ON THE VANISHING OF A (G, σ) PRODUCT IN A (G, σ) SPACE

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In this paper, we shall construct a vector space, called the (G, σ) space, which generalizes the tensor space, the Grassman space, and the symmetric space. Then we shall determine a necessary and sufficient condition that the (G, σ) product of the vectors $x_1, x_2, \ldots, x_n$ is zero.

1. Let G be a permutation group on $I = \{1, 2, \ldots, n\}$ and F, an arbitrary field. Let σ be a linear character of G, i.e., σ is a homomorphism of G into the multiplicative group F^* of F.

For each $i \in I$, let V_i be a finite-dimensional vector space over F. Consider the Cartesian product $W = V_1 \times V_2 \times \ldots \times V_n$.

1.1. *Definition.* W is called a G-set if and only if $V_i = V_{g(i)}$ for all $i \in I$, and for all $g \in G$.

1.2. *Definition.* A mapping f: $W \rightarrow U$, where U is a vector space over F, is called (G, σ) if and only if $(w_1, w_2, \ldots, w_n)f = \sigma(g)(w_{g(1)}, w_{g(2)}, \ldots, w_{g(n)})f$ for all $g \in G$, and $w_i \in V_i$, $i = 1, 2, \ldots, n$.

1.3. *Definition.* A vector space T over F is called a (G, σ) space of W if and only if there exists a mapping τ of W into T such that:

(i) τ is multilinear and (G, σ),

(ii) T has a "universal mapping property", i.e., if U is any vector space over F and f is any multilinear and (G, σ) mapping of W into U, then there exists a unique linear transformation $\bar{f}$ of T into U such that $\tau\bar{f} = f$.

1.4. THEOREM. *Given G, σ, and a G-set W, there exists a (G, σ) space over an arbitrary field F. Any two (G, σ) spaces are isomorphic as vector spaces.*

Proof. Let $F(W)$ denote the free vector space generated by W over an arbitrary field F. Let Ω be the smallest subspace of $F(W)$ generated by the elements of the form

$$(w_1, \ldots, \alpha w_i + \beta w_i', \ldots, w_n) - \alpha(w_1, \ldots, w_i, \ldots, w_n) - \beta(w_1, \ldots, w_i', \ldots, w_n)$$

and $(w_1, w_2, \ldots, w_n) - \sigma(g)(w_{g(1)}, w_{g(2)}, \ldots, w_{g(n)})$, for all i, $i = 1, 2, \ldots, n$,

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and all $g \in G$. Let $T = F(W)/\Omega$ be the quotient space and η the natural linear transformation of $F(W)$ onto T. If we take τ to be the restriction of η to W, then one can easily verify that T is a (G, σ) space, with τ as a multilinear and (G, σ) mapping of W into T.

The uniqueness of T, up to isomorphism, follows easily from the definition of a (G, σ) space.

In view of 1.4, we shall call T, the (G, σ) space of W and denote it by $P(W, G, \sigma)$.

1.5. By taking particular values for G and σ, we obtain the classical spaces; for instance:

If $G = \{e\}$, then

$$P(W, G, 1) = \bigotimes_{i=1}^{n} V_i,$$

the tensor space;

If $G = S_n$, and $\sigma(g) = 1$ if g is an even permutation and -1 if g is an odd permutation, then $P(W, G, \sigma) = \bigwedge^n V$, the Grassman space;

If $G = S_n$ and $\sigma(g) = 1$ for all $g \in G$, then $P(W, G, \sigma) = V_{(n)}$, the symmetric space.

1.6. *Notation.* If $(w_1, w_2, \ldots, w_n) \in W$, we shall denote its image $(w_1, w_2, \ldots, w_n)\tau$ under τ by $w_1 \triangle w_2 \triangle \ldots \triangle w_n$ and call it the (G, σ) product of $w_1, w_2, \ldots, w_n$.

1.7. We shall now determine a necessary and sufficient condition that $w_1 \triangle w_2 \triangle \ldots \triangle w_n = 0$. The conditions are known for the classical spaces (see [**1**; **2**; **3**; **4**]). For the symmetric space $V_{(n)}$, this result is derived in [**2**] under the restriction that V is an n-dimensional unitary space.

