*Proceedings of the Edinburgh Mathematical Society* (1979), **22**, 233–240 ©

# SEMI-EMBEDDINGS OF BANACH SPACES†

*by* HEINRICH P. LOTZ, N. T. PECK, and HORACIO PORTA
(Received 15th March 1978)

It is a most implausible fact that a one-to-one operator from $c_0$ into a Banach space which maps the unit ball of $c_0$ onto a closed set is necessarily an isomorphism.

In this paper the term *semi-embedding* denotes a one-to-one operator from one Banach space into another, which maps the closed unit ball of the domain onto a closed set. In the first section we study semi-embeddings in conjunction with weak compactness; in the second section we apply our results to the case of semi-embeddings defined on $C(X)$, $X$ compact.

In Propositions 2 and 3 we construct semi-embeddings which are not isomorphisms. The main result of Section 1 (Proposition 4) states that an operator can be factored through a dual space provided that there exists a semi-embedding defined on the codomain such that the composition of the two operators is weakly compact. This implies that the domain of a weakly compact semi-embedding must be a dual space (Corollary 6). In the case when the domain is $C(X)$, the existence of a weakly compact semi-embedding is equivalent to $X$ being hyperstonian and satisfying the countable chain condition.

In Section 2, after some technical lemmas, we strengthen an argument of Pelczynski and Semadeni. Then we give the main result of the paper: a compact space $X$ is scattered if and only if every semi-embedding of $C(X)$ is an isomorphism. This is even true for any equivalent norm on $C(X)$ (Corollary 12). In particular, every semi-embedding of $c_0$ is an isomorphism.

Theorem 11 also allows us to answer a question raised by Kalton and Wilansky (question 6.4 in (**5**)). Indeed, our results show (see Corollary 14) that a compact space $X$ is scattered if and only if all one-to-one Tauberian operators from $C(X)$ into arbitrary Banach spaces are isomorphisms.

Finally, we prove that every semi-embedding of $C[0, 1]$ is an isomorphism on a complemented subspace isomorphic to $C[0, 1]$.

We use standard terminology throughout (see (**1**)). An *operator* is a bounded linear operator from one Banach space into another; an *embedding* of a Banach space in another is an operator which is an isomorphism onto a closed subspace of its codomain. Finally, a compact space is *scattered* if it contains no perfect subsets.

## 1. Semi-embeddings and weak compactness

**Definition 1.** An operator $T$ from a Banach space $E$ into a Banach space $F$ is called a *semi-embedding* if $T$ is one-to-one and maps the closed unit ball of $E$ onto a closed subset of $F$.

†This research was partially supported by N.S.F. grants.

If $T$ is a semi-embedding of $E$ and if $E$ is given a new equivalent norm, $T$ may fail to be a semi-embedding of $E$ in the new norm. For example, let $E = l_\infty$ with the usual supremum norm and define $T: E \to c_0$ by $T(\alpha_i) = (\alpha_i/i)$. It is easy to see that $T$ is a semi-embedding. If $E$ is renormed by $|||(\alpha_i)||| = max\{\|(\alpha_i)\|, 2 \limsup |\alpha_i|\}$, where $\| \|$ is the usual supremum norm, then $||| \ |||$ and $\| \|$ are equivalent. However, let $(\alpha_i)_n$, $n = 1, 2, \ldots$ be the sequence whose $i$-th term is 1 if $i \leq n$, 0 if $i > n$; then $|||(\alpha_i)_n||| = 1$ and $T(\alpha_i)_n \underset{n}{\to} y$, $y$ not in the image under $T$ of the unit ball.

For $\lambda \geq 1$, a $\lambda$ *semi-embedding* of a Banach space $E$ into a Banach space $F$ is a one-to-one operator $T: E \to F$ such that $\overline{T(U)} \subset \lambda T(U)$, where $U$ is the unit ball of $E$. Evidently the property of being a $\lambda$ semi-embedding for some $\lambda$ is preserved when the domain is given an equivalent norm. A number of our results, including the main theorem, remain true or have valid analogues when "semi-embedding" is replaced by "$\lambda$ semi-embedding" in the statements of the results. We comment more on this in the remarks at the end.

