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# A MODEL OF UNIVERSAL TEICHMÜLLER SPACE AND ITS APPLICATION

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**Abstract**

In this paper, one model of the universal Teichmüller space is studied. By the method of construction, the lower bound of the inner radius of univalency by the Pre-Schwarzian derivative of quasidisks with infinity as an inner point (such as domains bounded by ellipses) is obtained.

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## 1. Preliminary and introduction

Let $D$ be a quasidisk in the complex plane $\mathbb{C}$. Here $\eta_D$ denotes its Poincaré density with Gaussian curvature $-4$. Furthermore, let $D^* = \overline{\mathbb{C}} \setminus \overline{D}$, $\triangle = \{z : |z| < 1\}$. So $\triangle^* = \overline{\mathbb{C}} \setminus \overline{\triangle}$ and $D^*$ is a quasidisk in $\overline{\mathbb{C}}$ with $\infty \in D^*$.

For a holomorphic and locally univalent function $f$ in domain $D$, define

$$T_f = \frac{f''}{f'} = (\log f')',$$

which is called the Pre-Schwarzian derivative of $f$, and

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2,$$

which is called the Schwarzian derivative of $f$. The Pre-Schwarzian derivative of $f$ vanishes in $D$ if and only if $f$ is a similarity. The Schwarzian derivative of $f$ vanishes in $D$ if and only if $f$ is a Möbius transformation. Moreover, $T_f$ is holomorphic at a finite point $z_0$ if and only if $f$ is holomorphic and injective around $z_0$. Similarly, $S_f$ is holomorphic at $z_0$ if and only if $f$ is holomorphic (or meromorphic) and injective around $z_0$.

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In addition, we define the following two Banach spaces of functions $\phi$ which are holomorphic in $D$:

$$B_1^*(D) = \left\{\phi : \phi \text{ is holomorphic in D with norm } \|\phi\|_D^* = \sup_{z\in D}\{|\phi(z)|\eta_D^{-1}(z)\} < \infty\right\},$$

$$B_2(D) = \left\{\phi : \phi \text{ is holomorphic in D with norm } \|\phi\|_D = \sup_{z\in D}\{|\phi(z)|\eta_D^{-2}(z)\} < \infty\right\}.$$

Let

$$\begin{aligned} S(D) &= \{S_f : f \text{ is holomorphic and univalent in } D\},\\ T(D) &= \{S_f : f \text{ is holomorphic and univalent in } D\\ &\qquad \text{and has quasiconformal extension to } \overline{\mathbb{C}}\},\\ S^*(D) &= \{T_f : f \text{ is holomorphic and univalent in } D\},\\ T^*(D) &= \{T_f : f \text{ is holomorphic and univalent in } D\\ &\qquad \text{and has quasiconformal extension to } \overline{\mathbb{C}}\}. \end{aligned}$$

It is known that

$$T(D) \subset S(D) \subset B_2(D) \quad \text{and} \quad T^*(D) \subset S^*(D) \subset B_1^*(D).$$

When $D = \triangle = \{z : |z| < 1\}$, $T(\triangle)$ is the famous Bers embedding model of the universal Teichmüller space. Here $T^*(\triangle)$ is an alternative model of the universal Teichmüller space introduced by Astala and Gehring [3] and Zhuravlev [9]. Zhuravlev proved in [9] that

$$T^*(\triangle) = \left\{\bigcup_{\theta\in[0,2\pi)} L_\theta\right\} \cup L,$$

where $L$ and $L_\theta$ are disconnected components of $T^*(\triangle)$ with $f$ bounded in $\triangle$ and $\lim_{z\to e^{i\theta}} f(z) = \infty$, respectively. Furthermore, some concrete properties of $L_\theta$ and $L$ were studied in [8]. It is easy to see that $\pi : \phi \to \phi' - \phi^2/2$ maps $T^*(D)$ onto $T(D)$ with $S_f = T_f' - (T_f)^2/2$, and it was also proved that $\pi$ is continuous.

