A CHARACTERIZATION OF LATTICE-ORDERED GROUPS BY THEIR CONVEX L-SUBGROUPS

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1. Introduction

In this paper we show that whether or not a group admits a lattice-order often depends upon whether or not it possesses a set of subgroups that satisfy certain algebraic conditions. Using these techniques we are able to determine large classes of groups that can be lattice-ordered.

There are many places in the literature where a class of groups that can be totally ordered has been characterized as those groups with a chain of subgroups that satisfy certain restrictions — for example see Iwasawa [6] and Neumann [8]. Malcev [7], Podderyugin [9] and Rieger [10] have derived conditions on a chain of subgroups that are necessary and sufficient for the ordering of a group (an account of part of their work can be found in Fuchs [5]). In our theory we recover the results of Podderyugin and Rieger.

In section 2 we recall some of the definitions and the theory of lattice-ordered groups ("l-groups") that we need. Most of these results are from [4]. We then prove the necessity of the conditions that are used in section 3 to characterize l-groups. Also we derive some new results. For example, Theorem 2.1 has proven to be extremely useful in this paper and also in other contexts.

In section 3 we determine all those groups that admit a lattice-order which is finite valued. We also determine those groups that admit a normal lattice-order. In fact, the techniques that we use can only describe normal lattice-orders, but since this includes all representable l-groups as well as all finite valued l-groups, this class is quite large.

The results in section 3 all have content if we restrict our attention to abelian groups. In section 4 we show that our theory fits in nicely with the representation theory for abelian l-groups that is developed in [2].

Throughout this paper we assume that the reader is familiar with the results in [4].

[1] This research was done while the author held a National Science Foundation senior postdoctoral fellowship.

Notation. If A and B are subgroups of a group G, then $N(A)$ will denote the normalizer of A in G, $[A, B]$ will denote the subgroup of G that is generated by all commutators $-a-b+a+b$ where $a \in A$ and $b \in B$, $A \lhd B$ will denote that A is a normal subgroup of B, and $A \backslash B$ will denote the set of all elements in A but not in B. We shall denote the null set by $\square$. We shall write l-group (o-group) for lattice-ordered group (totally ordered group). If x belongs to an l-group, G, then $G(x)$ will denote the subgroup $\{y \in G \mid |y| < n\,|x| \text{ for some } n > 0\}$. We shall denote the fact that $a, b \in G$ are not comparable by $a \parallel b$ and also that subsets X and Y are not comparable with respect to the inclusion relation by $X \parallel Y$.

2. Some properties of l-groups

Throughout this section G will denote an l-group. We first recall some definitions and results from [4] that we need. A subgroup C of G is called a *convex l-subgroup* if C is a sublattice of G such that $0 < g < c \in C$ implies $g \in C$. The set of all convex l-subgroups of G is a complete distributive sublattice of the lattice of all subgroups of G. A convex l-subgroup C is called *prime* (*regular*) if the convex l-subgroups of G that contain it form a chain (if it is maximal without containing some element $g \in G$). Theorem 3.2 in [4] establishes six equivalent definitions of a prime convex l-subgroup. In particular, each regular convex l-subgroup is prime. If C is a regular convex l-subgroup of G, then C is a proper subgroup of the intersection C^* of all convex l-subgroups that properly contain C. In fact, this property is equivalent to regularity. Also if C is a prime or a regular convex l-subgroup of G, then so is $-g+C+g$ for all $g \in G$.

Let C be a regular convex l-subgroup of G that is covered by the convex l-subgroup C^* and suppose that $C \lhd C^*$.

(a) $$[[N(C), N(C)], C^*] \subseteqq C.$$

Proof. Since C^* is the unique convex l-subgroup that covers C it follows that $N(C) \subseteqq N(C^*)$. Thus each $a \in N(C)$ induces an o-automorphism $\hat{a}$ of C^*/C

$$(C+x)\hat{a} = C-a+x+a.$$

But C^*/C is o-isomorphic to a naturally ordered additive group of real numbers. Thus the o-automorphisms of C^*/C are essentially multiplications by positive real numbers, and hence $\hat{a}\hat{b} = \hat{b}\hat{a}$ for all $a, b \in N(C)$, which is equivalent to property (a).

(b) *If $x \in C^* \backslash C$ and $a_1, \cdots, a_n \in N(C)$, then*

$$(-a_1+x+a_1)+\cdots+(-a_n+x+a_n) \notin C.$$

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PROOF. We may assume that $C+x > C$ and thus it follows that $C+(-a_i+x+a_i) > C$ for $i = 1, \cdots, n$, and hence

$$C+(-a_1+x+a_1)+\cdots+(-a_n+x+a_n) > C.$$

Note that (a) and (b) imply that C^*/C is an $N(C)$-group such that the elements of $N(C)$ commute as operators and the total order on C^*/C is an $N(C)$-order.

Let $\Gamma_1 = \Gamma_1(G)$ be an index set for the set of all regular convex l-subgroups G_γ of G. We shall frequently identify this set with Γ_1. Define $\alpha \leqq \beta$ in Γ_1 if $G_\alpha \subseteqq G_\beta$. Then Γ_1 is a *root system* (that is, Γ_1 is a partially ordered set and for each $\gamma \in \Gamma_1$, $\{a \in \Gamma_1 \mid \alpha \geqq \gamma\}$ is a chain). Let G^γ be the convex l-subgroup of G that covers G_γ $(\gamma \in \Gamma_1)$. If $g \in G^\gamma \backslash G_\gamma$, then γ is said to be a *value* of g, and if γ is the only value of g in Γ_1, then both γ and g are called *special*. It follows from Theorems 3.5 and 3.6 in [4] that if γ is special, then $G_\gamma \lhd G^\gamma$. A subset Δ of Γ_1 is called *plenary* if

(i) $\cap \{G_\gamma \mid \gamma \in \Delta\} = 0$, and

(ii) if $g \in G\backslash G^\gamma$ $(\gamma \in \Delta)$, then there exists a value α of g in Δ that exceeds γ.

If in addition, for each $\alpha \in \Delta$ and $g \in G$

(iii) $G_\alpha \lhd G^\alpha$, and

(iv) $-g+G_\alpha+g \in \{G_\gamma \mid \gamma \in \Delta\}$,

then we shall call Δ a *normal plenary subset* of Γ_1. Note that Γ_1 satisfies (i), (ii) and (iv) but that it need not satisfy (iii). We shall call the lattice-order of G *normal* if Γ_1 contains a normal plenary subset. Let Δ be the set of all special elements of Γ_1. Clearly for each $\gamma \in \Delta$, (iii) and (iv) are satisfied. Thus Δ is a normal plenary subset of Γ_1 provided that it is a plenary subset.

