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A REMARK ON TALENTI'S SEMIGROUP

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For $\alpha>0$ the Riemann-Liouville Integral $J(\alpha)$ is given for suitable functions g by

$$(1)\qquad [J(\alpha)g](t) = \int_0^t g(s)(t-s)^{\alpha-1}\, ds/\Gamma(\alpha) \qquad (0 \le t).$$

For a variety of function spaces (e.g., $C[0, 1]$ or $L_p(0, 1)$ with $p\geq 1$) this defines a C_0 semigroup which has been extensively studied (c.f., e.g., [3]). A generalization of this was given by G. Talenti [4], setting

$$(2)\qquad [I(\alpha)g](t) = \int_0^t g(s)\left[\int_s^t p(r)\, dr\right]^{\alpha-1} p(s)\, ds/\Gamma(\alpha)$$

with $p(\cdot)$ a specified positive continuous function; note that $J(\cdot)$ is the special case: $p\equiv 1$.

A. Chrysovergis [1] has shown that, for $\alpha>0$, $I(\alpha)$ is bounded on $C[0, 1]$ and depends continuously in norm on α. E. Hille [2] in reviewing [1], suggested the likelihood that, as with the Riemann-Liouville Integral J, this semigroup is holomorphic in α for $\mathcal{Re}\ \alpha>0$. Our present remark consists of the observation that Talenti's semigroup may be reduced to the Riemann-Liouville case ($p\equiv 1$) by an appropriate substitution.

We assume the functions are to be defined on $[0, 1]$ and set

$$P(t) = e^c \int_0^t p(s)\, ds$$

with

$$c = \log \int_0^1 p(s)\, ds.$$

Thus, P is strictly increasing (as p is positive) with $P(0)=0$, $P(1)=1$ and so is a homeomorphism of $[0, 1]$ to itself. Now, noting that

$$\int_s^t p(r)\, dr = e^c[P(t) - P(s)]$$

and setting $\sigma=P(s)$, $\tau=P(t)$ so $p(s)\, ds=e^c\, d\sigma$, one has from (2) that

$$(3)\qquad \begin{aligned}[I(\alpha)g](t) &= e^{-c\alpha}\int_0^\tau g(P^{-1}(\sigma))(\tau-\sigma)^{\alpha-1}\, d\sigma/\Gamma(\alpha)\\ &= e^{-c\alpha}[J(\alpha)(g\circ P^{-1})](P(t))\end{aligned}$$

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or

$$I(\alpha) = e^{-c\alpha} \cdot \pi J(\alpha)\pi^{-1} \tag{3'}$$

where π is the operator of composition with P, i.e.,

$$[\pi f](t) = f(P(t))., \qquad \pi : f \mapsto f \circ P;$$

For any suitable function space, then, the properties of $I(\cdot)$ are immediately deducible from the (known) properties of $J(\cdot)$ provided π is non-singular. For $C[0, 1]$ it is sufficient to require that $p \in L_1(0, 1)$ with $p>0$ a.e.; for $L_p(0, 1)$ one requires $0<p$, $1/p \in L_\infty(0, 1)$ although if one only had $0<p \in L_1(0, 1)$ one could consider $I(\cdot)$ acting on a space with an L_p norm weighted by p. In particular we see that $I(\alpha)$ is, indeed a C_0 semigroup on $C[0, 1]$, holomorphic in the right halfplane.

References

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2. E. Hille, review of [1] in Math. Rev., **46** (1973), rev. 9247.

3. E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloq. Publ., v. 31.

4. G. Talenti, *Sul Problema di Cauchy per le Equazioni a Derivate Parziali*, Ann. Mat. Pura Appl., LXVII (1965), pp. 365–394.

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