Let $g : \triangle \to D$ be a conformal mapping. Because $D$ is a quasidisk in $\mathbb{C}$, we know that $g$ has quasiconformal extension, so it is concluded that

$$S_g \in T(\triangle) \quad \text{and} \quad T_g \in T^*(\triangle).$$

For any $f$ holomorphic in $\triangle$, we have

$$\begin{aligned} \|S_f - S_g\|_\triangle &= \|S_{f\circ g^{-1}}\|_D,\\ \|T_f - T_g\|_\triangle^* &= \|T_{f\circ g^{-1}}\|_D^*, \end{aligned}$$

which implies that the map $\Phi : S_f \to S_{f\circ g^{-1}}$ is an isometric homeomorphism of $T(\triangle)$ onto $T(D)$, which carries the point $S_g \in T(\triangle)$ to the origin of $T(D)$. At the same time, the map $\Psi : T_f \to T_{f\circ g^{-1}}$ is also an isometric homeomorphism of $T^*(\triangle)$ onto $T^*(D)$ which carries the point $T_g$ to the origin of $T^*(D)$.

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We define the inner and outer radii of univalency by the Schwarzian derivative of domain $D$, denoted by $\sigma_I(D)$ and $\sigma_O(D)$, respectively, as

$$\sigma_I(D) = \sup\{\sigma \geq 0 : \|S_f\|_D \leq \sigma \Rightarrow f \text{ is univalent in } D\},$$
$$\sigma_O(D) = \sup\{\|S_f\|_D : f : D \to \overline{\mathbb{C}} \text{ is univalent}\}.$$

From the analysis above, we know that

$$\sigma_I(D) = \inf_{S_h \in \partial T(D)} \|S_h\|_D = \inf_{S_f \in \partial T(\triangle)} \|S_f - S_g\|_\triangle,$$

where $g : \triangle \to D$ is conformal. Hence we conclude that $\sigma_I(D)$ is the distance from the point $S_g$ to the boundary of $T(\triangle)$.

Similarly, we define the inner and outer radii of univalency by the Pre-Schwarzian derivative of domain $D$, denoted by $\sigma_I^*(D)$ and $\sigma_O^*(D)$, respectively, as

$$\sigma_I^*(D) = \sup\{\sigma \geq 0 : \|T_f\|_D \leq \sigma \Rightarrow f \text{ is univalent in } D\},$$
$$\sigma_O^*(D) = \sup\{\|T_f\|_D : f : D \to \mathbb{C} \text{ is univalent}\}.$$

So

$$\sigma_I^*(D) = \inf_{T_h \in \partial T^*(D)} \|T_h\|_D^* = \inf_{T_f \in \partial T^*(\triangle)} \|T_f - T_g\|_\triangle^*,$$

where $g : \triangle \to D$ is conformal. We can also conclude that $\sigma_I^*(D)$ is the distance from the point $T_g$ to the boundary of $T^*(\triangle)$.

In [3, 7], it was proved that $\sigma_I(D) > 0$ or $\sigma_I^*(D) > 0$ if and only if $D$ is a quasidisk. From the above point of view, we establish some formulas to compute the inner radius of univalency of domain $D$. As we know, a locally univalent function $f$ in $D$ is not always wholly univalent in $D$. With the aid of the inner radius of univalency, we can judge that a function $f$ is univalent in $D$ when $\|T_f\|_D^* < \sigma_I^*(D)$ or $\|S_f\|_D < \sigma_I(D)$. Some results have been obtained about the inner radius of univalency by the Schwarzian derivative such as those for the unit disk (or the half plane), angular domains, rhombus domains, regular polygons and so on. However, for the inner radius of univalency by the Pre-Schwarizian derivative, there are a many things that are unknown. Although the Schwarzian derivative plays an important role in the study of the universal Teichmüller space, and many important results have been obtained, it is not easy to deal with the Schwarzian derivative, in general, because of its complicated form. Therefore, some attempts to replace it by the Pre-Schwarzian derivative have been made [3]. Although the Pre-Schwarzian derivative is sometimes called the 'poor man's model', since it does not have much invariance, this model is interesting when considering geometric function theory. Sometimes the Pre-Schwarzian derivative model could be more useful although it is less investigated.

In the discussion above, the domain $D$ belongs to $\mathbb{C}$. In other words, the infinity point $\infty \notin D$. In the study of $T(D)$ which is the Teichmüller space of domain $D$ by the Schwarzian derivative, whether or not $\infty \in D$ is always ignored because the Schwarzian derivative is invariant under Möbius transformations. However, in the

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study of $T^*(D)$, which is the Teichmüller space of domain $D$ by the Pre-Schwarzian derivative, the two cases when $\infty \in D$ and $\infty \notin D$ must be considered separately because the Pre-Schwarzian derivative is not invariant under Möbius transformations.

