Can. J. Math., Vol. XXIX, No. 3, 1977, pp. 480–488

SYMMETRIZABLE, $\mathscr{F}$-, AND WEAKLY FIRST COUNTABLE SPACES

R. M. STEPHENSON, JR.

A number of results are given concerning the character and cardinality of symmetrizable and related spaces. An example is given of a symmetrizable Hausdorff space containing a point that is not a regular G_δ, and a proof is given that if a point p of a symmetrizable Hausdorff space has a neighborhood base of cardinality $\aleph_1$, then p is a G_δ. It is shown that for each cardinal number m there exists a locally compact, pseudocompact, Hausdorff $\mathscr{F}$-space X with $|X| \geqq m$. Several questions of A. V. Arhangel'skii and E. Michael are partially answered.

In [**6**] Peter W. Harley, III and I introduced and studied a family of spaces, called $\mathscr{F}$-spaces (defined below). A number of topologists have found the family $\mathscr{F}$ to be of interest since $\mathscr{F}$ contains: all symmetrizable spaces; the Sorgenfrey line; the Michael line; Alexandrov's double interval; and the top and bottom of the lexicographically ordered square. Some of the properties of $\mathscr{F}$ derived in [**6**] are these. An $\mathscr{F}$-space is (a) Lindelöf if it is $\aleph_1$-compact, and (b) hereditarily Lindelöf if each of its discrete subspaces is countable. A compact Hausdorff $\mathscr{F}$-space is first countable (and hence is a neighborhood $\mathscr{F}$-space). A neighborhood $\mathscr{F}$-space is hereditarily Lindelöf if and only if it is hereditarily separable. A Hausdorff $\mathscr{F}$-space X is symmetrizable if and only if $X \times \{0, 1, 1/2, 1/3, \ldots\}$ is an $\mathscr{F}$-space. If X is a symmetrizable space in which every point is a regular G_δ, then X is locally compact if and only if for every symmetrizable Hausdorff space Y, the product space $X \times Y$ is symmetrizable. One has the implications

$$\begin{array}{lcl}
\text{semimetrizable} & \Rightarrow & \text{symmetrizable} \\
\Downarrow & & \Downarrow \\
\text{neighborhood } \mathscr{F} & \Rightarrow & \mathscr{F} \\
\Downarrow & & \Downarrow \\
\text{first countable} & \Rightarrow & \text{weakly first countable} \\
\Downarrow & & \Downarrow \\
\text{Fréchet} & \Rightarrow & \text{sequential}
\end{array}$$

and for Hausdorff Fréchet spaces,

$$\begin{array}{lcl}
\text{symmetrizable} & \Rightarrow & \text{semimetrizable} \\
\Downarrow & & \Downarrow \\
\mathscr{F} & \Rightarrow & \text{neighborhood } \mathscr{F} \\
\Downarrow & & \Downarrow \\
\text{weakly first countable} & \Rightarrow & \text{first countable}
\end{array}$$

Received March 3, 1976 and in revised form, December 10, 1976. This research was supported by a grant from the University of South Carolina.

https://doi.org/10.4153/CJM-1977-052-4 Published online by Cambridge University Press

In this paper I would like to consider some questions concerning these concepts.

Definitions. Following A. V. Arhangel'skii [**1**], a space X will be called *weakly first countable* if there exists a mapping $B : \mathbf{N} \times X \rightarrow \mathscr{P}(X)$, where $\mathscr{P}(X)$ denotes the power set of X, such that

(i) for each $n \in \mathbf{N}$ and $x \in X$, $B(n + 1, x) \subset B(n, x)$, and $\{x\} = \bigcap \{B(i, x) : i \in \mathbf{N}\}$; and

(ii) a subset V of X is open if and only if for each $v \in V$ there exists n with $B(n, v) \subset V$.