G is said to be *representable* if there exists an l-isomorphism of G into a cardinal sum of o-groups. Byrd [1] has shown that G is representable if and only if for each $\gamma \in \Gamma_1$ and $g \in G$, $-g+G_\gamma+g$ and G_γ are comparable, and if G is representable, then Γ_1 is normal.

We shall call a subset $\{g_\lambda \mid \lambda \in \Lambda\}$ of G *disjoint* if each $g_\lambda > 0$ and $g_\lambda \wedge g_{\lambda'} = 0$ for $\lambda \neq \lambda'$. A convex l-subgroup C of G is called *$\mathscr{L}$-closed* if $\bigvee g_\lambda \in C$ for each disjoint subset $\{g_\lambda \mid \lambda \in \Lambda\}$ of C for which $\bigvee g_\lambda$ exists.

LEMMA 2.1. *If $\gamma \in \Gamma_1$ is special, then G_γ is $\mathscr{L}$-closed.*

PROOF. Suppose (by way of contradiction) that $0 < g = \bigvee g_\lambda \notin G_\gamma$, where $\{g_\lambda \mid \lambda \in \Lambda\}$ is a disjoint subset of G_γ. Let α be the value of g such that $\alpha \geqq \gamma$, and pick a special element $O < y \in G^\gamma \backslash G_\gamma$. Let $x = y \wedge g$. Then $x \in G^\gamma \backslash G_\gamma$ is special, and

$$x = g \wedge x = (\bigvee g_\lambda) \wedge x = \bigvee (g_\lambda \wedge x)$$

where each $g_\lambda \wedge x$ belongs to $G_\gamma \cap G(x)$. But $G(x)$ is a lexicographic extension of $G_\gamma \cap G(x)$ ([4] Theorem 3.6) and hence for each positive integer n, $x > n(g_\lambda \wedge x)$ and it follows that

$$x \geqq \bigvee n(g_\lambda \wedge x) = n \bigvee (g_\lambda \wedge x) = nx$$

for all $n > 0$ which is impossible.

LEMMA 2.2. *If* $g = \bigvee a_\alpha = \bigvee b_\beta$, *where* $\{a_\alpha \mid a \in A\}$ *and* $\{b_\beta \mid \beta \in B\}$ *are disjoint sets of special elements, then* $\{a_\alpha \mid \alpha \in A\} = \{b_\beta \mid \beta \in B\}$.

PROOF. Let δ be *the* value of b_β. It is easy to show that $\bigvee_{\lambda \neq \beta} b_\lambda$ exists and is disjoint from b_β, and hence $g = b_\beta + \bigvee_{\lambda \neq \beta} b_\lambda$. Thus since G_δ is prime, $\bigvee_{\lambda \neq \beta} b_\lambda \in G_\delta$ and hence $g \in G^\delta \backslash G_\delta$. Now by Lemma 2.1, there exists $a_\alpha \notin G_\delta$ and since $\{a_\alpha \mid \alpha \in A\}$ is a disjoint set this a_α is unique. If $\gamma \leqq \delta$, then $a_\alpha, b_\beta \notin G_\gamma$, and hence $\bigvee_{\mu \neq \alpha} a_\mu$ and $\bigvee_{\lambda \neq \beta} b_\lambda$ must belong to G_γ. Thus $G_\gamma + a_\alpha = G_\gamma + g = G_\gamma + b_\beta$. If $\gamma \nleqq \delta$, then $a_\alpha, b_\beta \in G_\gamma$. Therefore $G_\gamma + a_\alpha = G_\gamma + b_\beta$ for all $\gamma \in \Gamma_1$ and hence $a_\alpha = b_\beta$.

COROLLARY. *If* $b_\beta \in G^\delta \backslash G_\delta$, *then* $g \in G^\delta \backslash G_\delta$.

THEOREM 2.1. *Let* Δ *be the set of special elements in* Γ_1. *Then the following are equivalent.*

(a) *Each* $0 < g \in G$ *is the join of disjoint special elements in* G.

(b) Δ *is a plenary subset of* Γ_1.

If this is the case, then Δ *is the set of all regular* $\mathscr{L}$*-closed convex l-subgroups of* G *and* Δ *is a normal plenary subset of* Γ_1. *Moreover, the representation* $g = \bigvee g_\lambda$ *of* g *as the join of disjoint special elements is unique and the values of* g *in* Δ *are precisely the values of the* g_λ.

PROOF. Consider $O < g \in G$ and let Δ_g be the set of all values of g in Δ.

(a) $\rightarrow$ (b). $g = \bigvee g_\lambda$, where $\{g_\lambda \mid \lambda \in \Lambda\}$ is a disjoint set of special elements. If δ is the value of g_λ, then by the above corollary $\delta \in \Delta_g$. Thus $\cap \{G_\gamma \mid \gamma \in \Delta\} = O$. If $g \notin G_\alpha$ $(\alpha \in \Delta)$, then by Lemma 2.1 there exists $g_\lambda \notin G_\alpha$ and so *the* value of g_λ is $\geqq \alpha$ and it is also a value of g. Thus Δ is a plenary subset of Γ_1.

(b) $\rightarrow$ (a). Let $\delta \in \Delta_g$ and pick $O < x \in G^\delta \backslash G_\delta$ so that x is special and $G_\delta + x > G_\delta + g$. If $\delta \ngeqq \alpha \in \Delta$, then $x \in G_\alpha$ and hence $G_\alpha + x \wedge g = G_\alpha + x \wedge G_\alpha + g = G_\alpha$. If $\delta \geqq \alpha \in \Delta$, then since the right cosets of G_α are totally ordered and $G_\delta + x > G_\delta + g$ it follows that $G_\alpha + x > G_\alpha + g$. Thus $G_\alpha + x \wedge g = G_\alpha + x \wedge G_\alpha + g = G_\alpha + g$. If we set $g(\delta) = x \wedge g$, then we have shown that

$$G_\alpha + g(\delta) = \begin{cases} G_\alpha + g & \text{for all } \delta \geqq \alpha \in \Delta \\ G_\alpha & \text{for all } \delta \ngeqq \alpha \in \Delta. \end{cases}$$

Clearly δ is the only value of $g(\delta)$ in Δ and hence ([4] Proposition 3.11) δ is the only value of $g(\delta)$ in Γ_1. Thus $g(\delta)$ is special. For each $\delta \in \Delta_g$ define

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a $g(\delta)$ as above. Then the $g(\delta)$ are special and pairwise disjoint ([4] Proposition 3.10). In order to complete the proof it suffices to show that $g = \bigvee g(\delta)$ $(\delta \in \Delta_g)$. Clearly g exceeds each $g(\delta)$. Suppose that $h \in G$ and $h \geqq g(\delta)$ for all $\delta \in \Delta_g$ and let α be a value of $g-h$ in Δ. If $g \in G_\alpha$, then $G_\alpha+g-h = G_\alpha-h < G_\alpha$ since $h \geqq 0$, and if $g \notin G_\alpha$, then $\alpha \leqq \delta \in \Delta_g$ and hence $G_\alpha \neq G_\alpha+g-h = G_\alpha+g(\delta)-h < G_\alpha$. Thus $g-h \leqq O$ and hence $g = \bigvee g(\delta)$. Therefore (a) and (b) are equivalent.