In [5], Becker investigated the Teichmüller space of the domain $\triangle^* = \overline{\mathbb{C}} \setminus \overline{\triangle}$ by the Pre-Schwarzian derivative. Let $\Sigma$ be the set of functions $F$ that are meromorphic and univalent in $\triangle^*$ normalized by $F(z) = z + b_0 + b_1/z + \cdots$. For any $F(z) = z + b_0 + b_1/z + \cdots$, $T_F = (2b_1/z^3) + \cdots = O(z^{-3})$. The norm of the function $T_F$ is defined by

$$\|T_F\|^*_{\triangle^*} = \sup_{z \in \triangle^*} \{|zT_F(z)|\eta^{-1}_{\triangle^*}(z)\},$$

which seems to be more natural.

Define

$$B^*_1(\triangle^*) = \{\phi : \phi \text{ is meromorphic in } \triangle^* \text{ with } \|\phi\|^*_{\triangle^*} < \infty\},$$

and a topological model of the universal Teichmüller space

$$T^*(\triangle^*) = \{T_F : F \in \Sigma \text{ and has quasiconformal extension to } \overline{\mathbb{C}}\}.$$

Moreover, in [5] it was proved that the model $T^*(\triangle^*)$ is homeomorphic to the model $T(\triangle^*)$.

It is known that the map $\pi(\phi) = \phi' - \phi^2/2 : B^*_1(\triangle^*) \to B_2(\triangle^*)$ is continuous. By definition, $\pi(S^*(\triangle^*)) = S(\triangle^*)$ and $\pi(T^*(\triangle^*)) = T(\triangle^*)$. Since in [1], Ahlfors has proved that $T(\triangle^*)$ is an open subset of $B_2(\triangle^*)$ and $T^*(\triangle^*) = \pi^{-1}(T(\triangle^*))$, the set $T^*(\triangle^*)$ is also an open subset of $B^*_1(\triangle^*)$. Moreover, in [4, 6] it was proved that $\sigma^*_O(\triangle^*) = 6$ and $\sigma^*_I(\triangle^*) = 1$.

The introduction of $T^*(\triangle^*)$ raises some questions. Is the introduction of the universal Teichmüller space of domain $D^*$ with $\infty \in D^*$ denoted by $T^*(D^*)$ necessary? How should the norm of functions in $T^*(D^*)$ be defined? With the analysis above, we know that the answers to these questions will help us to compute the inner radius of univalency of domain $D^*$ by the Pre-Schwarzian derivative. Let $g : \triangle^* \to D^*$ be conformal in $D^* \setminus \{\infty\}$ with $g(\infty) = \infty$, $g'(\infty) = 1$. We define

$$\begin{aligned} B^*_1(D^*) = \Big\{ &\phi : \phi \text{ is meromorphic in } D^* \text{ with} \\ &\|\phi\|^*_{D^*} = \sup_{z \in D^*} |g^{-1}(z)\phi(z)|\eta^{-1}_{D^*}(z) < \infty \Big\}, \\ T^*(D^*) = \Big\{ &T_f : f \text{ is meromorphic and univalent in } D^* \text{ with } \lim_{z \to \infty} \left(\frac{f(z)}{z} - 1\right) = 0 \\ &\text{and } f \text{ has quasiconformal extension to } \overline{\mathbb{C}} \Big\}. \end{aligned}$$

So for any $F \in \Sigma$,

$$\|T_F - T_g\|^*_{\triangle^*} = \|T_{F \circ g^{-1}}\|^*_{D^*}.$$

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It is concluded that $\phi : T_F \to T_{F \circ g^{-1}}$ is an isometric homeomorphism of $T^*(\triangle^*)$ onto $T^*(D^*)$. Furthermore,

$$\sigma_I^*(D^*) = \inf_{T_h \in \partial T^*(D^*)} \|T_h\|_{D^*}^* = \inf_{T_F \in \partial T^*(\triangle^*)} \|T_F - T_g\|_{\triangle^*}^*.$$