A space X is called an $\mathscr{F}$*-space* if there exists such a mapping B which also satisfies

(iii) for each closed set F and point $x \notin F$ there exists $i \in \mathbf{N}$ so that for every point $y \in B(i, x)\backslash\{x\}$, there exists $k \in \mathbf{N}$ with $\{x, y\} \not\subset B(k, F) \equiv \bigcup \{B(k, f) : f \in F\}$.

In this case B is said to be an $\mathscr{F}$*-system for* X.

A space X is said to be *symmetrizable* if there exists a mapping $d : X \times X \rightarrow [0, \infty)$ such that:

(a) for all $x, y \in X$, $d(x, y) = d(y, x)$;

(b) for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$; and

(c) for each set $F \subset X$, F is closed if and only if for every point $x \notin F$, $0 < d(x, F) \equiv \inf \{d(x, f) : f \in F\}$. The mapping d is said to be a *symmetric for* X.

It can be shown that a space X is symmetrizable if and only if X has an $\mathscr{F}$-system B satisfying:

(iii)′ for each closed set F and point $x \notin F$ there exists k with $x \notin B(k, F)$.

If B is an $\mathscr{F}$-system for X (d is a symmetric for X), and if $x \in$ interior of $B(n, x)$ ($\{y : d(x, y) < 1/n\}$) for each $n \in \mathbf{N}$ and $x \in X$, then X is said to be a *neighborhood* $\mathscr{F}$*-space* (*semimetrizable space*), and B (d) is called a *neighborhood* $\mathscr{F}$*-system* (*semimetric*) *for the space* X.

Although the $\mathscr{F}$-system condition (iii) and the symmetric condition (iii)′ are formally similar, the family of all $\mathscr{F}$-spaces is quite different from the family of all symmetrizable spaces, since, for example, the Michael line, the Sorgenfrey line, Alexandrov's double interval, and the top and bottom of the lexicographically ordered square are all neighborhood $\mathscr{F}$-spaces none of which is symmetrizable. For a detailed discussion of differences and similarities between the two families, see [**6**].

Let us now consider some questions concerning these spaces.

QUESTION 1. (E. Michael) *Is every point of a symmetrizable Hausdorff space a* G_δ?

In [**3**], D. Bonnett partially answered this question by constructing an involved example of a symmetrizable but non-Hausdorff space in which one

https://doi.org/10.4153/CJM-1977-052-4 Published online by Cambridge University Press

point fails to be a G_δ. Last year Peter W. Harley, III [**6**] gave a more simple example of a non-Hausdorff, symmetrizable space in which no point is a G_δ.

As far as I know, Michael's question has not yet been answered. Perhaps one reason no one has succeeded (in obtaining a negative answer) is given by the following result. It shows that one cannot produce a symmetrizable Hausdorff space which contains a non-G_δ point by adding one point to a first countable symmetrizable space.

LEMMA 2. *Let X be a symmetrizable Hausdorff space, and let p be a point of X. If the space $X\backslash\{p\}$ is perfect, then p must be a G_δ in X (and hence X must be perfect).*

Proof. Let d be a symmetric for X, and for each $n \in \mathbf{N}$ let $F_n = (X\backslash\{p\}) \cap \mathrm{Cl}\ \{y : d(p, y) < 1/n\}$. Then each F_n is a closed subset of the perfect space $X\backslash\{p\}$, so there is a sequence $\{U_{mn} : m \in \mathbf{N}\}$ of open subsets of $X\backslash\{p\}$ with $F_n = \bigcap \{U_{mn} : m \in \mathbf{N}\}$. Now define $V_{mn} = \{p\} \cup U_{mn}$, and note that for each point $x \in V_{mn}$ there exists $e > 0$ with $\{y : d(x, y) < e\} \subset V_{mn}$. Thus each V_{mn} is an open subset of X, and so $\{p\} \subset \bigcap \{V_{mn} : m, n \in \mathbf{N}\} \subset \bigcap \{\mathrm{Cl}\ \{y : d(p, y) < 1/n\} : n \in \mathbf{N}\} \subset \bigcap \{\bar{V} : V$ is an open neighborhood of p in $X\} = \{p\}$, which shows that p is a G_δ.