**Proposition 2.** *Let E be a Banach space which has a separable infinite-dimensional quotient space. Then there is a Banach space G and a semi-embedding of $E^*$ into G which is not an embedding.*

**Proof.** Let $F$ be a subspace of $E$ such that $E/F$ is separable and infinite-dimensional, and let $\pi$ be the quotient map of $E$ onto $E/F$. Let $(x_i)$ be a sequence of norm one elements of $E$ such that the linear span of $(\pi(x_i))$ is dense in $E/F$. Define $S: l_1 \to E$ by $S((\lambda_i)) = \Sigma_i \lambda_i x_i / 2^i$; then $S$ is a compact operator.

Next, define $T: F \oplus l_1 \to E$ by $T(y, z) = y + S(z)$. Now let $G = F^* \oplus l_\infty$ and let $T^*: E^* \to G$ be the adjoint map. It is easily seen that the range of $T$ is dense in $E$, so $T^*$ is one-to-one. Moreover, the image of the unit ball of $E^*$ under $T^*$ is $w^*$-compact, so $T^*$ is a semi-embedding.

The space $F^\perp$, the annihilator of $F$ in $E^*$, is infinite-dimensional, and the restriction of $T^*$ to $F^\perp$ coincides with the restriction of $S^*$ to $F^\perp$. Since $S^*$ is compact, it follows that $T^*$ is not an embedding. This completes the proof.

Note that if $E$ itself is separable, then we may take $F = \{0\}$ in the above argument. So $T^* = S^*$ is a compact semi-embedding of $E^*$ into $l_\infty$. This observation will be needed in the proof of Proposition 3.

**Corollary.** *Let E be an infinite-dimensional Banach space such that $E^*$ has the Radon-Nikodym property. Then $E^{**}$ admits a semi-embedding which is not an embedding.*

**Proof.** Let $F$ be any separable infinite-dimensional subspace of $E$. By a result of Stegall (9), $F^*$ is separable. Also, $F^*$ is a quotient of $E^*$; hence the proposition applies to $E^*$.

**Proposition 3.** *Let E be a Banach space. If E has an infinite dimensional reflexive subspace, then there exists a semi-embedding of E into some Banach space which is not an embedding.*

https://doi.org/10.1017/S0013091500016394 Published online by Cambridge University Press

**Proof.** Assume that $H$ is an infinite dimensional, separable, reflexive subspace of $E$. According to Proposition 2, there exists a compact semi-embedding $S_0$ of $H$ in $l_\infty$. Let $S: E \to l_\infty$ be an extension of $S_0$. Define now $T: E \to (E/H) \times l_\infty$ by $Tx = (\hat{x}, Sx)$ where $x \to \hat{x}$ is the quotient map $E \to E/H$. Obviously $T$ is one-to-one and, since its restriction to $H$ is compact, $T$ is not an embedding. In order to show that $T$ is a semi-embedding, consider a sequence $\{x_n\}$ in $E$ such that $\|x_n\| \leqq 1$ and $\{Tx_n\}$ is convergent to a limit $(\hat{x}, z) \in (E/H) \times l_\infty$. Since $\hat{x} = \lim \hat{x}_n$ there is a sequence $\{h_n\}$ in $H$ such that $\lim (x_n - x - h_n) = 0$, and therefore $\limsup \|x + h_n\| \leqq 1$. Since the subspace generated by $H$ and $x$ is also reflexive and separable, by passing to a subsequence if necessary, we may assume that $\{x + h_n\}$ has a weak limit $x + h_\infty$ with $h_\infty \in H$ and $\|x + h_\infty\| \leqq 1$. Obviously $(x + h_\infty)^\wedge = \hat{x}$ and $S(x + h_\infty)$ is the weak limit of $S(x + h_n)$. Since $\lim S(x_n - x - h_n) = 0$ we also have $z = \lim Sx_n = S(x + h_\infty)$ and therefore $T(x + h_\infty) = (\hat{x}, z)$.

**Remark.** If $E$ contains isometrically the dual of a separable Banach space, a slight modification of the above proof shows that there is a $\lambda$ semi-embedding of $E$ into some Banach space which is not an embedding. The constant $\lambda$ can be taken to equal 3.