Now suppose that (a) and (b) hold and that γ is a value of $O < g \in G$ in $\Gamma_1 \backslash \Delta$. $g = \bigvee g_\lambda$, where $\{g_\lambda \mid \lambda \in \Lambda\}$ is a disjoint set of special elements. If $g_\lambda \notin G_\gamma$, then $\bigvee_{\lambda' \neq \lambda} g_{\lambda'} \in G_\gamma$ and thus γ is a value of g_λ, but this means that $\gamma \in \Delta$, a contradiction. Therefore $\{g_\lambda \mid \lambda \in \Lambda\} \subseteqq G_\gamma$ and $\bigvee g_\lambda \notin G_\gamma$, and hence G_γ is *not* $\mathscr{L}$-closed. Thus by Lemma 2.1 Δ is the set of all $\mathscr{L}$-closed regular convex l-subgroups of G. The rest of the theorem follows from our previous results.

An element $O < g \in G$ is said to be *irreducible* if

$$g = a+b \text{ and } a \wedge b = O \text{ imply } a = O \text{ or } b = O.$$

If $O < g \in G$ is special with value γ and $g = a+b$, where $a \wedge b = O$, then one but not both of a and b must belong to G_γ. For if $a, b \notin G_\gamma$, then $O = a \wedge b \notin G_\gamma$ ([4] Theorem 3.2). If $a \in G_\gamma$, then clearly γ is the only value of $b-a$ and $G_\gamma+b-a = G_\gamma+b > G_\gamma$. Thus $b > a$ and hence $O = b \wedge a = a$. Therefore each special element of G is irreducible.

COROLLARY. *If the set Δ of all special elements of Γ_1 is a plenary subset of Γ_1, then $O < g \in G$ is special if and only if it is irreducible.*

PROOF. If g is not special, then by our theorem $g = \bigvee g_\lambda$, where $\{g_\lambda \mid \lambda \in \Lambda\}$ is a disjoint set of special elements. Therefore $g = g_\alpha + \bigvee_{\lambda \neq \alpha} g_\lambda$, and hence g is reducible.

3. Characterizations of lattice-ordered groups

Throughout this section we shall assume that G is a group with a set $\{G_\delta \mid \delta \in \Delta\}$ of proper subgroups such that

(1) $\cap \{G_\delta \mid \delta \in \Delta\} = O$

and such that for each $\delta \in \Delta$ the following properties are satisfied.

(2) $\{G_\alpha \mid \alpha \in \Delta \text{ and } G_\alpha \supset G_\delta\}$ is a chain whose intersection G^δ properly contains G_δ.

(3) If $g \in G \backslash G^\delta$, then there exists an $\alpha \in \Delta$ such that $G_\alpha \supseteqq G^\delta$ and $g \in G^\alpha \backslash G_\alpha$.

(4) If $(-a_1+x+a_1)+ \cdots +(-a_n+x+a_n) \in G_\delta$, where $x \in G^\delta$ and $a_1, \cdots, a_n \in N(G)_\delta$, then $x \in G_\delta$.

(5) $-g+G_\delta+g \in \{G_\delta \mid \delta \in \Delta\}$ for all $g \in G$.

(6) $[G^\delta, G^\delta] \subseteqq G_\delta$ and $[[N(G_\delta), N(G_\delta)], G^\delta] \subseteqq G_\delta$.

It follows from the material in the last section that if H is an l-group, then each normal plenary subset Δ of $\Gamma_1(H)$ satisfies (1) through (6).

If $O \neq g \in G$, then by (1) and (3), $g \in G^\alpha \backslash G_\alpha$ for some $\alpha \in \Delta$ and hence by (4), $ng \in G^\alpha \backslash G_\alpha$ for each positive integer n. Thus G is torsion free, and also if $ng \in G_\delta$, then $g \in G_\delta$, and hence the G_δ and the G^δ are pure subgroups of G.

(a) *If* $-g+G_\delta+g = G_\delta$, *then* $-g+G^\delta+g = G^\delta$, *and hence* $N(G_\delta) \subseteqq N(G^\delta)$.

Proof.
$$\begin{aligned}-g+G^\delta+g &= -g+ \cap \{G_\alpha \mid G_\alpha \supset G_\delta\}+g \\ &= \cap \{-g+G_\alpha+g \mid G_\alpha \supset G_\delta\} = \cap \{G_\beta \mid G_\beta \supset -g+G_\delta+g\} \\ &= \cap \{G_\beta \mid G_\beta \supset G_\delta\} = G^\delta.\end{aligned}$$

In particular, each element $a \in N(G_\delta)$ induces an automorphism $\hat{a}$ on the group G^δ/G_δ
$$(G_\delta+x)\hat{a} = G_\delta-a+x+a.$$

(b) *Each* G^δ/G_δ *is an abelian* $N(G_\delta)$*-group and the elements of* $N(G_\delta)$ *commute as operators. Also* G^δ/G_δ *is* $N(G_\delta)$*-torsion-free.*

Proof. By the first part of (6), G^δ/G_δ is an abelian group, and it is an immediate consequence of the second half of (6) that the elements in $N(G_\delta)$ commute as operators on G^δ/G_δ (see for example [5] p. 51). The content of (4) is that G^δ/G_δ is $N(G_\delta)$-torsion-free.

Thus by a theorem of Podderyugin [9] any partial $N(G_\delta)$-order of G^δ/G_δ can be extended to a total $N(G_\delta)$-order. In particular, the trivial partial order can be so extended and hence there exists a total $N(G_\delta)$-order for G^δ/G_δ. Now define α and β in Δ to be equivalent if $G_\alpha = -g+G_\beta+g$ for some $g \in G$, and in each equivalence class pick a group G^δ/G_δ and give it a total $N(G_\delta)$-order $\mathscr{P}_\delta$. Next define $X = -g+Y+g \in (-g+G^\delta+g)/(-g+G_\delta+g)$ to be positive if $Y \in \mathscr{P}_\delta$. Using the fact that $\mathscr{P}_\delta$ is an $N(G_\delta)$-order it follows that this definition is independent of the particular choice of g, and that this is a total $N(-g+G_\delta+g)$-order of $(-g+G^\delta+g)/(-g+G_\delta+g)$.