Using the next lemma, however, one can prove that there does exist a symmetrizable Hausdorff space containing a point that is not a regular G_δ. (A point p of a space X is called a *regular* G_δ if there exists a sequence $\{V_n : n \in \mathbf{N}\}$ of neighborhoods of p such that $\{p\} = \bigcap \{\mathrm{Cl}\ V_n : n \in \mathbf{N}\}$.)

Recall that a topological space is called *feebly compact* if every locally finite system of open sets is finite. (It is known that every countably compact space is feebly compact, and every normal feebly compact T_1-space is countably compact.)

LEMMA 3. *Let X be a topological space containing a dense feebly compact subspace Y. Then no point of $X\backslash Y$ is a regular G_δ in X.*

Proof. If $p \in X\backslash Y$ and $\{V_n : n \in \mathbf{N}\}$ is a descending sequence of open neighborhoods of p with $\{p\} = \bigcap \{\mathrm{Cl}\ V_n : n \in \mathbf{N}\}$, then $\{(V_n\backslash \mathrm{Cl}\ V_{n+1}) \cap X : n \in \mathbf{N}\}$ is an infinite locally finite family of open subsets of X.

LEMMA 4. *Let Y be a regular symmetrizable space containing a countably infinite closed discrete subset C indexed in a one-to-one way as $\{x_n : n \in \mathbf{N}\}$. Let d be any symmetric for Y, and let $X = Y \cup \{p\}$, where $p \notin Y$. Denote by d^* the function determined by the rule*

$$d^*(y, x) = d^*(x, y) = \begin{cases} d(x, y) & \text{if } x, y \in Y \\ 1 & \text{if } x = p \text{ and } y \in Y\backslash C \\ 1/n & \text{if } x = p \text{ and } y = x_n \\ 0 & \text{if } x = y = p. \end{cases}$$

Topologize X so that it has d^* as a symmetric, that is, call a subset V of X open if and only if for each point $v \in V$ there exists $e > 0$ with $\{x : d^*(x, v) < e\} \subset V$. Then X is a symmetrizable Hausdorff space containing Y as a dense subspace.

The proof is straightforward.

EXAMPLE 5. Let Y be the Isbell-Mrowka space ψ in [**5**, 5I]. Specifically, let $\mathscr{M}$ be a maximal infinite family of infinite subsets of $\mathbf{N}$ such that the intersection of any two members is finite, and denote by $\{p_M : M \in \mathscr{M}\}$ a set of distinct points not in $\mathbf{N}$. Let $Y = \mathbf{N} \cup \{p_M : M \in \mathscr{M}\}$, topologized as follows: each point of $\mathbf{N}$ is isolated; and a neighborhood of a point p_M is any set containing p_M and all but finitely many members of M. It is known (and not difficult to prove) that Y is feebly compact, regular, and symmetrizable but not countably compact, so we may let d be any symmetric for Y and C any countably infinite subset of $\{p_M : M \in \mathscr{M}\}$ and take X to be the space produced by applying Lemma 4 to Y, d, and C. Then it follows from Lemmas 2, 3, and 4 that *the space X is a feebly compact symmetrizable Hausdorff space in which the point p is a G_δ but not a regular G_δ.*

The space X in Example 5 is also of interest for another reason. Let W denote the set $\{0\} \cup \mathbf{N} \cup \{m + 1/n : m, n \in \mathbf{N}\}$, topologized so that it has as a symmetric the mapping p: $W \times W \to [0, \infty)$ determined by the rule

$$p(x, y) = p(y, x) = \begin{cases} 0 & \text{if } x = y \\ |x - y| & \text{if } xy \neq 0 \text{ and } x \neq y \\ 1 & \text{if } x = 0,\ y \neq 0, \text{ and } y \notin \mathbf{N} \\ 1/y & \text{if } x = 0 \text{ and } y \in \mathbf{N}. \end{cases}$$

In [**6**, Lemma 6.11] we obtained the following result.