**Proposition 4.** *Let $E$, $F$, and $G$ be Banach spaces, let $T: E \to F$ be a semi-embedding and let $S: G \to E$ be an operator such that $TS$ is weakly compact. Then there exists a factorisation of $S$ through a dual Banach space $G \xrightarrow{S_2} B^* \xrightarrow{S_1} E$ such that $S_1$ is one-to-one, $TS_1$ is weak*-weak continuous, and $\|S_1\|\,\|S_2\| = \|S\|$.*

**Proof.** Denote the unit balls of $E$ and $G$ by $U$ and $W$, respectively, and denote the closure of $TS(W)$ by $A$. Then $A$ is weakly compact and moreover $A \subset \|S\| T(U)$, since $T$ is a semi-embedding. The linear hull $F_A$ of $A$ with $A$ as unit ball is the dual $B^*$ of a Banach space $B$ and the canonical map of $B^*$ into $F$ is weak*-weak continuous (3, Ch. I, p. 104). Now $TS$ considered as a map from $G$ into $F_A$ defines an operator $S_2: G \to B^*$ with $\|S_2\| \leqslant 1$. Since $F_A$ is contained in the range of $T$, we can define $S_1: B^* \to E$ by $S_1 y = T^{-1} y$ for all $y \in F_A$, and clearly $\|S_1\| \leqslant \|S\|$.

Finally, it is obvious that $TS_1$ is the canonical inclusion of $F_A$ in $F$.

**Corollary 5.** *Under the hypotheses of Proposition 4, the operator $S$ has an extension $\tilde{S}: G^{**} \to E$ with $\|\tilde{S}\| = \|S\|$.*

**Proof.** Take $\tilde{S} = S_1 Q^* S_2^{**}$ where $Q$ is the canonical map from $B$ into $B^{**}$.

**Corollary 6.** *Let $T$ be an operator from $E$ into $F$. Then the following are equivalent:*

a) *$T$ is a weakly compact semi-embedding;*

b) *$E$ is a dual space and $T$ is a one-to-one, weak*-weak continuous operator.*

**Proof.** b) $\Rightarrow$ a) is trivial and a) $\Rightarrow$ b) follows from Proposition 4 applied to the case $G = E$, $S$ the identity operator. ($B^*$ is isometric to $E$ since $S_1$ is one-to-one.)

**Corollary 7.** *Let $X$ be a compact space. Then the following are equivalent:*
a) *$X$ is hyperstonian and satisfies the countable chain condition;*
b) *there exists a weakly compact semi-embedding of $C(X)$ in some Banach space.*

**Proof.** Assume that $T: C(X) \to F$ is a weakly compact semi-embedding. It follows from Corollary 6 that $C(X)$ is a dual, that is, $X$ is hyperstonian (2). Now $T^*$ maps the unit ball of $F^*$ onto a weakly compact set $A$ in $C(X)^*$. It follows that there is a positive measure $\mu_0 \in C(X)^*$ such that all $\mu \in A$ are absolutely continuous with respect to $\mu_0$. Since $T$ is one-to-one, the support of $\mu_0$ is $X$. This implies that $X$ satisfies the countable chain condition. Thus b) $\Rightarrow$ a).

Assume now that a) holds; then there exists a strictly positive normal measure $\mu$ on $X$ and $C(X) = L^\infty(\mu)$ (*loc. cit.*). Since the inclusion map $L^\infty(\mu) \to L^p(\mu)$, $1 \leq p < +\infty$ is a weakly compact semi-embedding, b) follows.

## 2. Semi-embeddings of $C(X)$

In order to prove the main result of this section we need several preliminary results:

**Lemma 8.** *Let $E$ and $F$ be Banach spaces and $T: E \to F$ be a one-to-one operator. If $T$ is not an embedding then the restriction of $T$ to some infinite dimensional subspace of $E$ is compact.*

