ON CERTAIN DIRECT SUM DECOMPOSITIONS OF L^1 SPACES

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1.

Let L^1 denote temporarily the usual Lebesgue space over the circle group (equivalently: the additive group of real numbers modulo 2π), and let H^1 denote the Hardy space comprised of those f in L^1 whose complex Fourier coefficients vanish for all negative frequencies, so that

$$f(x) \sim \sum_{n=0}^{\infty} c_n e^{inx}.$$

D. J. Newman has settled a conjecture by showing [1] that there exists no continuous projection of L^1 onto H^1, i.e. that there exists in L^1 no topological complement to H^1.

In this note we aim to present other instances of this and related types of phenomena arising in connection with fairly general orthogonal families of functions and the corresponding generalised Fourier expansions.

Whilst Newman's proof depends ultimately on the special properties of Fourier series of power-series type and their relationship with functions of a complex variable, the instances we have in mind hinge upon the special properties of lacunary series. In functional analytic terms one may say that the non-existence of topological complements in the cases we consider reflects the fact that, whilst every closed vector subspace of L^1 inherits the latter's property of being weakly sequentially complete, this property is in general forfeited when one passes from L^1 to a quotient space thereof.

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2. Generalities and assumptions

The circle group figuring in the background of Newman's result will be replaced initially by an arbitrary locally compact space X, and Lebesgue measure by a fixed positive Radon measure μ on X. Later on the situation will be specialised to that in which X is a compact group and μ its Haar

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measure. For brevity we write L^p for $L^p(X, \mu)$, the usual Lebesgue space relative to X and μ, and we discard any notational distinction between a function on X and its class modulo the relation of equality a.e. (or locally a.e.) for μ. In place of the complex exponentials we assume given a fairly general orthogonal family $(u_t)_{t \in T}$ of functions on X. For our first theorem we shall require two hypotheses which we proceed to display in detail.

(I) *Each u_t is non-negligible, bounded, continuous, and integrable (and therefore belongs to every L^p), the family (u_t) is orthogonal, and* $\lim_t u_t = 0$ *weakly in L^∞.*

The final clause in (I) means that, given f in L^1 and any $\varepsilon > 0$, the inequality $|\int f\bar{u}_t d\mu| < \varepsilon$ is satisfied for all t in T save perhaps those belonging to a finite subset of T (which may depend on f and on ε).

In view of (I) the generalised Fourier coefficients

$$\hat{f}(t) = \int f\bar{u}_t d\mu$$

are well defined for any function f lying in at least one L^p space and for all t in T.

The second condition makes reference to a variable subset Q of T.

(II_Q) *Q is an infinite subset of T, and there exists in L^1 a sequence (f_n) for which* $\lim_n \int f_n \bar{g} d\mu$ *exists finitely for each bounded, continuous function g on X, and for which furthermore*

$$\limsup_{t \in Q, t \to \infty} |\lim_n \hat{f}_n(t)| > 0.$$

The subspaces of L^1 to be considered are those of the type $L^1[P]$, where $L^p[P]$ denotes, for any subset P of T, the set of f in L^p for which $\hat{f}$ vanishes at all points of T/P.

In Theorem 1 the set Q appearing in (II_Q) will belong to a very special type which we shall term *Szidon subsets* of T. A subset Q of T is termed a Szidon subset (relative to the family (u_t)) if each g in $L^\infty[Q]$ is equal locally a.e. to a bounded, continuous function on X. The terminology is, of course, suggested by a weakened form of a theorem of Szidon about the behaviour of bounded functions on the circle group whose complex Fourier series are lacunary; see, for example, [2], p. 247, Theorem (6.1).

3.

It is now possible to state and prove an analogue of Newman's result.

THEOREM 1. *The notations and assumptions being as in § 2 above, suppose that Q is an infinite Szidon subset of T and that* (I) *and* (II_Q) *are satisfied. Then, putting $P = T\backslash Q$, it is the case that $L^1[P]$ admits no topological complement in L^1, i.e. there exists no continuous projection of L^1 onto $L^1[P]$.*

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REMARK. The proof will make it plain that it is sufficient if the third clause of (I) is satisfied when t is restricted to Q.

PROOF. This proceeds by contradiction. Suppose, if possible, that L^1 is the topological direct sum of $L^1[P]$ and a vector subspace M of L^1. M is necessarily closed, and M and $L^1/L^1[P]$ are isomorphic Banach spaces. Since L^1, and therefore also its closed vector subspace M, is weakly sequentially complete, the same is true of $L^1/L^1[P]$. We show that this, together with (II_Q) and the assumption that Q is a Szidon subset of T, leads to conflict with (I).