2. Let U be a vector space over F, such that

$$\dim U \geqq \max\{\dim V_i, 1 \leqq i \leqq n\}.$$

Consider $W' = U \times U \times \ldots \times U$ (n copies) and $P(W', G, \sigma)$. If $f_i\colon V_i \to U$, $1 \leqq i \leqq n$, are monomorphisms with $f_i = f_j$, whenever $V_i = V_j$, then they induce an embedding of $P(W, G, \sigma)$ into $P(W', G, \sigma)$, such that the product $w_1 \triangle w_2 \triangle \ldots \triangle w_n$ in $P(W, G, \sigma)$ is mapped into the product $f_1(w_1) \triangle f_2(w_2) \triangle \ldots \triangle f_n(w_n)$ in $P(W', G, \sigma)$. Therefore, without any restriction on the generality of the problem, we can assume that $V_1 = V_2 = \ldots = V_n = V$ (say). Then $W = V \times V \times \ldots \times V$ (n copies). Let $\dim V = m$ and let $\{y_1, y_2, \ldots, y_m\}$ be a basis of V.

3. Some definitions.

3.1. An element $(w_1, w_2, \ldots, w_n) \in W$ is called a (G, σ) element if and only if there exists $g \in G$ such that $\sigma(g) \neq 1$ and $w_i, w_{g(i)}$ are linearly dependent for all $i \in I$.

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3.2. Two elements $(w_1, w_2, \ldots, w_n)$ and $(w_1', w_2', \ldots, w_n')$ in W are said to be G-related if and only if there exists $g \in G$ such that $w_i' = w_{g(i)}$ for all $i \in I$.

3.3. An element $(w_1, w_2, \ldots, w_n) \in W$ is said to have the property P, if and only if for each $i \in I$ and $i \geqq 2$, either $w_i \in \{w_1, w_2, \ldots, w_{i-1}\}$ or w_i is independent of the set $\{w_1, w_2, \ldots, w_{i-1}\}$.

3.4. An element $(w_1, w_2, \ldots, w_n) \in W$ is called a trivial element if and only if $w_i = 0$ for some i.

4. If $(w_1, w_2, \ldots, w_n) \in W$, is a trivial element, then clearly

$$w_1 \triangle w_2 \triangle \ldots \triangle w_n = 0.$$

4.1. THEOREM. *If $(w_1, w_2, \ldots, w_n) \in W$ is a non-trivial element, then it can be expressed in the form*

$$(w_1, w_2, \ldots, w_n) = \omega + \sum_{i=1}^{k} \alpha_i T_i$$

for some non-negative integer k, where $\omega \in \Omega$, $\alpha_i \in F$, $T_i \in W$, and for each i, T_i has the property P *and if $i \neq j$, then T_i and T_j are not G-related.*

Proof. Let $\{y_1, y_2, \ldots, y_m\}$ be a basis of V. For each $i \in I$, let $w_i = \sum_{j=1}^{m} b_{i,j} y_j$, and set $A_i = \{j | 1 \leqq j \leqq m \text{ and } b_{ij} \neq 0\}$.

Since $(w_1, w_2, \ldots, w_n)$ is non-trivial, A_i will be non-empty. Let $S = A_1 \times A_2 \times \ldots \times A_n$ (Cartesian product). If $s \in S$ and

$$s = (s_1, s_2, \ldots, s_n),$$

let $b_s = b_{1,s_1} b_{2,s_2} \ldots b_{n,s_n}$. Clearly $b_s \neq 0$. Define an equivalence relation on S as follows. If $s, t \in S$, then $s \sim t$ if and only if there exists $g \in G$ such that $t_i = s_{g(i)}$ for all $i \in I$. Let $A(s)$ denote the equivalence class containing s and let E be a set consisting of representatives of each of the equivalence classes $\{A(s)\}$. Now