Define $\alpha < \beta$ in Δ if $G^\alpha \subseteqq G_\beta$ or equivalently if $G_\alpha \subset G_\beta$. If $g \in G^\delta \backslash G_\delta$, then we shall say that δ is a *value* of g. Note that each $0 \neq g \in G$ has at least one value and that the set of all values of g is a trivially ordered subset of Δ which we shall denote by Δ_g. Define $O \neq g \in G$ to be *strictly positive* (notation $g > O$) if $G_\delta+g > G_\delta$ in G^δ/G_δ for all $\delta \in \Delta_g$.

Proposition 3.1. *G is a semiclosed po-group and the G_δ and the G^δ are pure convex subgroups of G. We shall call a partial order of G that is defined in this way a Δ-order.*

PROOF. Let $P = \{g \in G \mid g > O\}$, and consider $a, b \in P$ and $\alpha \in \Delta_a$. Then $\alpha \in \Delta_{-a}$ and $G_\alpha - a < G_\alpha$, and so $P \cap -P = \square$. If $a+b = O$, then $\alpha \in \Delta_b$ and $G_\alpha + a = G_\alpha - b < G_\alpha$, which contradicts the fact that $a \in P$. Thus $a+b \neq O$ and hence there exists a value γ of $a+b$. Clearly $\gamma \in \Delta_a \cup \Delta_b$ and $a, b \in G^\gamma$. Thus $G_\gamma + a + b = G_\gamma + a + G_\gamma + b > G_\gamma$ and hence $P+P \subseteq P$. If $-g+a+g \in G^\beta \backslash G_\beta$, then $a \in (g+G^\beta-g)\backslash(g+G_\beta-g)$ and hence $(g+G_\beta-g)+a$ is positive. Thus $G_\beta - g + a + g = -g + (g+G_\beta-g)+a+g$ is positive and hence $-g+a+g \in P$. Therefore G is a *po*-group.

If $ng \in P$, where $g \in G$ and $n > 0$, and $\gamma \in \Delta_g$, then $\gamma \in \Delta_{ng}$ and $n(G_\gamma + g) = G_\gamma + ng > G_\gamma$ in the o-group G^γ/G_γ. Thus $G_\gamma + g > G_\gamma$ and it follows that $g \in P$ and thus the *po* is semiclosed.

If $0 < x < g \in G_\delta$ and $x \notin G_\delta$, then there exists $\delta \leqq \beta \in \Delta_x$ and since $g \in G_\beta$, $\beta \in \Delta_{x-g}$. Thus $G_\beta < G_\beta + x - g < G_\beta$, which is impossible. Therefore $x \in G_\delta$ and hence G_δ is convex. Moreover, since G^δ is the intersection of convex subgroups it must also be convex.

We have not yet made use of our hypothesis that Δ is a root system, but we shall make repeated use of it in determining when a Δ-order is a lattice-order. Note that if Δ is a chain, then each Δ-order of G is a total order. Thus *a group H can be totally ordered if and only if it contains a chain of subgroups $\{H_\delta \mid \delta \in \Delta\}$ that satisfy* (1) *through* (6). This is entirely similar to the results of Podderyugin [9] and Rieger [10] about o-groups.

We shall say that $\delta \in \Delta_g$ is a *positive* (*negative*) *value of* g if $G_\delta + g > G_\delta$ ($G_\delta + g < G_\delta$). For each subset Π of Δ let Π' be the *ideal* determined by Π,

$$\Pi' = \{\delta \in \Delta \mid \delta \leqq \gamma \quad \text{for some} \quad \gamma \in \Pi\}.$$

PROPOSITION 3.2. *If Π is the set of positive values* of $g \in G$ *in a Δ-order of G and if $h \in \cap\, G_\gamma$ $(\gamma \in \Delta\backslash\Pi')$, and $h - g \in \cap\, G_\gamma$ $(\gamma \in \Pi')$ then $h = g \vee O$.*

PROOF. We first show that $\Pi = \Delta_h$. If $\delta \in \Pi$, then $G_\delta + h = G_\delta + g > G_\delta$ and hence $h \in G^\delta \backslash G_\delta$. Conversely suppose that $\delta \in \Delta_h$. If $\delta \in \Delta\backslash\Pi'$, then $h \in G_\delta$ and if $\delta \in \Pi'\backslash\Pi$, then $\delta < \gamma \in \Pi \subseteq \Delta_h$, both of which are impossible. Thus $\delta \in \Pi$ and hence $\Pi = \Delta_h$. In particular, h has only positive values and hence $h \geqq O$. In order to prove that $h \geqq g$ we must show that each value γ of $h-g$ is positive. If $\gamma \in \Pi'$, then $h - g \in G_\gamma$, a contradiction. If $\gamma \in \Delta\backslash\Pi'$, then $h \in G_\gamma$ and $G_\gamma + h - g = G_\gamma - g > G_\gamma$ (because γ must be a negative value of g). Therefore $h \geqq g$ and O.

Finally suppose that $c \in G$ and $h \neq c \geqq g$ and O, and consider $\gamma \in \Delta_{c-h}$. If $h \in G_\gamma$, then $G_\gamma + c - h = G_\gamma + c > G_\gamma$. If $h \notin G_\gamma$, then there exists $\gamma \leqq \beta \in \Delta_h = \Pi$ and hence $\gamma \in \Pi'$. Thus $G_\gamma + h - c = G_\gamma + g - c$ and hence $G_\gamma + c - h = G_\gamma + c - g > G_\gamma$ because $c \geqq g$. Thus all the values of $c - h$ are positive and hence $c \geqq h$. Therefore $h = g \vee O$.

In more detail the hypothesis of this proposition is

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(i) $\Delta_h \subseteqq \Delta_g$,
(ii) if γ is a negative value of g, then $h \in G_\alpha$ for all $\gamma \geqq \alpha \in \Delta$, and
(iii) if γ is a positive value of g, then $h-g \in G_\alpha$ for all $\gamma \geqq \alpha \in \Delta$.

Proposition 3.9 in [4] is the corresponding result for l-groups.

THEOREM 3.1. *A Δ-order of G is a lattice-order and the G_δ are prime convex l-subgroups if and only if*

(7) *for each $g \in G$, there exists $h \in \cap G_\gamma$ ($\gamma \in \Delta \backslash \Pi'$) such that $h-g \in \cap G_\gamma$ ($\gamma \in \Pi'$), where Π is the set of positive values of g.*

If this is the case, then $h = g \vee O$.

