ± 1.

Because of the relations (10.71), the system of superdifferential relations specifies the supersymmetric transformations within the supermultiplet.

Although there exist supermultiplets that do not satisfy these requirements, all worldline off-shell supermultiplets may be constructed starting with such "*adinkraic*" supermultiplets [284, 143]. *Adinkras* for a few such supermultiplets for small N (in the variant where neither ∂_τ- nor ∂_τ^{-1}-prefactors were used) are



etc. (10.121)

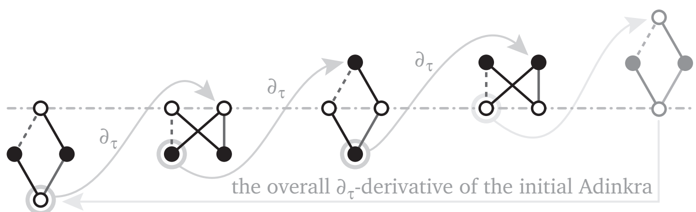
It should now be clear that there exist a combinatorially (hyper-exponentially) growing number of different node-height arrangements in Adinkras with growing N. Every new node-height arrangement corresponds to a new application of ∂_τ- and ∂_τ^{-1}-prefactors, which then specifies a new supermultiplet, which in turn results in a number of different supermultiplets that grows combinatorially with a growing N.

In turn, the matrices $\mathbb{L}_I$ and $\mathbb{R}_I$ in the equations (10.120) satisfy the relations

$$(L_I)_i{}^{\alpha}\,(R_J)_\alpha{}^{k} + (L_J)_i{}^{\alpha}\,(R_I)_\alpha{}^{k} = 2\delta_{IJ}\,\delta_i^k, \tag{10.122}$$

$$(R_I)_\alpha{}^{j}\,(L_J)_j{}^{\beta} + (R_J)_\alpha{}^{j}\,(L_I)_j{}^{\beta} = 2\delta_{IJ}\,\delta_\alpha^\beta, \tag{10.123}$$

which define a double cover of the Clifford algebre $\mathfrak{Cl}(0,N)$.[15] In the original articles [195, 197, 196, 198, 199, 194, 193] the algebra (10.123) was denoted $\mathcal{GR}(d,N)$, where it is assumed that, as needed, the superfields $\boldsymbol{\Phi}_i, \boldsymbol{\Psi}_\alpha$ may be replaced by their ∂_τ-derivatives. Indeed, this formal ∂_τ-mapping connects all supermultiplets with the same "chromo-topology" [139]. For the relatively simple case of quantum-mechanical $N = 2$ supersymmetry, iterations of such ∂_τ-mapping yield the cyclic sequence



(10.124)

[15] The double-covered Clifford algebra is obtained by identifying $\mathbb{L}_I, \mathbb{R}_I \xmapsto{2\text{–}1} e_I$.

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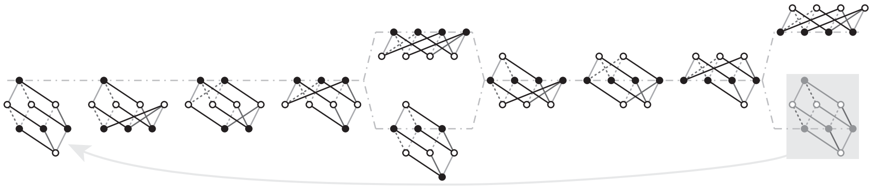
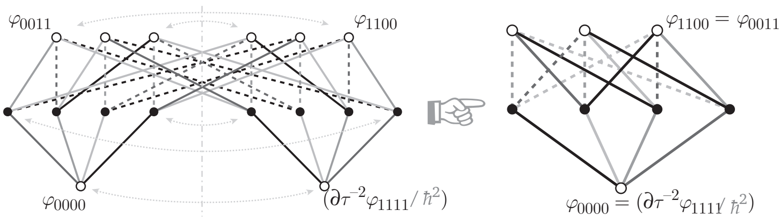
For quantum-mechanical $N = 3$ supersymmetry, the analogous cyclic sequence is



(10.125)

where the gray-highlighted Adinkra (bottom, right) is identical (up to the overall level, indicated by the gray dot-n-dash line) to the initial one, at far left, thus repeating the cycle. These illustrations show that for the (even just adinkraic) finite-dimensional representations of quantum-mechanical N-extended supersymmetries the number of possible node-height arrangements – and so the number of different supermultiplets – grows combinatorially with the growing number of supersymmetries, N.

In addition, starting with $N = 4$, there emerges a new possibility – "projections" – of which more in the next section. May it suffice here to show but one example:



(10.126)

The dashed double-ended arrows indicate some of the pairs of component fields in the left-hand supermultiplet that are identified so as to obtain the component fields of the right-hand supermultiplet. The naming convention of the labeled component fields is explained in the next section.