LEMMA 6. *Let X be a symmetrizable space in which each point is a regular G_δ. Then the following are equivalent.*

(a) *X is first countable.*

(b) *W cannot be embedded homeomorphically in X as a closed subset.*

The space X in Example 5 can be used to show that the regular G_δ requirement in Lemma 6 is necessary.

EXAMPLE 7. *Let X be the symmetrizable Hausdorff space in Example 5. Then X fails to be first countable, but the space W cannot be embedded homeomorphically in X as a closed subset.*

Proof. If $h : W \to X$ is an embedding, it is easy to see that $h(0) = p$, $h(n) \in C$ for all but finitely many $n \in \mathbf{N}$, and for each $m \in \mathbf{N}$, there exists $M \in \mathscr{M}$ with $h(m + 1/n) \in M$ for all but finitely many $n \in \mathbf{N}$. Next, by induction one can find an infinite subset I of $\mathbf{N}$ with $\emptyset \neq \bar{I} \backslash I \subset X \backslash h(W)$ and $I \subset h(W)$. Thus $h(W)$ is not a closed subset of X.

The next two theorems show that for a large number of cases, Michael's question does have an affirmative answer.

A space is called $\aleph_1$-*compact* if every uncountable subset has a limit point.

LEMMA 8. *Let Y be an $\aleph_1$ compact, weakly first countable space which contains at most one non-isolated point. Then Y is countable.*

Proof. It follows easily from condition (ii) of the definition of weakly first countable that the space Y must actually be first countable. Thus every point of Y is a G_δ, and so if Y were uncountable, then it would have an uncountable closed discrete subset.

THEOREM 9. *Let X be a Hausdorff $\aleph_1$-compact $\mathscr{F}$-space. Then every point of X is a regular G_δ.*

Proof. Let $p \in X$ and $Y = X\backslash\{p\}$. In order to prove that p is a regular G_δ, it suffices to prove that Y is a Lindlöf space. Since Y is an $\mathscr{F}$-space [**6**, Theorem 2.9], and every $\aleph_1$-compact $\mathscr{F}$-space is Lindelöf [**6**, Theorem 3.3], it will be enough to prove that Y is $\aleph_1$-compact.

Suppose that there exists an infinite subset I of Y which fails to have a limit point in Y. Then $\bar{I}$, like any closed subset of an $\mathscr{F}$-space, is also an $\mathscr{F}$-space [**6**, Theorem 2.9], and, except for p, each point of $\bar{I} = I \cup \{p\}$ is isolated, so by Lemma 8, I must be countable. Therefore, Y is $\aleph_1$-compact.

THEOREM 10. *Let X be a symmetrizable Hausdorff space, and let m be an infinite cardinal number and p be a point of X such that p has a neighborhood base of cardinality $\leqq m^+$. Then $\{p\}$ is an intersection of m or fewer open sets.*

Proof. Let us suppose that $\{V_a : a < m^+\}$ is a neighborhood base for p but $\{p\}$ is not an intersection of m or fewer open sets.

Let b be an ordinal number $< m^+$, and assume that we have chosen points $\{x_a : a < b\}$ and open sets $\{W_a : a < b\}$ such that for each $d < b$, $x_d \in W_d \cap (\bigcap \{V_a : a \leqq d\})$, and $p \notin \mathrm{Cl}\, W_d$, and for any $a < d$, $x_d \notin W_a$. Then we can select a point x in the nonempty set $(X\backslash\{p\}) \cap (\bigcap \{V_a : a \leqq b\}) \cap (\bigcap \{X\backslash \mathrm{Cl}\, W_a : a < b\})$ and define $x_b = x$. Next, choose an open set W with $x_b \in W$ and $p \notin \mathrm{Cl}\, W$, and set $W_b = W$. Then for each $d \leqq b$, $x_d \in W_d \cap (\bigcap \{V_a : a \leqq d\})$, and $p \notin \mathrm{Cl}\, W_d$, and for any $a < d$, $x_d \notin W_a$.