PROOF. First suppose that (7) is satisfied. Then by Proposition 3.2, $h = g \vee O$ exists for each $g \in G$ and so G is an l-group. Also if $g \in G_\delta$, then no value of $g \vee O$ exceeds δ and so $g \vee O \in G_\delta$. Therefore G_δ is an l-subgroup of G and hence by Proposition 3.1, G_δ is a convex l-subgroup of G. Suppose (by way of contradiction) that G_δ is not prime. Then there exists $a, b \in G^+ \backslash G_\delta$ such that $a \wedge b = O$. Let $\alpha(\beta)$ be the value of $a(b)$ that is $\geqq \delta$ and without loss of generality assume that $\alpha \geqq \beta$. If $\alpha = \beta$, then $G_\alpha + a$ and $G_\alpha + b$ are strictly positive in the o-group G^α/G_α and hence $G_\alpha = G_\alpha + a \wedge b = G_\alpha + a \wedge G_\alpha + b = \min [G_\alpha + a, G_\alpha + b] > G_\alpha$, a contradiction. If $\alpha > \beta$, then α is a positive value of $a-b$ and hence by Proposition 3.2, $G_\beta + a - b = G_\beta + (a-b) \vee O > G_\beta$ in the po set of right cosets of G_β. Thus $G_\beta + a > G_\beta + b$ and hence $G_\beta = G_\beta + a \wedge b = G_\beta + a \wedge G_\beta + b = G_\beta + b > G_\beta$, a contradiction. Therefore each G_δ is a prime convex l-subgroup of G.

Conversely suppose that G is an l-group and the G_δ are prime convex l-subgroups of G. Then it follows from the proof of Proposition 3.9 in [4] that (7) is satisfied.

COROLLARY. *If for each $g \in G$ and each subset Π of Δ_g there exists $h \in \cap G_\gamma$ ($\gamma \in \Delta \backslash \Pi'$) such that $h-g \in \cap G_\gamma$ ($\gamma \in \Pi'$), then every Δ-order of G is a lattice order for which the G_δ are prime convex l-subgroups.*

Theorem 3.6 provides a converse to this corollary. It is easy to show that a wreath product G of one torsion free abelian group by another has a natural set of subgroups $\{G_\delta \mid \delta \in \Delta\}$ that satisfy (1) through (6) and also satisfy the hypothesis of this corollary. Thus each Δ-order of G is a lattice-order. However, since G contains elements a and b such that $-a+b+a = -b$, it is clear that G does not admit a total order.

Suppose that we have a Δ-order for G that satisfies (7) and let Λ be the set of all regular convex l-subgroups M of G such that $M \supseteqq G_\delta$ for some $\delta \in \Delta$. Then it is easy to verify that Λ is a normal plenary subset of $\Gamma_1(G)$ and hence the Δ-order is a normal lattice order. We have proven the following theorem.

THEOREM 3.2. *A group H admits a normal lattice-order if and only if there exists a set $\{H_\delta | \delta \in \Delta\}$ of proper subgroups of H that satisfy* (1) *through* (7).

This result is not too satisfactory because whether or not (7) is satisfied depends upon the particular choice of the total order for the G^δ/G_δ. For the remainder of this section we derive conditions that insure that all the Δ-orderings of G satisfy (7). Note that the Corollary to Theorem 3.1 provides one such condition.

An element $\delta \in \Delta$ will be called *special* if there exist $g \in G^\delta \backslash G_\delta$ whose only value is δ. In this case g will be called *Δ-special*. Note that the hypothesis in the Corollary to Theorem 3.1 forces each $\delta \in \Delta$ to be special.

LEMMA 3.1. *If a and b are Δ-special with values $\alpha \parallel \beta$ respectively, then $a+b = b+a$ and $\Delta_{a+b} = \{\alpha, \beta\}$.*

PROOF. Let $C = \cap G_\gamma$ $(\gamma \leqq \alpha)$ and $D = \cap G_\gamma$ $(\gamma \leqq \beta)$. Then since b has only the value β it follows that $b \in C$, and similarly $a \in D$. Since $G_\alpha \lhd G^\alpha$ it follows that an inner automorphism of G by an element in G^α must induce a permutation on $\{\gamma \in \Delta | \gamma \leqq \alpha\}$. Therefore $C \lhd G^\alpha$ and similarly $D \lhd G^\beta$. $D+g = D+a+b = D+b$, where $g = a+b$, and hence $a+b = g = b+d$, where $d \in D$.

If $\gamma \leqq \beta$, then $G_\gamma+a = G_\gamma = G_\gamma+d$. If $\delta \nleqq \beta$, then $b \in G_\delta$. If $a \in G_\delta$, then $d = -b+a+b \in G_\delta$ and hence $G_\delta+a = G_\delta = G_\delta+d$. If $a \notin G_\delta$, then $\delta \leqq \alpha$ and hence $C \subseteqq G_\delta$. But then $C+a = C+a+b = C+b+d = C+d$ and hence $G_\delta+a = G_\delta+d$. Therefore $G_\delta+a = G_\delta+d$ for all $\delta \in \Delta$ and hence $a = d$.

Since $a \in G_\beta$, it follows that $\beta \in \Delta_g$ and similarly $\alpha \in \Delta_g$. Consider $\gamma \in \Delta_g$. If $a, b \notin G_\gamma$, then $\alpha, \beta \geqq \gamma$ and $\alpha \parallel \beta$, but this contradicts the fact that Δ is a root system. Thus exactly one of a and b belongs to G_γ and it follows that $\gamma = \alpha$ or $\gamma = \beta$.

THEOREM 3.3. *If Δ_g is finite for each $g \in G$, then the following are equivalent.*

(8) *Each $0 \neq g \in G$ has a representation $g = g_1 + \cdots + g_n$, where g_i is Δ-special with value δ_i and $\delta_i \parallel \delta_j$ if $i \neq j$.*

(b) *Each Δ-order of G is a lattice-order for which the G_δ are prime convex l-subgroups.*

(c) *There exists a Δ-order of G that is a lattice-order and such that the G_δ are prime convex l-subgroups.*

PROOF. (8) $\rightarrow$ (b). Select a Δ-order for G and consider $0 \neq g \in G$. Then $g = g_1 + \cdots + g_n$, where g_i is Δ-special with value δ_i and $\delta_i \parallel \delta_j$ if $i \neq j$. By Lemma 3.1 $\Delta_g = \delta_1, \cdots, \delta_n$ and $g_i+g_j = g_j+g_i$. Thus we may assume that $\Pi = \delta_1, \cdots, \delta_k$ $(k \leqq n)$ is the set of positive values of g. To complete the proof it suffices to show that $h = g_1 + \cdots + g_k \in \cap G_\gamma$ $(\gamma \in \Delta \backslash \Pi')$. and $h - g \in \cap G_\gamma$ $(\gamma \in \Pi')$. But this follows at once from the fact that $g_1, \cdots, g_k \in \cap G_\gamma$ $(\gamma \in \Delta \backslash \Pi')$ and $g_{k+1}, \cdots, g_n \in \cap G_\gamma$ $(\gamma \in \Pi')$.