In any case one may identify the dual of L^1 with L^∞ via the bilinear form

$$\langle f, g\rangle = \int f\bar{g}d\mu.$$

This having been done, the dual of $L^1/L^1[P]$ may be identified with the polar in L^∞ of $L^1[P]$. Thanks to orthogonality, this polar is seen to be contained in $L^\infty[Q]$ and to contain each u_t with t in Q.

Consider now the sequence (f_n) figuring in (II_Q). Since Q is a Szidon set, it appears from (II_Q) that the classes f_n^* modulo $L^1[P]$ form a weak Cauchy sequence in the quotient space $L^1/L^1[P]$, so that there exists an f in L^1 for which f^* is the weak limit of the classes f_n^*. This entails that $\lim_n \hat{f}_n(t) = \hat{f}(t)$ for each t in Q, and so (II_Q) yields the relation

$$\limsup_{t\in Q,\, t\to\infty} |\hat{f}(t)| > 0.$$

Q being infinite, this constitutes a contradiction of (I) (and even of (I) weakened as indicated in the remark preceding this proof).

4. Example

Consider the case in which X is a compact group and μ its Haar measure. It will be supposed that X is infinite and μ normalised by the condition $\mu(X) = 1$. (If X were finite, μ would be normalised so that each one-point set has unit measure. But this case is of no interest here.) An orthogonal system (u_t) results if one starts with unitary representations of X. More precisely, let Σ be a set of mutually inequivalent, continuous, irreducible, unitary representations U of X. Write $d(U)$ for the degree of the representation U. T will consist of triplets $t = (U, i, j)$, where $U \in \Sigma$ and $1 \leqq i$, $j \leqq d(U)$ and we define

$$u_t = U_{ij} \quad \text{for} \quad t = (U, i, j),$$

U_{ij} denoting the (i, j)-th entry in the matrix U. See [3], Chapters IV and V.

Condition (I) is satisfied in view of the Bessel inequality for orthogonal systems, together with the fact that the u_t are in this case uniformly bounded.

In order to satisfy (II_Q) we assume that X satisfies the first countability axiom and define the functions

$$f_n = \chi_{V_n}/\mu(V_n),$$

where (V_n) is a countable neighbourhood base at the neutral element e of X, formed of integrable sets, and χ_V denotes the characteristic function of V. Then

$$|\lim_n \hat{f}_n(t)| = |\overline{U_{ij}(e)}| = \delta_{ij}$$

whenever $t = (U, i, j)$. It appears therefore that

$$\lim\sup_{t\in Q, t\to\infty} |\lim_n \hat{f}_n(t)| = 1$$

provided Q contains infinitely many "diagonal elements", i.e. elements of the type (U, i, i) with $U \in \Sigma$ and $1 \leqq i \leqq d(U)$. If X is commutative this will be the case for any infinite Q, $d(U)$ being 1 for each U.

The case of ordinary Fourier series arises when X is the circle group, T is a subset of the set Z of all integers, and

$$u_t(x) = e^{itx}.$$

The values of

$$\hat{f}(t) = \int f(x) e^{-itx} d\mu(x) = (\tfrac{1}{2}\pi) \int_{-\pi}^{\pi} f(x) e^{-itx} dx$$

are just the ordinary Fourier coefficients of f. In this case the first examples of what we have termed Szidon sets appear when we take $T = Z$ and Q to be of the form $\{\pm n_k : k = 1, 2, \cdots\}$, where (n_k) is a sequence of natural numbers exhibiting Hadamard gaps, i.e.

$$\mathrm{Inf}_k\, n_{k+1}/n_k > 1.$$

For this example see Zygmund [2′], p. 220. Hewitt and Zuckerman ([4], Theorems 2.1, 3.2, 4.1 and 5.1) produce further examples. See also the discussion in Rudin [5], Section II, and [6], Section 5.7.

Examples for general compact X are given in [4], §§ 8, 9.

It should be noted that the conditions given in all these references ensure that Q is a Szidon set in a sense which is stronger than ours whenever $(u_t)_{t\in T}$ is constructed in the way described in this section.

5.

A problem closely related to that considered above concerns the possibility of decomposing $L^1[T]$ into the direct sum of $L^1[P]$ and $L^1[Q]$ when P and Q are complementary subsets of T. If, as we have supposed, each u_t is bounded, each of the subspaces just mentioned is closed in L^1. Consequent-

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ly the direct-sum decomposition, if valid in the algebraic sense, is valid also from the topological point of view.

At the expense of two further assumptions about the family (u_t) we shall in this section derive some conditions which are necessary in order that the decomposition be possible. The next section will indicate some instances in which the same conditions are also sufficient.

The first additional assumption is as follows.

(I′) *Each u_t belongs to the space $C_0(X)$ of continuous numerically-valued functions on X which tend to zero at infinity.*

This condition is automatically fulfilled when (I) holds and X is compact.

The second additional condition will in many cases appear as a corollary to the existence of a suitable summability method applying to expansions in terms of the u_t, but it is unnecessary to assume so much.