**Proof.** Suppose that $T$ is not an embedding. Then there is $x_1 \in E$ such that $\|x_1\| = 1$ and $\|Tx_1\| \leqq \frac{1}{4}$. Pick $f_1 \in E^*$ such that $\|f_1\| = 1$, $f_1(x_1) = 1$. Since the restriction of $T$ to the kernel of $f_1$ is not an embedding, there is $x_2$ in the kernel of $f_1$ with $\|x_2\| = 1$ and $\|Tx_2\| \leqq 2^{-4}$. Now pick $f_2 \in E^*$ such that $f_2(x_2) = 1$ and $\|f_2\| = 1$. Continuing in this way, we can define by induction sequences $\{x_n\}$ in $E$ and $\{f_n\}$ in $E^*$ such that for all $n$:

i) $\|x_n\| = 1$, $\|f_n\| = 1$ and $f_n(x_n) = 1$;
ii) $f_k(x_n) = 0$ if $1 \leqslant k < n$;
iii) $\|Tx_n\| \leqq 2^{-2n}$.

Suppose now that $\|\Sigma_{k=1}^n \alpha_k x_k\| \leqslant 1$. Applying $f_1$ we conclude that $|\alpha_1| \leqq 1$, hence $\|\Sigma_{k=2}^n \alpha_k x_k\| \leqq 2$. Applying $f_2$, we conclude that $|\alpha_2| \leqq 2$, hence $\|\Sigma_{k=3}^n \alpha_k x_k\| \leqq 4$. Proceeding in this way, we see that $|\alpha_k| \leqq 2^{k-1}$. Thus $\|T(\alpha_k x_k)\| \leqq 2^{-k+1}$. It follows that if $x = \Sigma_{k=N}^M \beta_k x_k$, then $\|Tx\| \leqq 2^{-N}\|x\|$ and so the restriction of $T$ to the closed linear hull of $\{x_n\}$ is compact.

**Lemma 9.** *Every scattered compact space is sequentially compact.*

**Proof.** Let $X$ be a scattered compact space and $\{x_n\}$ a sequence in $X$. Since the set of cluster points of $\{x_n\}$ can not be perfect, it contains an isolated point $x$. It is easy to see that a subsequence of $\{x_n\}$ converges to $x$.

The following result is proved with arguments similar to those of (7, Main Theorem, 4)$\to$5)). The conclusion is stronger in that the subspace obtained is complemented.

https://doi.org/10.1017/S0013091500016394 Published online by Cambridge University Press

**Proposition 10.** *Let $X$ be a sequentially compact space, let $\alpha$ be an ordinal and let $\varphi$ be a continuous map from $X$ onto $[0,\alpha]$. If $G$ is an infinite dimensional subspace of $C(X)$ consisting of functions of the form $f\circ\varphi$ with $f\in C([0,\alpha])$, then $G$ contains a closed subspace isomorphic to $c_0$ and complemented in $C(X)$.*

**Proof.** The Proposition is trivially true for finite $\alpha$. Assume now that $\alpha$ is an ordinal such that the Proposition holds for all ordinals $\beta<\alpha$. If $\alpha=\gamma+1$, $\alpha$ infinite, then $[0,\alpha]$ is homeomorphic to $[0,\gamma]$ so by induction it also holds for $\alpha$. Assume then that $\alpha$ is a limit ordinal.

For $\beta<\alpha$, let $P_\beta$ be the projection $f\to f\chi_\beta$ where $\chi_\beta$ is the characteristic function of $X_\beta=\varphi^{-1}[0,\beta]$. We identify the range of $P_\beta$ with $C(X_\beta)$. Now let $G_0$ be the subspace $G_0=\{f\in G;\ f=0 \text{ on } \varphi^{-1}(\alpha)\}$. Clearly $G_0$ is also infinite dimensional.

We now distinguish two cases:

a) there exists $\beta<\alpha$ such that $P_\beta|G_0$ is an isomorphism. In this case, by the induction hypothesis, $P_\beta(G_0)$ contains a subspace $H_0$ isomorphic to $c_0$ and which is the range of a projection $Q$ in $C(X_\beta)$. Then, with $T=P_\beta|G_0$, $T^{-1}QP_\beta$ is a projection from $C(X)$ onto $T^{-1}H_0$, and we are done.

b) for all $\beta<\alpha$, $P_\beta|G_0$ is not an isomorphism. In this case, a standard argument (see for example (7), p. 216, last paragraph) shows that there exist a sequence $\{f_n\}$ in $G_0$ and ordinals $\beta_1<\beta_2<\beta_3<\cdots<\alpha$ with the following properties:

i) $\|f_n\|=1$ for all $n$;

ii) the functions $g_n=(P_{\beta_{n+1}}-P_{\beta_n})f_n$ have norm one;

iii) $\Sigma\|f_n-g_n\|<2^{-4}$.