$$\begin{aligned}
(w_1, w_2, \ldots, w_n) &= \left(\sum_{s_1 \in A_1} b_{1,s_1} y_{s_1}, \sum_{s_2 \in A_2} b_{2,s_2} y_{s_2}, \ldots, \sum_{s_n \in A_n} b_{n,s_n} y_{s_n}\right) \\
&= \Bigg[\left(\sum_{s_1 \in A_1} b_{1,s_1} y_{s_1}, \sum_{s_2 \in A_2} b_{2,s_2} y_{s_2}, \ldots, \sum_{s_n \in A_n} b_{n,s_n} y_{s_n}\right) \\
&\quad - \sum_{s_1 \in A_1} b_{1,s_1}\left(y_{s_1}, \sum_{s_2 \in A_2} b_{2,s_2}, \ldots, \sum_{s_n \in A_n} b_{n,s_n} y_{s_n}\right)\Bigg] \\
&\quad + \sum_{s_1 \in A_1} b_{1,s_1}\Bigg[\left(y_{s_1}, \sum_{s_2 \in A_2} b_{2,s_2} y_{s_2}, \ldots, \sum_{s_n \in A_n} b_{n,s_n} y_{s_n}\right) \\
&\quad - \sum_{s_2 \in A_2} b_{2,s_2}\left(y_{s_1}, y_{s_2}, \ldots, \sum_{s_n \in A_n} b_{n,s_n} y_{s_n}\right)\Bigg] \\
&\quad + \ldots + \sum_{\substack{s_i \in A_i;\\ 1 \leqq i \leqq n}} b_{1,s_1} b_{2,s_2} \ldots b_{n,s_n} (y_{s_1}, y_{s_2}, \ldots, y_{s_n}).
\end{aligned}$$

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Since each term in the square bracket is in Ω, we have

$$(w_1, w_2, \ldots, w_n) = \omega_0 + \sum_{s\in S} b_s(y_{s1}, y_{s2}, \ldots, y_{sn}),$$

where $\omega_0 \in \Omega$ and $b_s \in F$. Now if $t \in A(s)$, then $t_i = s_{g(i)}$ for some $g \in G$ and all $i \in I$, or equivalently $s_i = t_{g^{-1}(i)}$. Hence

$$\begin{aligned}(y_{t1}, y_{t2}, \ldots, y_{tn}) = {} & [(y_{t1}, y_{t2}, \ldots, y_{tn}) - \sigma(g^{-1})(y_{tg^{-1}(1)}, \ldots, y_{tg^{-1}(n)})] \\ & + \sigma(g^{-1})(y_{tg^{-1}(1)}, y_{tg^{-1}(2)}, \ldots, y_{tg^{-1}(n)}) \\ = {} & \omega(t, g^{-1}) + \sigma(g^{-1})(y_{s1}, y_{s2}, \ldots, y_{sn}),\end{aligned}$$

where $\omega(t, g^{-1})$ is equal to the term within the square brackets and is in Ω. Therefore

$$\begin{aligned}(w_1, w_2, \ldots, w_n) = {} & \omega_0 + \sum_{s\in E} \sum_{t\in A(s)} b_t\omega(t, g^{-1}) \\ & + \sum_{s\in E} \sum_{t\in A(s)} \sigma(g^{-1}) b_t(y_{s1}, y_{s2}, \ldots, y_{sn}).\end{aligned}$$

Set

$$\omega = \omega_0 + \sum_{s\in E} \sum_{t\in A(s)} b_t w(t, g^{-1}), \qquad b_s' = \sum_{s\in E} \sum_{t\in A(s)} \sigma(g^{-1}) b_t,$$

and

$$T_s = (y_{s1}, y_{s2}, \ldots, y_{sn});$$

we then have $(w_1, w_2, \ldots, w_n) = \omega + \sum_{s\in E} b_s'T_s$, which is the required form, satisfying the conditions stated in the theorem.