(III) *There exists a directed family (c_k) of numerical functions on T, each having a finite support, such that $\lim_k c_k = 1$ pointwise on T, and such that for each f in $L^1[T]$ the family of sums $\sum_{t\in T} c_k(t)\hat{f}(t)u_t/||u_t||_2^2$ is weakly bounded in L^1 when k varies.*

One would speak of a summability method in any case in which the sums $\sum_{t\in T} c_k(t)\hat{f}(t)u_t/||u_t||_2^2$ converge weakly in L^1 to f. For our purposes it is pointless to assume that this is the case.

THEOREM 2. *Assume that conditions* (I), (I′), *and* (III) *are satisfied and that P and Q are complementary subsets of T. In order that $L^1[T]$ be the direct sum of $L^1[P]$ and $L^1[Q]$, it is necessary that the following two conditions be fulfilled.*

(IV_P) *If ξ is a function on P, which is the pointwise limit on P of a sequence $(\hat{f}_n|P)$, the sequence (f_n) being bounded in $L^1[T]$, then there exists a bounded Radon measure λ on X such that $\hat{\lambda}(t) = \int \bar{u}_t d\lambda$ coincides on P with ξ and is zero on Q.*

(IV_Q) *As* (IV_P), *but with P and Q interchanged.*

PROOF. Assuming that the direct-sum decomposition obtains, let π be the projection of $L^1[T]$ onto $L^1[P]$ and parallel to $L^1[Q]$. π is continuous and $\pi(u_t)$ is u_t if t belongs to P and is 0 if t belongs to Q. Continuity of π combines with condition (III) to show that the sums

$$\pi_k(f) = \sum_{t\in P} c_k(t)\hat{f}(t)u_t/||u_t||_2^2$$

are weakly bounded in L^1 for each f in $L^1[T]$.

Since it is evident that each π_k is continuous from $L^1[T]$ into L^1 (actually into $L^1[P]$), the weak boundedness of the π_k at each point of $L^1[T]$ entails that they are equicontinuous. In other words, there exists a number c such that

$$||\sum_{t\in P} c_k(t)\hat{f}(t)u_t/||u_t||_2^2||_1 \leqq c \cdot ||f||_1$$

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for each k and each f in $L^1[T]$, c being independent of k.

Applying this last inequality to the sequence (f_n) specified in the hypothesis of (IV_P), and then letting n tend to infinity, it appears that

$$||\Sigma_{t \in P} c_k(t)\xi(t)u_t/||u_t||_2^2||_1 \leqq c',$$

where c' is independent of k. Putting

$$h_k = \sum_{t \in P} c_k(t)\xi(t)u_t/||u_t||_2^2,$$

it follows that (h_k) has a limiting point λ relative to the weak topology on the dual of $C_0(X)$. Thus λ is a bounded Radon measure on X and, for each t in T, $\hat{\lambda}(t) = \int \bar{u}_t d\lambda$ is a limiting point of the family $\int h_k \bar{u}_t d\mu = \hat{h}_k(t)$ (indexed by k). This entails that $\hat{\lambda}(t) = \lim_k c_k(t)\xi(t) = \xi(t)$ for t in P, so that (IV_P) is established.

A similar argument (or an appeal to symmetry with respect to P and Q) shows that (IV_Q) also follows from the assumed existence of the direct-sum decomposition.

6. The group case

Let us return to the situation considered in § 4. The hypotheses of Theorem 2 are fulfilled if we take Σ to be a subset of a complete set Σ_0 of mutually inequivalent, continuous, irreducible, unitary representations of X, T being the set of triplets $t = (U, i, j)$ with $U \in \Sigma$ and $1 \leqq i, j \leqq d(U)$.

A suitable choice of the summability factors c_k arises from selecting a countable neighbourhood base (V_k) at the neutral element e of X, and then for each k constructing a continuous, positive, positive definite, central function p_k which vanishes outside V_k and satisfies $\int p_k d\mu = 1$. Regarding the existence of such functions, see [3], pp. 85—86. Since p_k is central and U is irreducible, the matrix Fourier transform $\hat{p}_k(U) = \int p_k(x) \cdot U(x)^* d\mu(x)$ takes the form $c_k(U) \cdot I_U$, $c_k(U)$ being a positive number and I_U denoting the unit $d(U) \times d(U)$ matrix. The Fourier expansion of $p_k * f(x)$ is

$$\sum_{U \in \Sigma_0} d(U)c_k(U) \sum_{i,j=1}^{d(U)} U_{ij}(x) \cdot \int f\bar{U}_{ij} d\mu.$$

By the analogue of Bochner's Theorem about continuous, positive definite functions, $\sum_{U \in \Sigma_0} d(U)^2 c_k(U) < +\infty$. Moreover it is easily verified that the inner sum above is in modulus not greater than $d(U) \cdot ||f||_1$. It follows that the demands of (III) are met, it being evident that $\lim_k p_k * f = f$ in L^1.

