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THE CONVEX FUNCTION DETERMINED BY A MULTIFUNCTION

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We shall show how each multifunction on a Banach space determines a convex function that gives a considerable amount of information about the structure of the multifunction. Using standard results on convex functions and a standard minimax theorem, we strengthen known results on the local boundedness of a *monotone* operator, and the convexity of the interior and closure of the domain of a *maximal monotone* operator. In addition, we prove that any point surrounded by (in a sense made precise) the convex hull of the domain of a maximal monotone operator is automatically in the interior of the domain, thus settling an open problem.

INTRODUCTION

We shall assume throughout this paper that E is a nontrivial Banach space. We shall show how each multifunction $S : E \to 2^{E^*}$ with $D(S) \neq \emptyset$ determines a convex function $\chi_S : E \to \mathbb{R} \cup \{\infty\}$, and we shall also show that χ_S gives a considerable amount of information about the structure of S.

We define χ_S in Definition 2. Lemma 3 contains a technical result which will be useful later in the paper, and Lemma 4 is our main result about χ_S. Our first application of Lemma 4 is in Theorem 6, in which we give a sufficient condition for S to be locally bounded at a point of E.

We next discuss the concept of an element x of E being "surrounded" by a subset A of E. This concept is related to x being an "absorbing point" of A, but differs in that we do not require that $x \in A$ (see [**5**, Definition 2.27(b), p.28]). Among other things, this difference will enable us to strengthen (in Theorem 12(b)) the result of Borwein and Fitzpatrick (see [**1**]) on the local boundedness of *monotone* operators.

Lemma 13(b) contains a result on the existence of elements of E^*, which we apply to *maximal monotone* operators in Theorem 14. Rockafellar proved in [**7**, Theorem 1, p.398] (see also [**6**, Theorem 1.9, p.6] that if S is maximal monotone and $\operatorname{int}(\operatorname{co} D(S)) \neq \emptyset$, then $\operatorname{int} D(S)$ and $\overline{D(S)}$ are both convex. (As usual, "co" stands for "convex hull of".) In Theorem 14, we give more explicit results and prove that, in fact,

$$\operatorname{int} D(S) = \operatorname{int}(\operatorname{dom} \chi_S) \quad \text{and} \quad \overline{D(S)} = \overline{\operatorname{dom} \chi_S}.$$

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The first of these results is true even if $\operatorname{int}(\operatorname{co} D(S)) = \emptyset$.

The equation "$\operatorname{int} D(S) = \operatorname{sur}(\operatorname{co} D(S))$" in Theorem 14 means the following: if $x \in E$ and,

$$\text{for all } w \in E \setminus \{0\}, \text{ there exists } \delta > 0 \text{ such that } x + \delta w \in \operatorname{co} D(S)$$

then $x \in \operatorname{int} D(S)$. This result answers in the affirmative a question raised by Phelps (see [**5**, p.29] and [**6**, p.8]).

The analysis in this paper gives insight into the "relative difficulty" of the results on the convexity of $\operatorname{int} D(S)$ and $\overline{\operatorname{dom} \chi_S}$ on the one hand, and the results on local boundedness on the other. The former use Lemma 4 in full generality, while the latter use Lemma 4 only for $m = 1$.

A word about tools. In Lemma 4 we use the standard result that a proper convex lower semicontinuous function on E is continuous on the interior of its domain. In Lemma 13(b), we use a minimax theorem. In fact, we could have used the Hahn-Banach theorem or a sandwich theorem instead, but a minimax theorem gives the fastest proof. We use the following classical minimax theorem, which can be deduced from more general results of Fan (see [**2**]) or Sion (see [**9**]). Fan's proof used a separation theorem for sets in finite dimensional spaces, and Sion's proof used the KKM theorem, but Theorem 1 can easily be proved without any functional analysis or fixed-point related concepts. See, for instance, the proof of Sion's theorem given by Komiya in [**4**].