4.2. In this section, we shall investigate the coefficients b_s', occurring in 4.1. Consider the $n \times m$ matrix $M = (b_{i,j})$, where $w_i = \sum_{j=1}^{m} b_{i,j}y_j$, for $i = 1, 2, \ldots, n$ and $w_i \neq 0$ for all i. For each $s \in S$, $s = (s_1, s_2, \ldots, s_n)$, we define an $n \times n$ matrix $M_s = (b_{i,s_j})$, obtained from M. Define

$$H_s = \{g \mid g \in G, \sigma(g) = 1 \text{ and } s_i = s_{g(i)} \text{ for all } i \in I\}.$$

Clearly H_s is a subgroup of G. Then the following propositions can be easily proved.

4.3. PROPOSITION. *If s and t are in S and $s \sim t$, then H_s and H_t are conjugate in G. In fact, if $t_i = s_{g(i)}$ for some $g \in G$ and all $i \in I$, then $H_t = g^{-1}H_s g$.*

4.4. PROPOSITION. *Let s and t be in S and $s \sim t$. Let g and h be in G such that $t_i = s_{g(i)}$ and $t_i = s_{h(i)}$ for all $i \in I$. If $(y_{s1}, y_{s2}, \ldots, y_{sn})$ is not a (G, σ) element, then $\sigma(g) = \sigma(h)$.*

For each $s \in S$, we have associated a matrix M_s and a subgroup H_s of G. Consider the coset decomposition of G with respect to H_s and let G_s be a set of representatives of these cosets. Let $\mathscr{S} = \{M_s \mid s \in S\}$. Define $D: \mathscr{S} \to F$, as

$$D(M_s) = \sum_{h\in G_s} \sigma(h^{-1}) b_{1,sh(1)} b_{2,sh(2)} \cdots b_{n,sh(n)}.$$

It can be easily verified that D is well-defined; i.e., it is independent of the set of representatives G_s.

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4.5. PROPOSITION. *Following the notation of* 4.1, *if* $s \in E$, *then* $b_s' = D(M_s)$. *Further, if* $(y_{s1}, y_{s2}, \ldots, y_{sn})$ *is not a* (G, σ) *element, and* $t \sim s$, *then* $D(M_t) = \sigma(g)D(M_s)$, *where* $g \in G$, *such that* $t_i = s_{g(i)}$ *for all* $i \in I$.

Proof. If $s = (s_1, s_2, \ldots, s_n) \in S$ and $g \in G$, we shall write

$$s_g = (s_{g(1)}, s_{g(2)}, \ldots, s_{g(n)}).$$

Moreover, $A(s) = \{t | t \in S, t \sim s\} = \{s_g | s_g \in S, g \in G\}$. But if s_g and s_h are in $A(s)$, and $H_s g = H_s h$, then $s_{g(i)} = s_{h(i)}$ for all $i \in I$, since $gh^{-1} \in H_s$. Therefore $A(s) = \{s_g | s_g \in S, g \in G_s\}$. Now

$$\begin{aligned} b_s' &= \sum_{t \in A(s)} \sigma(g^{-1}) b_t \\ &= \sum_{t \in A(s)} \sigma(g^{-1}) b_{1,t_1} b_{2,t_2}, \ldots, b_{n,t_n}, \end{aligned}$$

where $t = (t_1, t_2, \ldots, t_n)$

$$= \sum_{\substack{s_g \in S; \\ g \in G_s}} \sigma(g^{-1}) b_{1,s_{g(1)}} b_{2,s_{g(2)}} \ldots b_{n,s_{g(n)}}.$$

However,

$$\begin{aligned} & b_{1,s_{g(1)}} b_{2,s_{g(2)}} \ldots b_{n,s_{g(n)}} = 0 \\ & \Leftrightarrow b_{i,s_{g(i)}} = 0 \quad \text{for some } i \in I \\ & \Leftrightarrow s_{g(i)} \notin A_i \quad \text{for some } i \\ & \Leftrightarrow s_g \notin S. \end{aligned}$$