Because $D^*$ is a quasidisk, the conformal map $g : \triangle^* \to D^*$ has a quasiconformal extension and $T_g \in T^*(\triangle^*)$. Since $T^*(\triangle^*)$ is an open subset of $B_1^*(\triangle^*)$, we can conclude that

$$\sigma_I^*(D^*) = \text{dist}(T_g, \partial T^*(\triangle^*)) > 0.$$

Furthermore, $T^*(D^*)$ is an open subset of $B_1^*(D^*)$. In fact, for any $T_{\tilde{f}} \in T^*(D^*)$, we need to prove that there exists a constant $r > 0$, such that for any meromorphic and locally univalent function $h$ in $D^*$ with $\|T_h - T_{\tilde{f}}\|_{D^*}^* < r$, then $T_h \in T^*(D^*)$. Since $T_{\tilde{f} \circ g} \in T^*(\triangle^*)$ and $T^*(\triangle^*)$ is an open set, there exist a constant $r > 0$ such that $\{\phi : \|\phi - T_{\tilde{f} \circ g}\|_{\triangle^*}^* < r\} \subset T^*(\triangle^*)$. So for the $h$ above,

$$\|T_h - T_{\tilde{f}}\|_{D^*}^* = \|T_{h \circ g} - T_{\tilde{f} \circ g}\|_{\triangle^*}^* < r,$$

so that $h \circ g$ is univalent and has quasiconformal extension. So $h$ is univalent and also has quasiconformal extension, which implies that $T_h \in T^*(D^*)$.

## 2. A sufficient condition for quasiconformal extension

In the study of the universal Teichmüller space, it is important to judge whether a function $F \in \Sigma$ can be quasiconformally extended to $\overline{\mathbb{C}}$. To be specific, under what additional condition can $F$ be quasiconformally extended to $\overline{\mathbb{C}}$? In [5], Becker gave some answers to this question.

However, we can think over this question in a different point of view, it can be regarded as consisting of two parts.

(1) When does there exist a quasiconformal mapping $F^*$ of $\triangle$ such that $F$ and $F^*$ have equal continuous extensions to $|z| = 1$?

(2) If $F$ and $F^*$ have this property, when do they together form a locally homeomorphic mapping of $\overline{\mathbb{C}}$?

To the second question, there are some answers. For instance, if $F$ and $F^*$ extend to locally homeomorphic mappings of the closed domains $\overline{\triangle^*}$ and $\overline{\triangle}$, then they determine a global homeomorphism.

In 1974, Ahlfors [2] proved the theorem as follows.

THEOREM A. *If $f$ is holomorphic and locally univalent in $\triangle$, then the inequality*

$$\left|\sigma \frac{f''}{f'} + \sigma^2 - \sigma_z\right| \le k|\sigma_{\bar{z}}| \quad (0 \le k < 1),$$

*together with $1/\sigma = 0$ on $\partial\triangle$ and $\sigma_{\bar{z}}/\sigma^2$ in $\overline{\triangle}$, is sufficient to imply that $f$ has $(1+k)/(1-k)$-quasiconformal extension.*

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By the similar method, we can also get the following.

LEMMA 1. *If* $F \in \Sigma$ *and locally univalent in* $\triangle^*$*, then the inequality*

$$\left|\sigma \frac{F''}{F'} + \sigma^2 - \sigma_z\right| \le k|\sigma_{\bar{z}}|$$

*is sufficient to imply that* $F$ *has a* $(1+k)/(1-k)$*-quasiconformal extension where* $\sigma$ *satisfies* $1/\sigma = 0$ *on* $\{|z| = 1\}$ *and* $(\sigma_{\bar{z}}/\sigma^2) \ne 0$ *in* $\overline{\triangle^*}$.

For the completeness of the paper, we give the proof of Lemma 1.

PROOF. We can assume that $F$ is meromorphic and locally univalent in $\overline{\triangle^*}$. Define the extension

$$W(z) = \begin{cases} F(z), & z \in \overline{\triangle^*}, \\ G\left(\dfrac{1}{\bar{z}}\right), & z \in \overline{\triangle}, \end{cases}$$

where $G$ is sense-reserving and $k$-quasiconformal in $\triangle^*$ with $G = F$ on $\{|z| = 1\}$.