Theorem 1. *Let X and Y be nonempty compact convex subsets of topological vector spaces. Let $f : X \times Y \to \mathbb{R}$ be (separately) concave and upper semicontinuous on X and convex and lower semicontinuous on Y. Then*

$$\max_X \min_Y f = \min_Y \max_X f.$$

The convex function determined by a multifunction

Definition 2: If $m \geqslant 1$, let

$$\sigma_m := \{a = (a_1, \ldots, a_m) : a_1, \ldots, a_m \geqslant 0,\ a_1 + \cdots + a_m = 1\} \subset \mathbb{R}^m.$$

If $S : E \to 2^{E^*}$ and $D(S) \neq \emptyset$, we define $\chi_S : E \to \mathbb{R} \cup \{\infty\}$ by

$$\chi_S(w) := \sup_{m \geqslant 1,\ (y_1, y_1^*), \ldots, (y_m, y_m^*) \in G(S),\ a \in \sigma_m} \frac{\sum_i a_i \langle w - y_i, y_i^* \rangle}{1 + \left\| \sum_i a_i y_i \right\|}. \tag{2.1}$$

χ_S is clearly convex and lower semicontinuous. (Here $G(S)$ stands for the graph of S.)

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In [8], a function $\psi_S : E \to \mathbb{R} \cup \{\infty\}$ was defined by the formula

$$\psi_S(w) := \sup_{(y,y^*) \in G(S)} \frac{\langle w - y,\, y^* \rangle}{1 + \|y\|},$$

which is the $m = 1$ version of the formula used to define χ_S. This was adequate for proving the convexity of int $D(S)$ and $\overline{D(S)}$ in the reflexive case, but it seems that the more complicated function χ_S is required in the general case.

The results on translation contained in Lemma 3 will enable us to simplify the computations in Theorems 6 and 14 considerably.

Lemma 3. *Let $S : E \to 2^{E^*}$ with $D(S) \neq \emptyset$, and $z \in E$. Define $T : E \to 2^{E^*}$ by*

$$Tx := S(x + z).$$

Then:

(a) *For all $w \in E$,*

$$\frac{\chi_T(w)}{1 + \|z\|} \leqslant \chi_S(w + z) \leqslant (1 + \|z\|)\chi_T(w).$$

(b) $\operatorname{Dom} \chi_T = \operatorname{dom} \chi_S - z$.

(c) $D(T) = D(S) - z$.

(d) *If S is monotone or maximal monotone then so is T.*

Proof: In (a), we shall prove the second inequality — the first inequality follows by replacing z by $-z$ and interchanging the roles of S and T.

Let $m \geqslant 1$, $(y_1, y_1^*), \cdots, (y_m, y_m^*) \in G(S)$ and $a \in \sigma_m$. Then

$$(y_1 - z, y_1^*), \cdots, (y_m - z, y_m^*) \in G(T).$$

Thus, using the definition of $\chi_T(w)$,

$$\begin{aligned}
\sum_i a_i \langle (w + z) - y_i,\, y_i^* \rangle &= \sum_i a_i \langle w - (y_i - z),\, y_i^* \rangle \\
&\leqslant \Big(1 + \Big\|\sum_i a_i (y_i - z)\Big\|\Big)\chi_T(w) \\
&= \Big(1 + \Big\|\sum_i a_i y_i - z\Big\|\Big)\chi_T(w) \\
&\leqslant \Big(1 + \Big\|\sum_i a_i y_i\Big\| + \|z\|\Big)\chi_T(w) \\
&\leqslant \Big(1 + \Big\|\sum_i a_i y_i\Big\|\Big)(1 + \|z\|)\chi_T(w).
\end{aligned}$$

We obtain (a) by dividing by $\left(1+\left\|\sum_i a_i y_i\right\|\right)$, taking the supremum over m, (y_i, y_i^*) and a, and using the definition of $\chi_S(w+z)$.

(b) follows from (a), and (c) and (d) are immediate. □

Lemma 4. *Let $S : E \to 2^{E^*}$ with $D(S) \neq \emptyset$, and $0 \in \operatorname{int}(\operatorname{dom}\chi_S)$. Then there exist $\eta \in (0,1]$ and $P > 0$ such that*

$$m \geqslant 1,\ (y_1, y_1^*), \cdots, (y_m, y_m^*) \in G(S) \quad \text{and} \quad a \in \sigma_m$$

imply

$$\sum_i a_i \langle y_i,\, y_i^* \rangle \geqslant \eta \left\|\sum_i a_i y_i^*\right\| - P\left(1+\left\|\sum_i a_i y_i\right\|\right).$$