Hence

$$b_s' = \sum_{g \in G_s} \sigma(g^{-1}) b_{1,s_{g(1)}} b_{2,s_{g(2)}} \ldots b_{n,s_{g(n)}} = D(M_s),$$

which proves the first assertion. Since $t \in A(s)$, we have $t_i = s_{g(i)}$ for some $g \in G$ and all $i \in I$. Then

$$D(M_t) = \sum_{h \in G_t} \sigma(h^{-1}) b_{1,t_{h(1)}} b_{2,t_{h(2)}} \ldots b_{n,t_{h(n)}},$$

where G_t is a set of coset representatives of H_t in G. Since $t_i = s_{g(i)}$, we have $t_{h(i)} = s_{gh(i)}$ for all $i \in I$. Hence

$$D(M_t) = \sum_{h \in G_t} \sigma(h^{-1}) b_{1,s_{gh(1)}} b_{2,s_{gh(2)}} \ldots b_{n,s_{gh(n)}}. \tag{1}$$

By 4.3, $H_s = gH_t g^{-1}$, and therefore $[G:H_s] = [G:H_t]$. Also it can be easily shown that $G_s' = \{gh | h \in G_t\}$ is also a set of coset representatives of H_s in G. Therefore (1) becomes

$$\begin{aligned} D(M_t) &= \sum_{h \in G_t} \frac{\sigma(h^{-1}g^{-1})}{\sigma(g^{-1})} b_{1,s_{gh(1)}} b_{2,s_{gh(2)}} \ldots b_{n,s_{gh(n)}} \\ &= \frac{1}{\sigma(g^{-1})} \sum_{gh \in G_s'} \sigma(h^{-1}g^{-1}) b_{1,s_{gh(1)}} b_{2,s_{gh(2)}} \ldots b_{n,s_{gh(n)}} \\ &= \sigma(g) D(M_s), \end{aligned}$$

which completes the proof.

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We now restate the result in 4.1 as follows.

4.6. THEOREM. *If* $(w_1, w_2, \ldots, w_n)$ *is a non-trivial element in* W, *then* $(w_1, w_2, \ldots, w_n)$ *can be written in the form*

$$(2) \qquad (w_1, w_2, \ldots, w_n) = \omega + \sum_{s \in E} D(M_s)(y_{s_1}, y_{s_2}, \ldots, y_{s_n}),$$

where $\omega \in \Omega$, *and for each* $s \in E$, $(y_{s_1}, y_{s_2}, \ldots, y_{s_n})$ *has the property* P; *moreover, if* $s, t \in E$ *and* $s \neq t$, *then* $(y_{s_1}, y_{s_2}, \ldots, y_{s_n})$ *and* $(y_{t_1}, y_{t_2}, \ldots, y_{t_n})$ *are not G-related.*

We shall call (2) a representation of $(w_1, w_2, \ldots, w_n)$ with respect to the basis $\{y_1, y_2, \ldots, y_m\}$ of V.

4.7. *Remark* 1. If E' is another set of representatives of the equivalence classes, then

$$(w_1, w_2, \ldots, w_n) = \omega' + \sum_{s' \in E'} D(M_{s'})(y_{s_1'}, y_{s_2'}, \ldots, y_{s_n'}),$$

is another representation. By 4.5, if $s' \in A(s)$, and $(y_{s_1}, y_{s_2}, \ldots, y_{s_n})$ is not a (G, σ) element, then $D(M_s)$ and $D(M_{s'})$ are related by $D(M_{s'}) = \sigma(g)D(M_s)$, where $s_i' = s_{g(i)}$ for some $g \in G$ and all $i \in I$. Moreover, $\sigma(g)$ is uniquely determined by 4.4.

Remark 2. If $s = (s_1, s_2, \ldots, s_n) \in S$, then $(y_{s_1}, y_{s_2}, \ldots, y_{s_n})$ is a (G, σ) element if and only if $g \in G$ such that $\sigma(g) \neq 1$ and $s_i = s_{g(i)}$ for all $i \in I$.