Now for each $n$ pick $t_n\in X$ with $|g_n(t_n)|=1$. By passing to a subsequence we may assume that $\{t_n\}$ is convergent to a limit $t_\infty$. Since the functions $g_n$ are disjointly supported, ii) implies that the sequence $\{g_n\}$ is equivalent to the usual basis of $c_0$. Also, the map

$$Pf=\sum\,[f(t_n)-f(t_\infty)]\operatorname{sgn}[g_n(t_n)]g_n$$

is a projection of $C(X)$ onto the closed linear hull of $\{g_n\}$. Using iii) and I.1.7 in (6) we conclude that the closed linear hull of $\{f_n\}$ is also isomorphic to $c_0$ and complemented in $C(X)$.

This completes the proof of Proposition 10.

**Main Theorem 11.** *Let $X$ be a compact space. Then the following conditions are equivalent:*

a) *$X$ is scattered;*

b) *every semi-embedding of $C(X)$ in a Banach space is an embedding;*

c) *if $F$ is a Banach space and $T:C(X)\to F$ is a one-to-one operator which is not an embedding, then there is a complemented subspace $G$ of $C(X)$ such that $G$ is isomorphic to $c_0$ and $T|G$ is compact.*

d) *every infinite dimensional subspace of $C(X)$ contains a subspace isomorphic to $c_0$ and complemented in $C(X)$.*

**Proof.** d) $\Rightarrow$ c). This follows from Lemma 8. c) $\Rightarrow$ b). Suppose that there is a semi-embedding $T: C(X) \to F$ that is not an embedding. Let $G$ be a subspace of $C(X)$ as described in c). Then Corollary 5 implies that the inclusion map $G \to C(X)$ has an extension $G^{**} \to C(X)$. Since $G$ is complemented in $C(X)$, it follows that it is complemented in $G^{**}$, which is impossible since $G$ is isomorphic to $c_0$.

b) $\Rightarrow$ a). Suppose that $X$ is not scattered. Then $C(X)$ contains a subspace isometric to $C[0, 1]$ (7, Main Theorem) and *a fortiori*, an infinite dimensional reflexive subspace. But then Proposition 3 implies that b) can not hold.

a) $\Rightarrow$ d). The arguments for (1) $\Rightarrow$ (4) in the proof of the Main Theorem of (7) show that Proposition 10 can be applied.

**Remark.** H. P. Rosenthal kindly pointed out to us that (a) $\Rightarrow$ (c) also follows from Lemma 8, (0) $\Rightarrow$ (5) of the Main Theorem of (6), and the following unpublished result of his: *Let $X$ be a sequentially compact space and let $Z$ be a subspace of $C(X)$ isomorphic to $c_0$. Then $Z$ contains a subspace isomorphic to $c_0$ and complemented in $C(X)$.*

**Corollary 12.** *Let $T: E \to F$ be a semi-embedding and let $G$ be a complemented subspace of $E$ isomorphic to $C(X)$ where $X$ is scattered. Then $T|G$ is an embedding.*

**Proof.** This follows as in the proof of c) $\Rightarrow$ b) above by using c) and Corollary 5.

**Corollary 13.** *Every semi-embedding of $c_0$ is an embedding.*

Now we come to the relationship between semi-embeddings and Tauberian operators. A bounded linear operator $T: E \to F$ is *Tauberian* if $(T^{**})^{-1}(F) \subset E$.

These facts follow from the definitions:

(A) All quotient maps $P: E \to E/H$, where $H$ is a reflexive subspace, are Tauberian.

(B) If $T: E \to M \times L$ is defined as $Tx = (Px, Sx)$ with $P: E \to M$ Tauberian (and $S: E \to L$ just linear and bounded) then $T$ is Tauberian.