Proof: From [**5**, Proposition 3.3, p.39], there exist $\eta \in (0,1]$ and $P > 0$ such that

$$w \in E \ \text{ and } \ \|w\| \leqslant \eta \quad \Longrightarrow \quad \chi_S(w) \leqslant P.$$

Thus,

$$w \in E,\ \|w\| \leqslant \eta,\ m \geqslant 1,\ (y_1,\, y_1^*), \ldots, (y_m, y_m^*) \in G(S) \ \text{ and } \ a \in \sigma_m$$

imply that

$$\sum_i a_i \langle w - y_i,\, y_i^* \rangle \leqslant P\left(1+\left\|\sum_i a_i y_i\right\|\right),$$

that is to say,

$$\sum_i a_i \langle y_i,\, y_i^* \rangle \geqslant \sum_i a_i \langle w,\, y_i^* \rangle - P\left(1+\left\|\sum_i a_i y_i\right\|\right) = \left\langle w, \sum_i a_i y_i^* \right\rangle - P\left(1+\left\|\sum_i a_i y_i\right\|\right).$$

We complete the proof of Lemma 4 by taking the supremum of the right hand expression over all $w \in E$ such that $\|w\| \leqslant \eta$. □

Definition 5: Let $S : E \to 2^{E^*}$ with $D(S) \neq \emptyset$, and $x \in E$. Following [**6**, Definition 1.8, p.5] we say that S is *locally bounded* at x if there exist $\delta, Q > 0$ such that

$$(y, y^*) \in G(S) \ \text{ and } \ \|y - x\| < \delta \quad \Longrightarrow \quad \|y^*\| \leqslant Q.$$

Note that this definition does not require that $x \in D(S)$.

Theorem 6. *Let $S : E \to 2^{E^*}$ with $D(S) \neq \emptyset$. Then S is locally bounded at each point of* $\operatorname{int}(\operatorname{dom}\chi_S)$.

Proof: From the results on translation in Lemma 3, it suffices to prove that

$$0 \in \operatorname{int}(\operatorname{dom}\chi_S) \quad \Longrightarrow \quad S \text{ is locally bounded at } 0.$$

So suppose that $0 \in \operatorname{int}(\operatorname{dom}\chi_S)$. Let η and P be as in Lemma 4. From Lemma 4 with $m = 1$,

$$(y, y^*) \in G(S) \quad \Longrightarrow \quad \eta\,\|y^*\| \leqslant \langle y, y^*\rangle + P(1 + \|y\|).$$

So

$$(y, y^*) \in G(S) \ \text{ and } \ \|y\| \leqslant \frac{\eta}{2} \quad \Longrightarrow \quad \eta\,\|y^*\| \leqslant \frac{\eta}{2}\,\|y^*\| + P\Big(1 + \frac{\eta}{2}\Big)$$
$$\Longrightarrow \quad \|y^*\| \leqslant \frac{3P}{\eta}.$$

Thus Definition 5 is satisfied with $\delta := \eta/2$ and $Q := 3P/\eta$. ∎

Since Theorem 6 only uses the $m = 1$ version of Lemma 4, it could in fact be strengthened to give the result that S is locally bounded at each point of $\operatorname{int}(\operatorname{dom}\psi_S)$ — see the comment following Definition 2.

Surrounded points and surrounding sets

Definition 7: Let $x \in E$ and $A \subset E$. We say that A *surrounds* x if, for each $w \in E \setminus \{0\}$, there exists $\delta > 0$ such that $x + \delta w \in A$. Furthermore, we define

$$\operatorname{sur} A := \{x : x \in E,\ A \text{ surrounds } x\}.$$

We note that, in general, $\operatorname{sur} A \not\subset A$. (Consider, for example, the case where A is the circumference of a circle in the plane and x is the centre of A.)

Lemma 8 provides some general culture concerning surrounding sets.