(c) $\rightarrow$ (8). Select a Δ-order for G that is a lattice-order and such that each G_δ is a prime convex l-subgroup of G. Let Λ be the set of regular convex l-subgroups M of G such that $M \supseteqq G_\delta$ for some $\delta \in \Delta$. Then Λ is a plenary subset of $\Gamma_1(G)$ and each $g \in G$ has at most a finite number of values in Λ. (8) is now an immediate consequence of Theorem 3.7 in [4].

Remark. Suppose that G satisfies (8) and that we have selected a Δ-order for G. Then from the theory in [4], Theorem 2.1 and Theorem 3.3 we have the following results. The representation of g given in (8) is unique and $g \vee O$ is just the sum of the positive g_i. $\Gamma_1(G) = \Lambda$ and the lattice $\mathscr{M}$ of all convex l-subgroups of G is freely generated by Λ.

It is easy to show by an example that even if G is abelian and generated by its Δ-special elements, (8) need not be satisfied.

Theorem 3.4. *There exists a lattice-ordering of a group H such that each element has at most a finite number of values if and only if there exists a set $\{H_\delta | \delta \in \Delta\}$ of proper subgroups of H that satisfy properties* (1) *through* (6) *and* (8). *If this is the case, then each Δ-order of H is a lattice-order.*

Proof. Suppose there exists a lattice-order for H such that each element has at most a finite number of values. Then each element in $\Gamma_1(H)$ is special ([4] Theorem 3.9) and hence $H_\gamma \lhd H^\gamma$ for all $\gamma \in \Gamma_1(H)$. Thus $\{H_\gamma | \gamma \in \Gamma_1(H)\}$ satisfies (1) through (6), and (8) follows from Theorem 2.1 or from Theorem 3.7 in [4]. The converse is an immediate consequence of Theorem 3.3.

Theorem 3.5. *If there exists a Δ-order of G that is a lattice-order and for which the G_δ are prime convex l-subgroups, then the following are equivalent.*

(a) *G is representable* (*as a subdirect sum of a cardinal sum of* 0-*groups*).

(b) *For each $\delta \in \Delta$ and $g \in G$, $-g + G_\delta + g \subseteqq G_\delta$ or $\supseteqq G_\delta$.*

Proof. Byrd [1] has shown that if M is a prime convex l-subgroup of a representable l-group H, then $-h + M + h$ is comparable with M for all $h \in H$, and hence it follows that (a) implies (b). Conversely suppose that (b) is satisfied. Let $N_\delta = \cap \{-g + G_\delta + g | g \in G\}$. Then $N_\delta \lhd G$ and since N_δ is the intersection of a chain of prime convex l-subgroups, N_δ is a prime convex l-subgroup ([4] Theorem 3.2). Thus G/N_δ is an o-group. The natural homomorphism of G into the large cardinal sum of the G/N_δ is a representation of G.

Theorem 3.6. *If for each $\delta \in \Delta$ and $g \in G$, $-g + G_\delta + g \subseteqq G_\delta$ or $\supseteqq G_\delta$, then the following are equivalent.*

(a) *Each Δ-order of G is a lattice-order for which the G_δ are prime convex l-subgroups.*

(b) *If $g \in G^\gamma \backslash G_\gamma$ for all $\gamma \in \Pi \subseteqq \Delta$, then there exists $h \in \cap G_\gamma (\gamma \in \Delta \backslash \Pi')$ such that $h - g \in \cap G_\gamma$ $(\gamma \in \Pi')$.*

(c) *If Q_1 and Q_2 are dual ideals of Δ such that $Q_1 \cup Q_2 = \Delta$, then*

$$\bigcap_{Q_1 \cap Q_2} G_\delta = \bigcap_{Q_1} G_\delta \oplus \bigcap_{Q_2} G_\delta .$$

PROOF. (a) $\rightarrow$ (b). If α and β are values of g, then $G_\alpha \parallel G_\beta$ and hence G_α and G_β are not conjugate. Thus we can choose a Δ-order for G so that $G_\gamma + g > G_\gamma$ for all $\gamma \in \Pi \subseteqq \Delta_g$ and $G_\gamma + g < G_\gamma$ for all other values γ of g. Then by Theorem 3.1, $h = g \vee O$ satisfies (b).

(b) $\rightarrow$ (c). Since $Q_1 \cap Q_2 \subseteqq Q_1$ and Q_2

$$\bigcap_{Q_1 \cap Q_2} G_\delta \supseteqq \bigcap_{Q_1} G_\delta \text{ and } \bigcap_{Q_2} G_\delta$$

and since

$$\cap \{G_\delta | \delta \in \Delta = Q_1 \cup Q_2\} = 0$$

we have

$$(\bigcap_{Q_1} G_\delta) \cap (\bigcap_{Q_2} G_\delta) = 0.$$

In order to show that $\cap G_\delta$ $(\delta \in Q_1)$ is normal in $\cap G_\delta$ $(\delta \in Q_1 \cap Q_2)$ it suffices to show that if $g \in \cap G_\delta$ $(\delta \in Q_1 \cap Q_2)$, then the inner automorphism of G determined by g induces a mapping of Q_1 into itself. If there exists $\delta \in Q_1$ such that $-g + G_\delta + g = G_\beta$, where $\beta \notin Q_1$, then $\beta \in Q_2$ and $\delta > \beta$. Thus $\delta \in Q_1 \cap Q_2$ and hence $g \in G_\delta$, but then $-g + G_\delta + g = G_\delta$, a contradiction. Thus in general

$$\bigcap_{Q_1 \cap Q_2} G_\delta \supseteqq \bigcap_{Q_1} G_\delta \oplus \bigcap_{Q_2} G_\delta .$$

Consider $g \in \cap G_\delta$ $(\delta \in Q_1 \cap Q_2)$ and let $\Pi_1(\Pi_2)$ be the set of all values of g in $Q_1(Q_2)$, and let Π_1' and Π_2' be the corresponding ideals in Δ. Since $Q_1 \cup Q_2 = \Delta, \Pi_1 \cup \Pi_2$ is the set of all values of g.

$$\Pi_1' \subseteqq Q_1 \subseteqq \Delta \backslash \Pi_2' \text{ and } \Pi_2' \subseteqq Q_2 \subseteqq \Delta \backslash \Pi_1'.$$

For if $\alpha \in \Pi_1'$, then $\alpha \leqq \beta \in \Pi_1 \subseteqq Q_1$, and if $\alpha \notin Q_1$, then $\alpha \in Q_2$ and hence $\beta \in Q_1 \cap Q_2$. But then $g \in G_\beta$ which contradicts the fact that β is a value of g. If $\gamma \in Q_1 \cap \Pi_2'$, then $\gamma \leqq \delta \in \Pi_2 \subseteqq Q_2$ and hence $\delta \in Q_1 \cap Q_2$, which is impossible.