It has been proven that the number of such "*adinkraic*" off-shell supermultiplets grows fantastically fast with the number of supersymmetries, and one expects about 10^{47} distinct supermultiplets for $N \leqslant 32$, which are expected to form about 10^{12} equivalence classes [141, 142]. Finally, it has been shown that an infinite number of ever larger (and non-adinkraic) supermultiplets can be constructed as networks of adinkraic supermultiplets, connected by one-way supersymmetry transformations [284]; this is also the structure of some rather well-known supermultiplets of simple supersymmetry in 4-dimensional spacetime [190].

— ❦ —

For such a (worldline) supermultiplet to be the 1-dimensional "shadow" of a supermultiplet from a 4-dimensional supersymmetric field theory, it is necesary that both the component fields and the supercharge action are compatible with Poincaré symmetry in 4-dimensional spacetime. One expects this to be a rather nontrivial requirement [197, 157, 158, 191, 283], which should drastically reduce the number of possible supermultiplets in higher-dimensional spacetime, but this verification (dimensional reconstruction) is far from solved in general; see Refs. [157, 158, 191, 283, 409].

10.4.2 Supermultiplets and binary encryption

It is fascinating that the classification of off-shell quantum-mechanical supermultiplets [140, 142] is closely related to the classification of doubly even binary linear block codes, which may be used in error-detecting and error-correcting encryption [286].

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That is, in the quantum-mechanical N-extended supersymmetry (10.113) we have N real supercharges $\mathcal{Q}_I$, so that a supermultiplet may be identified – up to an application of ∂_τ- and/or ∂_τ^{-1}-prefactors – with a complete iterative application of all $\mathcal{Q}_I$'s upon some starting component field. The supermultiplet in Example 10.2 on p. 390 may be reconstructed also as

$$\{\,\phi,\;\; \psi_1 := \mathcal{Q}_1(\phi),\;\; \psi_2 := \mathcal{Q}_2(\phi),\;\; F := \mathcal{Q}_1\big(\mathcal{Q}_2(\phi)\big)\,\}, \tag{10.127}$$

and the supermultiplet in Example 10.3 on p. 391 as

$$\{\,\varphi_1,\;\; \chi_1 := \mathcal{Q}_1(\varphi_1),\;\; \chi_2 := \mathcal{Q}_2(\varphi_1),\;\; \varphi_2 := \partial_\tau^{-1}\big(\mathcal{Q}_1(\mathcal{Q}_2(\varphi_1))\big)\,\}. \tag{10.128}$$

As the defining relations of the supersymmetry algebra (10.113) imply that

$$\mathcal{Q}_I\mathcal{Q}_J = -\mathcal{Q}_J\mathcal{Q}_I, \qquad I \neq J, \tag{10.129a}$$

$$\big(\mathcal{Q}_I\big)^2 = H, \qquad I = 1,\dots,N, \tag{10.129b}$$

it follows that every formal $\mathcal{Q}$-monomial can be expressed as a linear combination of H-multiples of lexicographically ordered monomials from the basis

$$\Big\{\,\boldsymbol{\mathcal{Q}}^{\boldsymbol{b}} := \mathcal{Q}_1^{b_1}\mathcal{Q}_2^{b_2}\cdots\mathcal{Q}_N^{b_N},\;\;\; b_I = 0,1,\;\; I = 1,\dots,N\,\Big\}. \tag{10.130}$$

Evidently, there are $\sum_{k=0}^{N}\binom{N}{k} = 2^N$ so-ordered $\mathcal{Q}$-monomials and they are *unambiguously* encoded by the binary exponents b_I, which may be concatenated into a binary number of a formal binary exponent $\boldsymbol{b}$. Following the examples (10.127) and (10.128), we define

$$\begin{Bmatrix}\phi_{\boldsymbol{b}}\\ \psi_{\boldsymbol{b}}\end{Bmatrix} := \boldsymbol{\mathcal{Q}}^{\boldsymbol{b}}(\phi_{00\cdots}),\;\; \text{when } |\boldsymbol{b}| := \sum_{I=1}^{N} b_I \;\text{ is }\; \begin{cases}\text{even},\\ \text{odd}.\end{cases} \tag{10.131}$$

The field identification in the relation between the two Adinkras (10.126) requires the imposition of the operatorial conditions[16]

$$\mathcal{Q}_1\mathcal{Q}_2 \simeq +\mathcal{Q}_3\mathcal{Q}_4, \quad \mathcal{Q}_1\mathcal{Q}_3 \simeq -\mathcal{Q}_2\mathcal{Q}_4, \quad \mathcal{Q}_1\mathcal{Q}_4 \simeq +\mathcal{Q}_2\mathcal{Q}_3, \tag{10.132a}$$

in addition to the relations (10.113), i.e., (10.129). Indeed, acting (always only from the right!) by the operators $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3$ and $\mathcal{Q}_4$ on the relations (10.132a) produces

$$H\mathcal{Q}_1 \simeq +\mathcal{Q}_2\mathcal{Q}_3\mathcal{Q}_4, \quad H\mathcal{Q}_2 \simeq -\mathcal{Q}_1\mathcal{Q}_3\mathcal{Q}_4, \quad H\mathcal{Q}_3 \simeq +\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_4, \quad H\mathcal{Q}_4 \simeq -\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3, \tag{10.132b}$$

and then, finally, also

$$(H^2 = -\hbar^2\partial_\tau^2) \simeq -\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3\mathcal{Q}_4. \tag{10.132c}$$