Lemma 8. *Suppose that C is a nonempty, convex subset of E. Then:*

(a) *$\operatorname{sur} C$ is convex.*
(b) *$\operatorname{sur} C \subset C$.*
(c) *$x \in \operatorname{sur} C$ if and only if, for each $w \in E$ there exists $\delta > 0$ such that $x + [-\delta, \delta]w \subset C$, that is to say, x is a radial point of C, (see [3, p.14]).*
(d) *If $\operatorname{sur} C \neq \emptyset$ then $\overline{C} = \overline{\operatorname{sur} C}$.*

Proof: (a) Suppose that $x, y \in \operatorname{sur} C$ and $\theta \in [0, 1]$. Let $w \in E \setminus \{0\}$, and pick $\delta_1, \delta_2 > 0$ such that $x + \delta_1 w \in C$ and $y + \delta_2 w \in C$. Define $\delta := (1 - \theta)\delta_1 + \theta\delta_2$. Then, from the convexity of C,

$$[(1 - \theta)x + \theta y] + \delta w = (1 - \theta)(x + \delta_1 w) + \theta(y + \delta_2 w) \in C.$$

Since this holds for all $w \in E \setminus \{0\}$, $(1 - \theta)x + \theta y \in \operatorname{sur} C$, as required.

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(b) Suppose that $x \in \operatorname{sur} C$. Let $w \in E \setminus \{0\}$ and pick $\delta_1, \delta_2 > 0$ such that

$$x + \delta_1 w \in C \quad \text{and} \quad x - \delta_2 w \in C.$$

Since C is convex, $[x + \delta_1 w, x - \delta_2 w] \subset C$. In particular, $x \in C$.

(c) Suppose that $x \in \operatorname{sur} C$. Let $w \in E$. If $w = 0$ then, by (b), $x + [-1,1]w = \{x\} \subset C$. If $w \neq 0$, pick $\delta_1, \delta_2 > 0$ such that

$$x + \delta_1 w \in C \quad \text{and} \quad x - \delta_2 w \in C.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Since C is convex,

$$x + [-\delta, \delta]w = [x - \delta w, x + \delta w] \subset [x + \delta_1 w, x - \delta_2 w] \subset C.$$

The converse is immediate.

(d) Suppose that $x \in C$. Let $y \in \operatorname{sur} C$. We claim that

$$\theta \in (0,1] \quad \Longrightarrow \quad (1-\theta)x + \theta y \in \operatorname{sur} C.$$

So let $\theta \in (0,1]$. Let $w \in E \setminus \{0\}$, and pick $\rho > 0$ such that $y + \rho w \in C$. Define $\delta := \rho\theta$. Then, from the convexity of C,

$$[(1-\theta)x + \theta y] + \delta w = (1-\theta)x + \theta(y + \rho w) \in C.$$

Since this holds for all $w \in E \setminus \{0\}$, $(1-\theta)x + \theta y \in \operatorname{sur} C$, as required. It now follows by letting $\theta \to 0+$ that $x \in \overline{\operatorname{sur} C}$. So we have proved that $C \subset \overline{\operatorname{sur} C}$, from which it follows immediately that $\overline{C} \subset \overline{\operatorname{sur} C}$. The reverse inclusion follows from (b), and this completes the proof of (d). ∎

Let E be infinite dimensional. Then there exists a discontinuous linear functional $L : E \to \mathbb{R}$. Let $C := \{x \in E : |Lx| \leqslant 1\}$. Then C is convex and $0 \in \operatorname{sur} C$, but $0 \notin \operatorname{int} C$. The point of this simple example is to contrast the situation for general convex sets with that exhibited in Theorem 9.

Theorem 9. *Let $\emptyset \neq C \subset E$. Suppose that $\{F_n\}$ is an increasing sequence of closed convex sets such that $C = \bigcup_{n \geqslant 1} F_n$. Then $\operatorname{sur} C = \operatorname{int} C$.*

Proof: It suffices from a translation argument to show that

$$0 \in \operatorname{sur} C \Longrightarrow 0 \in \operatorname{int} C.$$

Since $0 \in \operatorname{sur} C$, $E = \bigcup_{k \geqslant 1} kC$. So $E = \bigcup_{k,n \geqslant 1} kF_n$. By the Baire category theorem, there exist $n, k \geqslant 1$ such that $\operatorname{int} kF_n \neq \emptyset$, from which $\operatorname{int} F_n \neq \emptyset$. Choose $x \in \operatorname{int} F_n$. If