At the same time, if one partitions Σ into two subsets Σ' and Σ'' and takes for P the set of all $t = (U, i, j)$ with U in Σ' and $1 \leqq i, j \leqq d(U)$, Q being defined likewise with respect to Σ'', consideration of the sequence $(f_n) = (p_n)$ shows that the function ξ, defined on P by the formula

$$\xi(t) = \delta_{ij} \quad \text{for} \quad t = (U, i, j) \in P,$$

satisfies the hypothesis of (IV_P). As a result Theorem 2 shows that, if $L^1[T]$ is the direct sum of $L^1[P]$ and $L^1[Q]$, then there exists a Radon measure λ on X whose matrix-valued Fourier transform

$$\hat{\lambda}(U) = \int U(x)^* d\lambda(x)$$

takes the value I_U for $U \in \Sigma'$ and the value 0 for $U \in \Sigma''$. The same must also be true when Σ' and Σ'' are interchanged. This adds nothing unforeseen, however, since it is clear that the measure $\nu = \varepsilon - \lambda$ (ε = the Dirac measure at e) satisfies $\hat{\nu}(U) = I_U$ for $U \in \Sigma''$ and $\hat{\nu}(U) = 0$ for $U \in \Sigma'$.

In the present case the converse of Theorem 2 is also true, for the convolution mapping

$$f \to \lambda * f$$

is a continuous projection of $L^1[T]$ onto $L^1[P]$ parallel to $L^1[Q]$, provided λ satisfies the condition stated above.

Since $L^1[T]$, $L^1[P]$ and $L^1[Q]$ respectively depend only upon Σ, Σ' and Σ'', we may make the suggested slight change of notation and formulate the resulting theorem in the following terms.

THEOREM 3. *Let X be an infinite compact group satisfying the first countability axiom, Σ_0 a complete set of mutually inequivalent, continuous, irreducible, unitary representations of X, Σ a subset of Σ_0, and Σ' and Σ'' complementary subsets of Σ. Denote by $L^1[\Sigma]$ the subspace of L^1 formed of those $f \in L^1$ for which*

$$\hat{f}(U) = \int f(x) U(x)^* d\mu(x)$$

is zero for $U \in \Sigma_0 \backslash \Sigma$, and by $L^1[\Sigma']$ and $L^1[\Sigma'']$ the corresponding subspaces associated with Σ' and Σ'' respectively. Then $L^1[\Sigma]$ is the direct sum of $L^1[\Sigma']$ and $L^1[\Sigma'']$ if and only if there exists a Radon measure λ on X for which

$$\hat{\lambda}(U) = \int U(x)^* d\lambda(x)$$

takes the value I_U when $U \in \Sigma'$ and the value 0 when $U \in \Sigma''$. In that case the measure $\nu = \varepsilon - \lambda$ is likewise related to Σ'', and the convolution maps defined by λ and ν are the continuous projections of $L^1[\Sigma]$ onto $L^1[\Sigma']$ and $L^1[\Sigma'']$ associated with the said direct sum decomposition.

Two remarks may be made by way of supplements to Theorem 3.

(a) If Σ satisfies a suitable type of lacunarity condition, the conditions of Theorem 3 will be fulfilled by arbitrary subsets Σ' and Σ'' of Σ. Indeed, for this to be the case it is evidently enough that each bounded function b on T be the restriction to T of the Fourier transform of some Radon measure β on X, i.e. that

$$b(t) = \int \overline{U_{ij}(x)} d\beta(x)$$

for $t = (U, i, j)$, $U \in \Sigma$, $1 \leqq i, j \leqq d(U)$. For examples of such sets Σ, see the references mentioned in § 4 above. When X is the circle group the desired condition on T is in fact fulfilled whenever T is a Szidon set in the more restrictive sense used in [5].

(b) The situation is simplified if, as at the end of § 4, we consider the classical case in which X is the circle group. Here, as whenever X is commutative, each irreducible representation is one-dimensional and can be identified by its character alone. In particular, if X is the circle group, each $U \in \Sigma$ is paired off with an integer t in such a way that

$$U_{ij}(x) = e^{itx} = u_t(x),$$

the sole admissible value for each of i and j being unity. The 1×1 matrices $\hat{f}(U)$ and $\hat{\lambda}(U)$ are accordingly identified with the numbers

$$\hat{f}(t) = \int e^{-itx} f(x) d\mu(x) = (\tfrac{1}{2}\pi) \int_{-\pi}^{\pi} f(x) e^{-itx} dx$$

and

$$\hat{\lambda}(t) = \int e^{-itx} d\lambda(x),$$

the usual Fourier coefficients of f and λ.

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