By (b) there exist elements h and k in G such that

$$h \in \bigcap_{\Delta \backslash \Pi'_1} G_\gamma \subseteqq \bigcap_{Q} G_\gamma \text{ and } h - g \in \bigcap_{\Pi'_1} G_\gamma$$

$$k \in \bigcap_{\Delta \backslash \Pi'_2} G_\gamma \subseteqq \bigcap_{Q_1} G_\gamma \text{ and } k - g \in \bigcap_{\Pi'_2} G_\gamma .$$

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To complete the proof it suffices to show that $g = k+h$. If $\gamma \in \Pi_1'$, then $k, h-g \in G_\gamma$ and if $\gamma \in \Pi_2'$, then $h, k-g \in G_\gamma$. If $\gamma \notin \Pi_1' \cup \Pi_2'$, then $h, k \in G_\gamma$, and if $g \notin G_\gamma$, then there exists a value β of g such that $\gamma \leqq \beta \in \Pi_1 \cup \Pi_2$ and so $\gamma \in \Pi_1' \cup \Pi_2'$, a contradiction. Thus using the fact that $h+k = k+h$ we have that $h+k-g \in \cap G_\gamma = 0$ $(\gamma \in \Delta)$ and hence $g = k+h$.

(c) $\to$ (a). We assume that G has a Δ-order and pick $0 \neq g \in G$. Let Π be the set of all positive values of g. By Theorem 3.1 it suffices to show that there exists $h \in \cap G_\gamma$ $(\gamma \in \Delta\backslash\Pi')$ such that $h-g \in \cap G_\gamma$ $(\gamma \in \Pi')$. Let

$$Q_1 = \{\delta \in \Delta \mid \delta \geqq \gamma \text{ for some } \gamma \in \Pi'\} \supseteqq \Pi' \text{ and } Q_2 = \Delta\backslash\Pi'.$$

Then Q_1 and Q_2 are dual ideal ideals of Δ and $Q_1 \cup Q_2 = \Delta$.

Suppose (by way of contradiction) that $g \notin G_\gamma$ for some $\gamma \in Q_1 \cap Q_2$. Then there exists a value β of g such that $\beta \geqq \gamma$. Since $\gamma \in Q_2$, $\gamma \notin \Pi'$ and hence β is a negative value of g. Since $\gamma \in Q_1$, $\beta \geqq \gamma \geqq \delta \in \Pi'$ and hence β is a positive value of g, which is impossible. Therefore

$$g = k+h \in \bigcap_{Q_1 \cap Q_2} G_\gamma = \bigcap_{Q_1} G_\gamma \oplus \bigcap_{Q_2} G_\gamma,$$

where $k \in \cap G_\gamma$ $(\gamma \in Q_1)$ and $h \in \cap G_\gamma$ $(\gamma \in Q_2 = \Delta\backslash\Pi')$. If $\gamma \in \Pi'$, then $k \in G_\gamma$ and hence $h-g \in \cap G_\gamma$ $(\gamma \in \Pi')$. This completes the proof of Theorem 3.6.

We can use Theorems 3.5 and 3.6 to characterize a class of groups that admit a representable lattice-order, but instead we shall only apply Theorem 3.6 to representable l-groups. Let H be a representable l-group and let Δ be a normal plenary subset of $\Gamma_1(H)$. Then $\{H_\gamma | \gamma \in \Delta\}$ satisfies properties (1) through (7) and $-h+H_\gamma+h$ is comparable with H_γ for all $\gamma \in \Delta$ and $h \in H$. Also since the given lattice-order is a Δ-order, we have the equivalence of (a), (b) and (c) in Theorem 3.6.

THEOREM 3.7. *If H is a representable l-group and Δ is a normal plenary subset of $\Gamma_1(H)$, then each of (a), (b) and (c) is equivalent to*

(d) *each $\delta \in \Delta$ is special and if $\bigvee a_\alpha$ exists, where $\{a_\alpha | \alpha \in A\}$ is a disjoint set of positive elements of H, then $\bigvee a_\beta$ exists for each set $\{a_\beta | \beta \in B\} \subseteqq \{a_\alpha | \alpha \in A\}$.*

PROOF. (b) $\to$ (d). Consider $0 \leqq h \in H^\delta\backslash H_\delta$, where $\delta \in \Delta$. By (b) there exists $k \in \cap H_\gamma$ $(\gamma \nleqq \delta)$ such that $k-h \in \cap H_\gamma$ $(\gamma \leqq \delta)$. In particular δ is the only value of k, and hence each $\delta \in \Delta$ is special. By Theorem 2.1, $h = \bigvee h_\lambda$, where $\{h_\lambda | \lambda \in \Lambda\}$ is a set of disjoint special elements. Moreover, the values of h in Δ are precisely the values of the h_λ. Let Φ be a subset of Λ and let Π be the set of all the values of the h_λ for $\lambda \in \Phi$. Then by (b) there exists $k \in \cap H_\gamma$ $(\gamma \in \Delta\backslash\Pi')$ such that $k-h \in \cap H_\gamma$ $(\gamma \in \Pi')$. Now by Theorem 2.1, k has a unique representation as the join of disjoint special elements. Thus it follows that $k = \bigvee h_\lambda$ $(\lambda \in \Phi)$, and hence it is clear that (d) is satisfied.

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(d) $\rightarrow$ (a). By Theorem 2.1, each positive element in H has a unique representation as the join of a set of disjoint special elements. Consider $h \in H$.

$$h = h \vee 0 - (-h \vee 0) = \bigvee_A a_\alpha - \bigvee_B b_\beta$$

where $\{a_\alpha \mid \alpha \in A\}$ and $\{b_\beta \mid \beta \in B\}$ are sets of disjoint special elements and $h \vee 0 = \bigvee a_\alpha$ and $-h \vee 0 = \bigvee b_\beta$. (If $h \vee 0 = 0$, then we let $A = \square$ and $\bigvee_A a_\alpha = 0$.) Without loss of generality we may assume that α is the value of a_α in Δ for each $\alpha \in A$ and that β is the value of b_β in Δ for each $\beta \in B$. In particular, $A \cup B$ is the set of values of h in Δ.