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$x = 0$ then $0 \in \operatorname{int} F_n \subset \operatorname{int} C$. If $x \neq 0$ then, since $0 \in \operatorname{sur} C$, there exists $p > 0$ such that $-x \in pC$, from which there exists $m \geqslant 1$ such that $-x \in pF_m$. Let $q = m \vee n$. Then

$$x \in \operatorname{int} F_q \quad \text{and} \quad \frac{-x}{p} \in F_q.$$

Using [**3**, 13.1(i), p.110], the convexity of F_q implies $0 \in \operatorname{int} F_q \subset \operatorname{int} C$. This completes the proof of Theorem 9. □

Corollary 10. *Let $f : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semicontinuous. Then* $\operatorname{sur}(\operatorname{dom} f) = \operatorname{int}(\operatorname{dom} f)$.

Proof: This follows from Theorem 9, with $F_n := E\{f \leqslant n\}$. □

Results for monotone operators

Lemma 11. *Let $S : E \to 2^{E^*}$ be monotone, with $D(S) \neq \emptyset$. Then:*

(a) $D(S) \subset \operatorname{co} D(S) \subset \operatorname{dom} \chi_S$.

(b) *Let $m \geqslant 1$, $\{(y_1, y_1^*), \cdots, (y_m, y_m^*)\} \subset G(S)$ and $a \in \sigma_m$. Then*

$$\sum_i a_i \langle y_i, y_i^* \rangle \geqslant \Big\langle \sum_i a_i y_i, \sum_j a_j y_j^* \Big\rangle.$$

Proof: (a) Since $\operatorname{dom} \chi_S$ is convex, it suffices to prove that

$$D(S) \subset \operatorname{dom} \chi_S \tag{11.1}$$

To this end, let $w \in D(S)$. Pick $w^* \in Sw$, and define $\beta := \langle w, w^* \rangle \vee \|w^*\|$. Let $m \geqslant 1$, $(y_1, y_1^*), \cdots, (y_m, y_m^*) \in G(S)$, and $a \in \sigma_m$. Then, since S is monotone,

$$\begin{aligned}
\sum_i a_i \langle w - y_i, y_i^* \rangle &\leqslant \sum_i a_i \langle w - y_i, w^* \rangle \\
&= \langle w, w^* \rangle - \Big\langle \sum_i a_i y_i, w^* \Big\rangle \\
&\leqslant \langle w, w^* \rangle + \Big\|\sum_i a_i y_i\Big\| \, \|w^*\| \\
&\leqslant \beta\Big(1 + \Big\|\sum_i a_i y_i\Big\|\Big).
\end{aligned}$$

Dividing by $1 + \|\sum_i a_i y_i\|$, we obtain

$$\frac{\sum_i a_i \langle \|w - y_i, y_i^* \rangle}{1 + \|\sum_i a_i y_i\|} \leqslant \beta.$$

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Taking the supremum over $m \geqslant 1$, $(y_1, y_1^*), \cdots, (y_m, y_m^*) \in G(S)$ and $a \in \sigma_m$ we see that $\chi_S(w) \leqslant \beta$, which implies that $w \in \operatorname{dom} \chi_S$. This completes the proof of (11.1), and hence that of Lemma 11(a).

(b) follows from the following relations:

$$\begin{aligned}\sum_i a_i \langle y_i, y_i^* \rangle - \Big\langle \sum_i a_i y_i, \sum_j a_j y_j^* \Big\rangle &= \sum_{i,j} a_i a_j \langle y_i, y_i^* \rangle - \sum_{i,j} a_i a_j \langle y_i, y_j^* \rangle \\ &= \sum_{i,j} a_i a_j \langle y_i, y_i^* - y_j^* \rangle \\ &= \sum_{i<j} a_i a_j \langle y_i, y_i^* - y_j^* \rangle + \sum_{j<i} a_i a_j \langle y_i, y_i^* - y_j^* \rangle \\ &= \sum_{i<j} a_i a_j \langle y_i, y_i^* - y_j^* \rangle + \sum_{i<j} a_i a_j \langle y_j, y_j^* - y_i^* \rangle \\ &= \sum_{i<j} a_i a_j \langle y_i - y_j, y_i^* - y_j^* \rangle \geqslant 0.\end{aligned}$$

□

THEOREM 12. *Let $S : E \to 2^{E^*}$ be monotone, with $D(S) \neq \emptyset$. Then:*

(a) $\operatorname{sur} D(S) \subset \operatorname{sur}(\operatorname{co} D(S)) \subset \operatorname{sur}(\operatorname{dom} \chi_S)$
$= \operatorname{int}(\operatorname{dom} \chi_S) \supset \operatorname{int}(\operatorname{co} D(S)) \supset \operatorname{int} D(S)$.