Thus by transfinite induction there exist points $\{x_a : a < m^+\}$ and open sets $\{W_a : a < m^+\}$ such that the sequence $\{x_a : a < m^+\}$ converges to p, and for all $a < d < m^+$, $x_a \in W_a$ and $x_d \notin W_a$. Let s be a symmetric for X. Passing to subsequences and appealing to the regularity of the cardinal number m^+, one can find an integer k and sequences $\{y_a : a < m^+\}$ of points and $\{T_a : a < m^+\}$ of open sets such that: $\{y_a : a < m^+\}$ converges to p; for all $a < d < m^+$, $y_a \in T_a$ and $y_d \notin T_a$; and for each $a < m^+$, $\{z : s(y_a, z) < 1/k\} \subset T_a$.

Since s is a symmetric for X, and since the set $Y = \{y_a : a < m^+\}$ is not closed (it has p as a limit point), there must exist a point q such that $q \notin Y$ and

$s(q, Y) = 0$. From the equation $s(q, Y) = 0$ it follows that $q \in \bigcup \{T_a : a < m^+\}$, for otherwise we would have $s(q, Y) \geqq 1/k$.

Denote by c the smallest ordinal for which $q \in T_c$. Then for every $d < c$, $q \notin T_d$ and hence $s(q, y_d) \geqq 1/k$. Because T_c is open and $q \in T_c$, there is a number $t > 0$ with $\{z : s(q, z) < t\} \subset T_c$. Thus for every $d > c$, $s(q, y_d) \geqq t$. Let $e = \min \{1/k, t, s(q, y_c)\}$. Then $s(q, Y) \geqq e > 0$, which is a contradiction.

COROLLARY 11. *Let X be a separable symmetrizable Hausdorff space in which each point has a neighborhood base consisting of regular open sets. If $\mathfrak{c} = \aleph_1$, then every point of X is a G_δ.*

Proof. Let D be a countable dense subset of X, and let $\mathscr{R} = \{V : V = \text{Int}\ (\text{Cl}\ V)\}$, i.e., let $\mathscr{R}$ be the family of all regular open subsets of X. Since the mapping $i : \mathscr{R} \to \mathscr{P}(D)$ given by $i(V) = V \cap D$ is known to be one-to-one, each point of X has a neighborhood base of cardinality $\leqq \aleph_1$, and so Theorem 10 may be applied.

Let us now turn to some other related problems.

QUESTION 12. (A. V. Arhangel'skii [**1**, p. 129]) *Is every weakly first countable compact Hausdorff space first countable?*

QUESTION 13. (A. V. Arhangel'skii [**1**, p. 129]) *Does every weakly first countable compact Hausdorff space have cardinality $\leqq \mathfrak{c}$?*

Of course, an affirmative answer to Question 12 would provide an affirmative answer to Question 13 by [**2**]. More generally, an immediate consequence of [**2**, Theorem 2] is the following partial answer to Question 13.

THEOREM 14. (A. V. Arhangel'skii) *If X is a sequential compact Hausdorff space in which each point is an intersection of $\mathfrak{c}$ or fewer open sets, then $|X| \leqq \mathfrak{c}$.*

Another special case is covered by the next result.

THEOREM 15. *Let X be a weakly first countable, completely normal, countably compact Hausdorff space. Then X is first countable.*

Proof. Suppose that X fails to be first countable. Then it cannot be a Fréchet space, so there must exist a point $p \in X$ and a set $A \subset X$ such that $p \in \bar{A}$ but no sequence in A converges to p.

Let $B = \bar{A} \backslash \{p\}$. Then A is a dense subset of B and every infinite subset of A has a limit point in B, so B is (easily seen to be) feebly compact. On the other hand, because $\bar{A}$ is a sequential space and B is not a closed subset of $\bar{A}$, there must exist a sequence in B which converges to p. Thus B is not countably compact but is feebly compact, whereas every normal feebly compact T_1-space is countably compact.