5. Let $(v_1, v_2, \ldots, v_n)$ be a non-trivial element of W. For each $i \in I$, let $v_i = \sum_{j=1}^{m} a_{ij} y_j$. Consider the sets A_i and S, as defined in 4.1. For each $s \in S$, define f_s: $W \to F$ as follows. If $(w_1, w_2, \ldots, w_n) \in W$ and $w_i = \sum_{j=1}^{m} b_{ij} y_j$, $i = 1, 2, \ldots, n$, set

$$(w_1, w_2, \ldots, w_n)f_s = \sum_{g \in G_s} \sigma(g^{-1}) b_{1, s_{g(1)}} b_{2, s_{g(2)}} \ldots b_{n, s_{g(n)}},$$

where G_s is a set of representatives of the cosets of H_s in G.

One can easily show that f_s is well-defined; i.e., it is independent of the choice of G_s. Then we have the following simple lemma.

5.1. LEMMA. f_s *is multilinear and* (G, σ).

6. We now come to our main problem stated in 1.7. We shall first prove a special case of the problem in the following lemma.

6.1. LEMMA. *Let* $(v_1, v_2, \ldots, v_n)$ *be a non-trivial element in* W *which has the property* P. *Then* $v_1 \triangle v_2 \triangle v_n = 0$ *if and only if* $(v_1, v_2, \ldots, v_n)$ *is a* (G, σ) *element.*

Proof (*"if" part*). $(v_1, v_2, \ldots, v_n)$ being a (G, σ) element implies that there exists $g \in G$ such that $\sigma(g) \neq 1$ and v_i and $v_{g(i)}$ are dependent for all $i \in I$.

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Moreover, since $(v_1, v_2, \ldots, v_n)$ has the property P, we have $v_i = v_{g(i)}$ for all $i \in I$. Hence

$$(1 - \sigma(g))(v_1, v_2, \ldots, v_n) = (v_1, v_2, \ldots, v_n) - \sigma(g)(v_1, v_2, \ldots, v_n)$$
$$= (v_1, v_2, \ldots, v_n) - \sigma(g)(v_{g(1)}, v_{g(2)}, \ldots, v_{g(n)}) \in \Omega$$

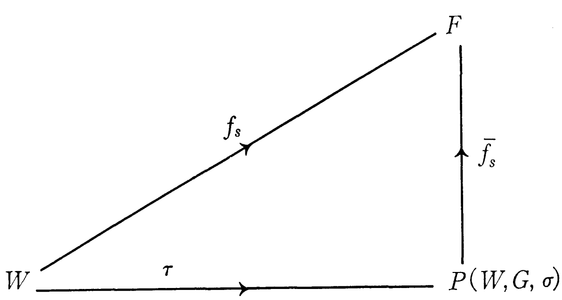
and since $\sigma(g) \neq 1$, we have $(v_1, v_2, \ldots, v_n) \in \Omega$ and hence

$$v_1 \triangle v_2 \triangle \ldots \triangle v_n = (v_1, v_2, \ldots, v_n)\tau = (v_1, v_2, \ldots, v_n)\eta = 0$$

(*"only if" part*). Suppose that the assertion is false. Choose $\alpha_1 = 1$ and α_i inductively as follows. α_2 is the first index j such that $v_j \neq v_1$; α_ν is the first index j such that v_j is not any one of $v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_{\nu-1}}$. If there are precisely k distinct vectors v_i, we have defined $1 = \alpha_1 < \alpha_2 < \ldots < \alpha_k \leqq n$. Clearly $\{v_1, v_{\alpha_2}, \ldots, v_{\alpha_k}\}$ is an independent set of vectors. Extend this to a basis $\{y_1, y_2, \ldots, y_m\}$ of V, such that $y_i = v_{\alpha_i}$, $i = 1, 2, \ldots, k \leqq m$. Then for each $i \in I$, if $i = \alpha_j$ for some j, $v_i = \sum_{l=1}^m a_{il} y_l$, where $a_{il} = 1$ if $l = \alpha_j$ and zero if $l \neq \alpha_j$. If $\alpha_j < i < \alpha_{j+1}$, then $v_i = v_{\alpha_{j'}}$ for some $j' \leqq j$. In this case, $v_i = \sum_{l=1}^m a_{il} y_l$, where $a_{il} = 1$ if $l = \alpha_{j'}$ and zero if $l \neq \alpha_{j'}$. And finally if $\alpha_n < i \leqq n$, then $v_i = v_{\alpha_{j'}}$, for some $j' \leqq k$ and $v_i = \sum_{l=1}^m a_{il} y_l$, where $a_{il} = 1$ if $l = \alpha_{j'}$ and zero if $l \neq \alpha_{j'}$. Thus in every case A_i is a singleton, i.e.,