(b) *S is locally bounded at each point of $\operatorname{sur}(\operatorname{co} D(S))$.*

PROOF: (a) It follows from Lemma 11(a) that $\operatorname{sur} D(S) \subset \operatorname{sur}(\operatorname{co} D(S)) \subset \operatorname{sur}(\operatorname{dom} \chi_S)$ and $\operatorname{int}(\operatorname{dom} \chi_S) \supset \operatorname{int}(\operatorname{co} D(S)) \supset \operatorname{int} D(S)$. Since $D(S) \neq \emptyset$, χ_S is proper so, from Corollary 10, $\operatorname{sur}(\operatorname{dom} \chi_S) = \operatorname{int}(\operatorname{dom} \chi_S)$.

(b) This is immediate from (a) and Theorem 6. □

LEMMA 13. *Let $S : E \to 2^{E^*}$ be monotone with $D(S) \neq \emptyset$, $0 \in \operatorname{int}(\operatorname{dom} \chi_S)$, and η and P be as in Lemma 4. Define $M := P/\eta$. Now let $m \geqslant 1$ and $(y_1, y_1^*), \cdots, (y_m, y_m^*) \in G(S)$. Then:*

(a) *For all $a \in \sigma_m$,*

$$\sum_i a_i \langle y_i, y_i^* \rangle + M \Big\| \sum_i a_i y_i \Big\| \geqslant 0. \tag{13.1}$$

(b) *There exists $z^* \in E^*$ such that*

$$\|z^*\| \leqslant M \text{ and, for all } i = 1, \ldots, m,\ \langle y_i, y_i^* - z^* \rangle \geqslant 0.$$

PROOF: (a) Let $a \in \sigma_m$. If $\|\sum_i a_i y_i^*\| > M$ then, since $M = P/\eta \geqslant P$,

$$\sum_i a_i \langle y_i, y_i^* \rangle + M\|\sum_i a_i y_i\| \geqslant \sum_i a_i \langle y_i, y_i^* \rangle + P\|\sum_i a_i y_i\|,$$

from Lemma 4,

$$\geqslant \eta\|\sum_i a_i y_i^*\| - P$$
$$= \eta(\|\sum_i a_i y_i^*\| - M) > 0,$$

and (13.1) follows. If, on the other hand, $\|\sum_i a_i y_i^*\| \leqslant M$ then, from Lemma 11(b),

$$\begin{aligned}\sum_i a_i \langle y_i, y_i^* \rangle + M\|\sum_i a_i y_i\| &\geqslant \langle \sum_i a_i y_i, \sum_i a_i y_i^* \rangle + M\|\sum_i a_i y_i\| \\ &\geqslant M\|\sum_i a_i y_i\| - \|\sum_i a_i y_i\|\, \|\sum_i a_i y_i^*\| \\ &= (M - \|\sum_i a_i y_i^*\|)\|\sum_i a_i y_i\| \geqslant 0,\end{aligned}$$

and (13.1) follows again. This completes the proof of Lemma 13(a).

(b) From Theorem 1,

$$\begin{aligned}\max_{\|z^*\|\leqslant M} \min_{a\in\sigma_m} \Big[\sum_i a_i \langle y_i, y_i^* - z^* \rangle\Big] &= \min_{a\in\sigma_m} \max_{\|z^*\|\in M} \Big[\sum_i a_i \langle y_i, y_i^* - z^* \rangle\Big] \\ &= \min_{a\in\sigma_m} \max_{\|z^*\|\leqslant M} \Big[\sum_i a_i \langle y_i, y_i^* \rangle - \langle \sum_i a_i y_i, z^* \rangle\Big] \\ &= \min_{a\in\sigma_m} \Big[\sum_i a_i \langle y_i, y_i^* \rangle - \min_{\|z^*\|\leqslant M} \langle \sum_i a_i y_i, z^* \rangle\Big] \\ &= \min_{a\in\sigma_m} \Big[\sum_i a_i \langle y_i, y_i^* \rangle + M\|\sum_i a_i y_i\|\Big] \geqslant 0,\end{aligned}$$