Let Π be the set of positive values of h in a *new* Δ-order of H. Then $\Pi = A^* \cup B^*$, where $A^* \subseteqq A$ and $B^* \subseteqq B$. It follows by an easy computation that

$$k = \bigvee_{A^*} a_\alpha - \bigvee_{B^*} b_\beta$$

satisfies property (7). Thus the new Δ-order of H is a lattice-order and the H_δ are prime convex l-subgroups. Therefore (a) is satisfied.

In the proof that (d) implies (a) we did not use the fact that H is a representable l-group, hence we have the following corollary.

COROLLARY. *If H is an l-group with a normal plenary subset Δ of $\Gamma_1(H)$ that satisfies* (d), *then each Δ-order of H is a lattice-order for which the H_δ are prime convex l-subgroups.*

4. Lattice-orderings of an abelian group

If G is an abelian group, then conditions (5) and (6) used in the definition of a Δ-order of G are trivially satisfied and condition (4) simply states that each group $G^\delta \backslash G_\delta$ is torsion free. The latter is equivalent to the hypothesis that G is torsion free and each of the G_δ is a pure subgroup of G. Thus for the remainder of this section we shall assume that *G is a torsion free abelian group with a set $\{G_\delta \mid \delta \in \Delta\}$ of proper pure subgroups of G that satisfies the conditions* (1), (2) *and* (3) from section 3.

Using the fact that the set consisting only of the zero subgroup satisfies our hypothesis, Theorem 3.2 simply states that an abelian group admits a lattice-order if and only if it is torsion free, and Theorem 3.5 states that each lattice-ordered abelian group is representable. The other theorems all establish significant relationships between a lattice-order for G and the subgroup structure of G.

Let $V = V(\Delta, G^\delta/G_\delta)$ be the set of all Δ-vectors $v = (\cdots, v_\delta, \cdots)$, where $v_\delta \in G^\delta/G_\delta$, for which the support $S_v = \{\delta \in \Delta \mid v_\delta \neq G_\delta\}$ contains no infinite ascending chains. Choose a Δ-order for G and define $O \neq v \in V$

to be positive if $v_\gamma > G_\gamma$ for every maximal element γ of S_v (such a v_γ will be called a *maximal component* of v). Then V is an l-group ([2] Theorem 2.2).

THEOREM 4.1. *If G is divisible, then there exists an o-isomorphism μ of G into $V(\Delta, G^\delta/G_\delta)$ such that δ is a value of g if and only if $(g\mu)_\delta$ is a maximal component of $g\mu$ and if this is the case $(g\mu)_\delta = G_\delta + g$.*

PROOF. G is a Γ-group, with $\Gamma = \Delta$ (see [3] p. 3) and hence by the embedding theorem in [3], there exists an isomorphism μ of G into V with the above property. It follows from the definition of the Δ-order for G and the lattice order for V that for $O \neq g \in G$ the following are equivalent.

(a) g is positive in G.

(b) $G_\delta + g > G_\delta$ for each $\delta \in \Delta_g$.

(c) All maximal components of $g\mu$ are positive.

(d) $g\mu$ is positive in V.

Therefore μ is an o-isomorphism of G into V, and we shall call such an o-isomorphism a *Δ-isomorphism.*

If we remove from (2) the assumption that $\{G_\alpha \mid \alpha \in \Delta \text{ and } G_\alpha \supset G_\delta\}$ is a chain, then $V(\Delta, G^\delta/G_\delta)$ is a *po*-group and Theorem 4.1 together with its proof remains valid.

As is usual when dealing with abelian *po*-groups we can dispense with the devisibility assumption. For let $\bar{G}$ be the *d-closure* of G (that is, the minimal divisible abelian group that contains G) and let $\bar{G}^\delta$ ($\bar{G}_\delta$) be the d-closure of G^δ (G_δ) in $\bar{G}$. Then $\bar{G}^+ = \{\bar{g} \in \bar{G} \mid n\bar{g} \in G^+ \text{ for some } n > 0\}$ is the minimal semiclosed *po* of $\bar{G}$ that contains the given Δ-order of G. The mapping τ of $G_\delta + g$ upon $\bar{G}_\delta + g$ is an isomorphism of G^δ/G_δ into $\bar{G}^\delta/\bar{G}_\delta$ and $\bar{G}^\delta/\bar{G}_\delta$ is the d-closure of $(G^\delta/G_\delta)\tau$. Thus there is a unique extension of the total order of $(G^\delta/G_\delta)\tau$ to a total order of $\bar{G}^\delta/\bar{G}_\delta$ and it is easy to verify that

$$\bar{G}^+ = \{\bar{g} \in \bar{G} \mid \bar{G}_\gamma + \bar{g} > \bar{G}_\gamma \text{ for all } \gamma \in \Delta_{\bar{g}}\}.$$

COROLLARY I. *There exists a Δ-isomorphism of G into $V(\Delta, \bar{G}^\delta/\bar{G}_\delta)$.*

COROLLARY II. *If G satisfies* (7), *then each Δ-isomorphism of G into $V(\Delta, \bar{G}^\delta/\bar{G}_\delta)$ is necessarily and l-isomorphism.*

PROOF. Let μ be a Δ-isomorphism of G into V and consider $g \in G$. It suffices to show that $(g \vee O)\mu = g\mu \vee O$. Let Π be the set of positive values of g in Δ and let $h = g \vee O$. If $\gamma \in \Delta \backslash \Pi'$, then (by (7)) $h \in G_\gamma$ and hence $(h\mu)_\gamma = \bar{G}_\gamma$. For if $(h\mu)_\gamma \neq \bar{G}_\gamma$, then there exists a maximum component $(h\mu)_\beta$ of $h\mu$ such that $\beta \geqq \gamma$ and hence $h \in G^\beta \backslash G_\beta$, a contradiction. If $\gamma \in \Pi'$ then $h - g \in G_\gamma$ and so $(h\mu)_\gamma - (g\mu)_\gamma = ((h-g)\mu)_\gamma = \bar{G}_\gamma$. Thus we have

$$(h\mu)_\gamma = \begin{cases} (g\mu)_\gamma \text{ if } \gamma \leqq \beta, \\ \qquad \text{where } (g\mu)_\beta \text{ is a maximal positive component of } g\mu \\ \bar{G}_\gamma \text{ otherwise,} \end{cases}$$

but this means that $(g \vee O)\mu = h\mu = g\mu \vee O$.

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