Other special cases for which Questions 12 and 13 have affirmative answers are all those compact Hausdorff weakly first countable spaces which are symmetrizable (they are metrizable by [**1**, p. 126]) or are $\mathscr{F}$-spaces (they are first countable by [**6**, Theorem 3.14]).

Since a space providing a negative answer to Question 13 would have to be sequential but not Fréchet, it is natural to ask if there even exists a compact Hausdorff sequential but not Fréchet space which is of cardinality $> \mathfrak{c}$. In [**4**, Example 7.1] S. P. Franklin noted that the Alexandroff one-point compactification ψ^* of the space ψ (which has cardinality $\leqq \mathfrak{c}$) is sequential but not Fréchet. One also has other such examples.

EXAMPLE 16. Let m be an uncountable cardinal number, let $\mathscr{M}_m$ be a maximal family of countably infinite subsets of m such that the intersection of any two members is finite. Denote by $\{p_M : M \in \mathscr{M}_m\}$ a set of distinct points not in m, and let $X_m = m \cup \{p_M : M \in \mathscr{M}_m\}$, topologized as follows: each point of m is isolated; and a neighborhood of a point p_M is any set containing p_M and all but finitely many points of M. Then one can prove the following. *The space X_m is a Hausdorff, feebly compact, locally compact, neighborhood $\mathscr{F}$-space such that $|X_m| \geqq m$. Furthermore, the Alexandroff one-point compactification $X_m{}^*$ of X_m is sequential but not Fréchet.*

By Lemma 8, none of the closed subspaces $\{p_M : M \in \mathscr{M}_m\}^*$ is weakly first countable, so none of the spaces $X_m{}^*$ is weakly first countable. By Theorem 17, no X_m is symmetrizable.

THEOREM 17. *Let X be a feebly compact symmetrizable space. Then the set I of isolated points of X is countable.*

Proof. If I is uncountable, let d be a symmetric for X and note that for some integer k there is an uncountable subset Y of I such that for each $y \in Y$, $\{y\} = \{z : d(z, y) < 1/k\}$. By the feeble compactness of X, the set Y cannot be closed. Thus there is a point $x \in X \backslash Y$ with $d(x, Y) = 0$, in contradiction of the fact that for any point t of X, there is at most one $y \in Y$ with $d(t, y) < 1/k$.

COROLLARY 18. *Let X be a symmetrizable space, and let D be a subset of X such that every infinite subset of D has a limit point in X. Then every discrete subspace of D is countable, and if X is semimetrizable, then $\bar{D}$ is separable.*

Proof. If I is a discrete subspace of D, then I is also the set of isolated points for the feebly compact symmetrizable space $\bar{I}$.

For the last statement, let d be a semimetric for X, and for each positive integer n let D_n be a maximal subset of D with $d(x, y) \geqq 1/n$ for all $x, y \in D_n$ with $x \neq y$. Then $\bigcup \{D_n : n \in \mathbf{N}\}$ is a countable dense subset of $\bar{D}$.

QUESTION 19. *Is every feebly compact symmetrizable space separable?*

In [**3**] Bonnett answered a question of Michael by giving an example of a symmetrizable Hausdorff space that is not perfect. He also raised the next problem.

QUESTION 20. (D. Bonnett [3]) *Is every regular symmetrizable space perfect?*

Peter W. Harley, III has pointed out that the following partial answer to Question 20 is an immediate consequence of Theorems 2, 3, and 4 of [7].

THEOREM 21. (Peter W. Harley, III) *If* $2^{\aleph_1} > \mathfrak{c}$ *then every separable normal symmetrizable space is hereditarily Lindelöf (and hence perfect).*

We will conclude by giving a result which shows that the converse of Theorem 10 is not true. As far as the author knows, an example has not been given previously of a weakly first countable Hausdorff space in which there is a point p which has no neighborhood base of cardinality $\leqq \mathfrak{c}$.