$$A_i = \begin{cases} \{j\} & \text{if } i = \alpha_j, \\ \{j'\} & \text{if } \alpha_j < i < \alpha_{j+1}, \quad \text{where } j' \leqq j, \\ \{j'\} & \text{if } \alpha_k < i \leqq n, \quad \text{where } j' \leqq k. \end{cases}$$

Therefore $S = A_1 \times A_2 \times \ldots \times A_n = \{s\}$ say, where $s_{\alpha_j} = j, j = 1, 2, \ldots, k$. Thus $(v_1, v_2, \ldots, v_n) = (y_{s_1}, y_{s_2}, \ldots, y_{s_n})$, and, by our assumption, is not a (G, σ) element. Therefore by Remark 2, $g \in G$ implies $\sigma(g) = 1$ or $s_i \neq s_{g(i)}$ for some $i \in I$. Define f_s: $W \to F$, as in § 5. Since f_s is multilinear and a (G, σ) mapping, we have by the universal mapping property, as defined in 1.3 (ii), a unique linear transformation $\bar{f}_s$ of $P(W, G, \sigma)$ into F, which makes the following diagram



commutative; i.e., $\tau\bar{f}_s = f_s$. Now

$$(v_1, v_2, \ldots, v_n)f_s = \sum_{h \in G_s} \sigma(h^{-1}) a_{1,sh(1)} a_{2,sh(2)} \ldots a_{n,sh(n)}.$$

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But if $h \in G_s$, and $h \notin H_s$, then either $\sigma(h) \neq 1$ or $s_i \neq s_{h(i)}$ for some $i \in I$, and since $(y_{s1}, y_{s2}, \ldots, y_{sn})$ is not a (G, σ) element, we have $s_i \neq s_{h(i)}$ for some $i \in I$. Therefore $s_{h(i)} \notin A_i$, since $A_i = \{s_i\}$. Hence, $a_{i,sh(i)} = 0$ and therefore $a_{1,sh(1)}a_{2,sh(2)} \ldots a_{n,sh(n)} = 0$. Thus

$$(v_1, v_2, \ldots, v_n)f_s = a_{1,sh(1)}a_{2,sh(2)} \ldots a_{n,sh(n)},$$

where $h \in G_s$ is the coset representative of H_s. But then $s_{h(i)} = s_i$ for all $i \in I$, and hence $(v_1, v_2, \ldots, v_n)f_s = a_{1,s1}a_{2,s2} \ldots a_{n,sn} = 1 \neq 0$. But since $\tau\bar{f}_s = f_s$, we have $(v_1, v_2, \ldots, v_n)\tau\bar{f}_s \neq 0$, i.e., $(v_1, v_2, \ldots, v_n)\tau \neq 0$, and hence $v_1 \triangle v_2 \triangle \ldots \triangle v_n \neq 0$, which is a contradiction. Therefore $(v_1, v_2, \ldots, v_n)$ is a (G, σ) element.

We shall now prove our main result.

6.2. THEOREM. *Suppose that* $(v_1, v_2, \ldots, v_n) \in W$ *is a non-trivial element. Let*

$$(3) \qquad (v_1, v_2, \ldots, v_n) = \omega + \sum_{s \in E} D(M_s)(y_{s1}, y_{s2}, \ldots, y_{sn})$$

be its representation with respect to a basis $\{y_1, y_2, \ldots, y_n\}$ *of* V. *Then a necessary and sufficient condition for* $v_1 \triangle v_2 \triangle \ldots \triangle v_n$ *to be zero is that for each* $s \in E$, *either* $(y_{s1}, y_{s2}, \ldots, y_{sn})$ *is a* (G, σ) *element or* $D(M_s) = 0$.