This last relation corresponds to the identification of the component fields:

$$\Big(H^2\phi_{0000} = -\hbar^2(\partial_\tau^2\phi_{0000})\Big) = \Big(-\mathcal{Q}_1\mathcal{Q}_2\mathcal{Q}_3\mathcal{Q}_4(\phi_{0000}) =: -\phi_{1111}\Big). \tag{10.133}$$

Similarly, other relations (10.132) encode all other identifications (10.126), and so also the projection of the bigger, left-hand side supermultiplet to the smaller, right-hand side supermultiplet. It is essential to note that the relations (10.132) do not impose any ∂_τ-differential equation upon any of the component fields, and each field – and so the entire supermultiplet – remains off-shell.

[16] By operatorial conditions/relations one implies conditions/relations between two operatorial expressions, and which conditions/relations must hold when the left-hand and the right-hand sides of the equality are applied on any object upon which the operation of the given operators is defined.

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Digression 10.14 Note that – *up to additional H-factors* – each of the eight relations (10.132) may be obtained from any other one. For example,

$$H\mathcal{Q}_2 \simeq -\mathcal{Q}_1\mathcal{Q}_3\mathcal{Q}_4 \quad \xrightarrow{\cdot\mathcal{Q}_3} \quad H\mathcal{Q}_2\mathcal{Q}_3 \simeq -\mathcal{Q}_1\mathcal{Q}_3\mathcal{Q}_4\mathcal{Q}_3 = +H\mathcal{Q}_1\mathcal{Q}_4. \tag{10.134a}$$

In that sense are the three relations (10.132a) "basic" since all other relations (10.132b)–(10.132c) follow with no additional H-factors, whereas the converse does not follow. Jointly, the relations (10.132a) may be written as

$$\mathcal{Q}_I\mathcal{Q}_J - \tfrac{1}{2}\varepsilon_{IJ}{}^{KL}\mathcal{Q}_K\mathcal{Q}_L \simeq 0, \tag{10.134b}$$

which indicates the need for the Levi-Civita symbol ε_{IJKL}, where all four indices have precisely one of the four possible values – corresponding to the binary number $\boldsymbol{b} = 1111$ [☞ Ref. [141] for analogous relations that correspond to other codes].

All the relations (10.132), and so also all the identifications (10.126) are almost unambiguously encoded by the binary number $\boldsymbol{b} = 1111$,[17] which generates a so-called "binary doubly even linear block code" d_4 [286], which is also the simplest such code. These codes are used in binary encryption that helps in communications by enabling the detection of transmission errors and even some corrections, and without re-transmitting the original message. Once projected, as in the example (10.126), the smaller supermultiplet may be connected with various "node-height rearrangements" by applying the formal ∂_τ- and ∂_τ^{-1}-prefactors, which then generates all possible supermultiplets with that chromo-topology [142].

Thus, the classification of off-shell worldline supermultiplets is closely related to the classification of "binary doubly even linear block codes," and gives a close relationship between supersymmetry and encryption – which is a fully unexpected and fascinating result in this research. Numerically, it is even more fascinating that there are at least $\sim 10^{47}$ such codes for $N \leqslant 32$ (which is a limit suggested by the M-theoretic extension of superstrings), and that they form at least $\sim 10^{12}$ equivalence classes; moreover, the number of supermultiplets of which the "chromo-topology" [142] is defined by any one such code itself grows combinatorially with N, which further increases the "menagerie."

The construction and classification of off-shell supermultiplets in higher-dimensional spacetimes starting from the so far discussed worldline off-shell supermultiplets is in progress [157, 158, 191, 283, 409]☝. In addition, other approaches and methods can complement these efforts, even if in more specific setting (such as for a fixed number of supersymmetries, N): see, for example, Refs. [281, 292, 47], to begin with.

10.4.3 Exercises for Section 10.4

✎ ***10.4.1*** *Prove that the Lagrangian terms (10.117) and (10.118) are invariant with respect to the supersymmetry transformations (10.114)–(10.116).*

✎ ***10.4.2*** *Derive and solve the equations of motion defined by the Lagrangian density (10.117).*

✎ ***10.4.3*** *Complete the Lagrangian term* $\mathscr{L}_3 = \omega(\varphi_1\dot{\varphi}_2 - \varphi_2\dot{\varphi}_1) + \cdots$ *so it is invariant with respect to the supersymmetry transformations (10.116).*

✎ ***10.4.4*** *Derive and solve the equations of motion defined by the Lagrangian density* $\mathscr{L}_1 + \mathscr{L}_3$, *as defined in the expression (10.117) and the solutions of Exercise 10.4.3.*

[17] Except for the choice of the relative sign in equation (10.134b), for cases with a total of $N = 4k$ supersymmetries.