THEOREM 22. *Let* m *be an infinite cardinal number. Then there exists a paracompact, zero-dimensional symmetrizable space* X *in which there is a* G_δ *point* p *such that every neighborhood base for* p *has cardinality* $> m$.

Proof. Let $T = \{0\} \cup \{1/n : n \in \mathbf{N}\}$ and $Y = m \times T$. Choose a point $p \notin Y$ and set $X = Y \cup \{p\}$.

Let $\{B_n : n \in \mathbf{N}\}$ be a descending sequence of subsets of $m \times \{0\}$ such that $\bigcap \{B_n : n \in \mathbf{N}\} = \emptyset$ and each set $B_i \backslash B_{i+1}$, $i \in \mathbf{N}$, has cardinality m. Define $d : X \times X \rightarrow [0, \infty)$ as follows: for any $x, y \in X$,

$$d(x, y) = d(y, x) = \begin{cases} 0 & \text{if } x = y \\ 1/(n+1) & \text{if } x = p \text{ and } y \in B_n \backslash B_{n+1} \\ 1 & \text{if } x = p \text{ and } y \in Y \backslash B_1 \\ 1 & \text{if } x = (a, s), y = (b, t) \text{ and } a \neq b \\ |s - t| & \text{if } x = (a, s) \text{ and } y = (a, t) \text{ for some } a \in m. \end{cases}$$

Topologize X so that it has d as a symmetric. Then it is easy to see that p is a G_δ and X is paracompact, zero-dimensional, and symmetrizable.

Let $\mathscr{V}$ be a family of open neighborhoods of p with $0 < |\mathscr{V}| \leqq m$. We will prove that there exists a neighborhood W of p which contains no member of $\mathscr{V}$.

For each $n \in \mathbf{N}$, let $\mathscr{V}_n = \{V \in \mathscr{V} : V \supset \{x : d(x, p) < 1/n\}\}$, and if $\mathscr{V}_n \neq \emptyset$, then index its members as $\{V_{na} : (a, 0) \in B_n \backslash B_{n+1}\}$, where repetitions are permitted. Choose an integer k such that each $\mathscr{V}_n$, $n \geqq k$, is nonempty. For each integer $n \geqq k$ and each $a \in m$ with $(a, 0) \in B_n \backslash B_{n+1}$, find the largest integer i with $(a, 1/i) \notin V_{na}$, and let $S_a = \{(a, x) \in Y : |x| < 1/(i+1)\}$; if no such i exists then let $S_a = \{a\} \times (T \backslash \{1\})$. Then the set $W = \{p\} \cup (\bigcup \{S_a : (a, 0) \in B_n \backslash B_{n+1}, n \geqq k\})$ is a neighborhood of p which contains no member of $\mathscr{V}$.

REFERENCES

1. A. V. Arhangel'skii, *Mappings and spaces*, Uspehi Mat. Nauk *21* (1966), no. 4 (*130*), 133–184 Russian Math. Surveys *21* (1966), no. 4, 115–162.
2. ——— *On the cardinality of bicompacta satisfying the first axiom of countability*, Soviet Math. Dokl *10* (1969), 951–955.
3. D. A. Bonnett, *A symmetrizable space that is not perfect*, Proc. Amer. Math. Soc. *34* (1972), 560–564.

4. S. P. Franklin, *Spaces in which sequences suffice, II*, Fund. Math. *57* (1967), 51–56.
5. L. Gillman and M. Jerison, *Rings of continuous functions* (Van Nostrand, Princeton, N.J., 1960).
6. Peter W. Harley, III and R. M. Stephenson, Jr., *Symmetrizable and related spaces*, Trans. Amer. Math. Soc. *217* (1976), to appear.
7. S. Nedev, *Symmetrizable spaces and final compactness*, Soviet Math. Dokl. *8* (1967), 890–892.

University of South Carolina,
Columbia, South Carolina