Also, using the equivalence "(a) $\Leftrightarrow$ (b)" on page 251 of (5), we conclude:

(C) All one-to-one Tauberian operators are semi-embeddings.

With this we can state:

**Corollary 14.** *Let $X$ be a compact space. Then $X$ is scattered if and only if all one-to-one Tauberian operators from $C(X)$ into arbitrary Banach spaces are embeddings.*

**Proof.** If $X$ is scattered, Theorem 11 and (C) apply to obtain the desired conclusion. Suppose now that $X$ is not scattered. Then the operator $T$ defined in the proof of Proposition 3 is Tauberian by (A) and (B). But it was shown in Proposition 3 that $T$ is not an embedding, and this concludes the proof of the corollary.

We remark that (A), (B) and (C) actually provide an alternative proof that the operator $T$ of Proposition 3 is indeed a semi-embedding.

Consider now the case $X = [0, 1]$. We know that there are semi-embeddings of

$C(X)$ which are not embeddings. However, we have:

**Proposition 15.** *Let $T$ be an operator from $C[0,1]$ into some Banach space. Assume that $T$ is a semi-embedding for some norm on $C[0,1]$ equivalent to the sup norm. Then there is a complemented subspace $C_1$ of $C[0,1]$ isomorphic to $C[0,1]$ such that the restriction of $T$ to $C_1$ is an embedding.*

**Proof.** Since $C[0,1]$ is isomorphic to $C(\Delta)$, ($\Delta$ is the Cantor set) we will prove the Proposition for $C(\Delta)$. We shall use the dyadic decomposition of $\Delta$ as $\Delta = \Delta_1 \cup \Delta_2$, $\Delta_1 = \Delta_3 \cup \Delta_4$, $\Delta_2 = \Delta_5 \cup \Delta_6, \ldots$ where $\Delta_1 = \Delta \cap [0, 1/3]$, $\Delta_2 = \Delta \cap [2/3, 1]$, $\Delta_3 = \Delta \cap [0, 1/9]$, $\Delta_4 = \Delta \cap [2/9, 1/3]$, $\Delta_5 = \Delta \cap [2/3, 7/9]$, etc. If $G_n$ is the subspace consisting of the functions vanishing off $\Delta_n$, then clearly $C(\Delta) = G_1 \oplus G_2$, $G_1 = G_3 \oplus G_4$, etc.

Now, if $T|G_n$ is an isomorphism for some $n$, we are done with $C_1 = G_n$. On the other hand, if $T|G_n$ is not an isomorphism for each $n$, then we can pick $f_n \in G_n$ with $\|f_n\| = 1$ and $\|Tf_n\| \leq 1/n$. But then the closed linear span of $\{f_2, f_4, f_8, \ldots\}$ is isomorphic to $c_0$ and Corollary 12 is violated.

**Remarks.** We comment here on the effect of replacing "semi-embedding" by "$\lambda$ semi-embedding" in the statements of our results.

Proposition 4 remains true except that the statement $\|S_1\|\,\|S_2\| = \|S\|$ has to be replaced by $\|S_1\|\,\|S_2\| \leq \lambda\|S\|$. Similarly, Corollaries 5 and 6 remain valid, with minor modifications. Of course, the validity of the Main Theorem for $\lambda$ semi-embeddings is implied by Corollary 12.

In Corollary 7, a)$\Rightarrow$b) obviously remains valid if "semi-embedding" is replaced by "$\lambda$ semi-embedding". Also, b) still implies that $X$ satisfies the countable chain condition; all that was used here is that $T$ is one-to-one and weakly compact. However, b) no longer implies that $X$ is hyperstonian. For example, let $Y = \beta N$ and let $X$ be the quotient space of $Y$ obtained by identifying two points of $\beta N \sim N$. The spaces $C(X)$ and $C(Y)$ are isomorphic, so by Proposition 2 there is a weakly compact $\lambda$ semi-embedding of $C(X)$ into some Banach space. However, $X$ is not Stonian. (For details, see (**8**, Theorem 2.6).)

In connection with condition d) in the proof of the Main Theorem, we remark that J. Hagler (**4**) has constructed an example of a separable Banach space $B$ such that $B^*$ is not separable and $B$ is hereditarily $c_0$.

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