using (a). Thus there exists $z^* \in E^*$ such that $\|z^*\| \leqslant M$ and

$$\text{for all } a \in \sigma_m, \ \sum_i a_i \langle y_i, y_i^* - z^* \rangle \geqslant 0.$$

We complete the proof of Lemma 13(b) by letting a run through the vertices of σ_m. □

THEOREM 14. *Let $S : E \to 2^{E^*}$ be maximal monotone. Then:*

(a) $\operatorname{Sur} D(S) = \operatorname{sur}(\operatorname{co} D(S)) = \operatorname{sur}(\operatorname{dom}\chi_S)$
$= \operatorname{int}(\operatorname{dom}\chi_S) = \operatorname{int}(\operatorname{co} D(S)) = \operatorname{int} D(S)$.

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(b) *If* $\operatorname{sur}(\operatorname{co} D(S)) \neq \emptyset$ *then*

$$\begin{aligned}\overline{D(S)} = \overline{\operatorname{co} D(S)} = \overline{\operatorname{dom} \chi_S} = \overline{\operatorname{sur} D(S)} = \overline{\operatorname{sur}(\operatorname{co} D(S))} \\ = \overline{\operatorname{sur}(\operatorname{dom} \chi_S)} = \overline{\operatorname{int}(\operatorname{dom} \chi_S)} = \overline{\operatorname{int}(\operatorname{co} D(S))} = \overline{\operatorname{int} D(S)}.\end{aligned}$$

Proof: (a) We first prove that

$$\operatorname{int}(\operatorname{dom} \chi_S) \subset D(S). \tag{14.1}$$

We can suppose that $\operatorname{int}(\operatorname{dom} \chi_S) \neq \emptyset$, for otherwise there is nothing to prove. From the results on translation in Lemma 3, it suffices to prove that

$$0 \in \operatorname{int}(\operatorname{dom} \chi_S) \implies 0 \in D(S). \tag{14.2}$$

So suppose that $0 \in \operatorname{int}(\operatorname{dom} \chi_S)$. Let M be as in Lemma 13. Then, for each finite subset F of $G(S)$, the set

$$\bigcap_{(y,y^*) \in F} \{z^* : z^* \in E^*,\ \|z^*\| \leqslant M,\ \langle y, y^* - z^* \rangle \geqslant 0\}$$

is nonempty. As F runs, these sets are $w(E^*, E)$-compact and directed downwards, hence their intersection is nonempty. It follows that there exists $z^* \in E^*$ such that

$$\text{for all } (y, y^*) \in G(S), \quad \langle y, y^* - z^* \rangle \geqslant 0.$$

Since S is maximal monotone, this implies that $z^* \in S0$, from which $0 \in D(S)$. This establishes (14.2), and hence (14.1). From (14.1), $\operatorname{int}(\operatorname{dom} \chi_S) \subset \operatorname{int} D(S) \subset \operatorname{sur} D(S)$. The result follows from Theorem 12(a).

(b) From Lemma 11(a), $\overline{D(S)} \subset \overline{\operatorname{co} D(S)} \subset \overline{\operatorname{dom} \chi_S}$. From (a), $\operatorname{int}(\operatorname{dom} \chi_S) \neq \emptyset$. Thus, from [**3**, 13.1(i)] again, with $C := \operatorname{dom} \chi_S$, and a second application of (a),

$$\overline{\operatorname{dom} \chi_S} = \overline{\operatorname{int}(\operatorname{dom} \chi_S)} = \overline{\operatorname{int} D(S)} \subset \overline{D(S)}.$$

Thus we have proved that

$$\overline{D(S)} = \overline{\operatorname{co} D(S)} = \overline{\operatorname{dom} \chi_S} = \overline{\operatorname{int} D(S)}.$$

The result now follows from a third application of (a). ∎

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