Proof. Let $E' = \{s | s \in E, (y_{s1}, y_{s2}, \ldots, y_{sn})$ is not a (G, σ) element$\}$; E' may be an empty set. Then (3) becomes

$$(4) \qquad (v_1, v_2, \ldots, v_n) = \omega + \sum_{s \in E-E'} D(M_s)(y_{s1}, y_{s2}, \ldots, y_{sn}) + \sum_{s \in E'} D(M_s)(y_{s1}, y_{s2}, \ldots, y_{sn}).$$

We shall prove the sufficiency first. $E - E'$ is the index set that selects the non-vanishing terms in the sum (4). Thus

$$(5) \qquad (v_1, v_2, \ldots, v_n) = \omega + \sum_{s \in E-E'} D(M_s)(y_{s1}, y_{s2}, \ldots, y_{sn}).$$

Now if $s \in E - E'$, then $(y_{s1}, y_{s2}, \ldots, y_{sn})$ is a (G, σ) element. Moreover, it has the property P. Therefore by 6.1, $y_{s1} \triangle y_{s2} \triangle \ldots \triangle y_{sn} = 0$. Thus on applying η to (5), we obtain $v_1 \triangle v_2 \triangle \ldots \triangle v_n = 0$.

To prove the necessity, we assume it to be false; i.e., suppose that there exists $s \in E'$ such that $D(M_s) \neq 0$. Define f_s on W into F, as in § 5. Then by the universal mapping property, there exists a unique linear transformation $\bar{f}_s$ on $P(W, G, \sigma)$ into F, such that $\tau\bar{f}_s = f_s$. Now in (4), for each $s \in E - E'$, we have $y_{s1} \triangle y_{s2} \triangle \ldots \triangle y_{sn} = 0$ by 6.1. Hence, on applying η to (4) we obtain

$$(6) \qquad 0 = \sum_{s \in E'} D(M_s)(y_{s1}, y_{s2}, \ldots, y_{sn})\eta.$$

Now we calculate each term of this sum. First we choose $s \in E'$, for which

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$D(M_s) \neq 0$. We know that such an s exists by our assumption. Then

$$(y_{s1}, y_{s2}, \ldots, y_{sn})f_s = \sum_{h \in G_s} \sigma(h^{-1}) c_{s1,sh(1)} c_{s2,sh(2)} \ldots c_{sn,sh(n)},$$

where $c_{si,sh(i)} = 1$ if $s_i = s_{h(i)}$, and zero otherwise. Now since $s \in E'$, $(y_{s1}, y_{s2}, \ldots, y_{sn})$ is not a (G, σ) element. Thus

$$c_{s1,sh(1)} c_{s2,sh(2)} \ldots c_{sn,sh(n)} = 1$$

if h is a coset representative of H_s, and zero otherwise. Therefore

$$(y_{s1}, y_{s2}, \ldots, y_{sn})f_s = 1. \tag{7}$$

Next, for any $t \in E'$, if $t \neq s$, then t and s are not equivalent; thus for any $h \in G$ and in particular in G_s, $t_i \neq s_{h(i)}$ for some $i \in I$. Therefore

$$\begin{aligned}(y_{t1}, y_{t2}, \ldots, y_{tn})f_s &= \sum_{h \in G_s} \sigma(h^{-1}) c_{t1,sh(1)} c_{t2,sh(2)} \ldots c_{tn,sh(n)} \\ &= 0.\end{aligned} \tag{8}$$

However, from (6), we have

$$\begin{aligned} 0 &= \sum_{s \in E'} D(M_s)(y_{s1}, y_{s2}, \ldots, y_{sn})\tau \\ &= \sum_{s \in E'} D(M_s)(y_{s1}, y_{s2}, \ldots, y_{sn})\tau \bar{f}_s \\ &= \sum_{s \in E'} D(M_s)(y_{s1}, y_{s2}, \ldots, y_{sn})f_s \\ &= D(M_s), \text{ using (7) and (8),} \end{aligned}$$

which contradicts the fact that $D(M_s) \neq 0$, and this completes the proof.

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