If we set $G = F + u$ with $u = 0$ on $|z| = 1$, then $u$ should satisfy

$$|F' + u_z| \le k|u_{\bar{z}}|. \tag{1}$$

In addition, we should require that $|u_{\bar{z}}| \ne 0$ in $\overline{\triangle^*}$. Under these conditions it is evident that $W$ will be a $k$-quasiconformal extension of $F$.

If we choose

$$u = \frac{F'}{\sigma},$$

with $1/\sigma = 0$ on $|z| = 1$ and $(\sigma_{\bar{z}}/\sigma^2) \ne 0$ in $\overline{\triangle^*}$, then (1) becomes

$$\left|\sigma \frac{F''}{F'} + \sigma^2 - \sigma_z\right| \le k|\sigma_{\bar{z}}|,$$

which implies that $F$ has a quasiconformal extension.

When $F$ is not meromorphic and locally univalent on $\{|z| = 1\}$, we can change $W(z)$ into the limitation of $W_r(z)$ as $r \to 1+$, where $W_r(z)$ is the extension function of $F_r(z) = F(rz)$. □

## 3. Estimation for the inner radius of univalency of domains with infinity as an inner point

As an application of Lemma 1 and the definition of $\|T_f\|^*_{D^*}$, we can estimate the inner radius of univalency of domain $D^* = \overline{\mathbb{C}} \setminus \overline{D}$ with $\infty \in D^*$.

THEOREM 2. *Let $D^*$ be a quasidisk in $\overline{\mathbb{C}}$ with $\infty \in D^*$, and let $\lambda$ be the quasiconformal reflection across $\partial D^*$, then*

$$\sigma_I^*(D^*) \geq \inf_{w\in D^*} \frac{(|\lambda_{\overline{w}}(w)| - |\lambda_w(w)|)|g^{-1}(w)|}{|w-\lambda(w)|\eta_{D^*}(w)}.$$

PROOF OF THEOREM 2. Let $w = g(z)$ be the conformal mapping from $\triangle^*$ onto $D^*$, so we construct

$$\tau = \frac{\lambda \circ g}{g} \quad \text{and} \quad \sigma = -\frac{g'}{g}\frac{1}{1-\tau} = -\frac{g'}{g-\lambda\circ g}. \tag{2}$$

We know that

$$\frac{1}{\sigma} = \frac{g-\lambda\circ g}{g} = 0 \quad \text{on } |z| = 1$$

and

$$\sigma_{\overline{z}} = -\frac{g'}{g}\frac{\tau_{\overline{z}}}{(1-\tau)^2},$$

where

$$\tau_{\overline{z}} = \frac{\lambda_w g_{\overline{z}} + \lambda_{\overline{w}}\overline{g}_{\overline{z}}}{g} = \frac{\lambda_{\overline{w}}\overline{g}_{\overline{z}}}{g} \neq 0 \quad \text{in } \overline{\triangle^*}.$$

Let $F$ be meromorphic and locally univalent in $\triangle^*$; by Lemma 1 and replacing (2) into (1), we have

$$\left| -\frac{g'}{g}\frac{1}{1-\tau}\frac{F''}{F'} + \frac{(g')^2}{g^2}\frac{1}{(1-\tau)^2} + \frac{g''}{g}\frac{1}{1-\tau} - \frac{(g')^2}{g^2}\frac{1}{1-\tau} + \frac{g'}{g}\frac{\partial_z\tau}{(1-\tau)^2}\right| \\ \leq \frac{\partial_{\overline{z}}\tau}{(1-\tau)^2},$$

that is

$$\left| -(1-\tau)\left(\frac{F''}{F'} - \frac{g''}{g'}\right) + \frac{g'}{g}\tau + \partial_z\tau \right| \leq k|\partial_{\overline{z}}\tau|. \tag{3}$$

The inequality implies that $F$ has a $(1+k)/(1-k)$-quasiconformal extension. It follows from (2) that

$$\left| \frac{z}{\eta_{\triangle^*}}\left(\frac{F''}{F'} - \frac{g''}{g'}\right) - \frac{z\partial_z(g\tau)}{g(1-\tau)\eta_{\triangle^*}} \right| \leq k\frac{|z||\partial_{\overline{z}}(g\tau)|}{|g(1-\tau)\eta_{\triangle^*}|}. \tag{4}$$

It is easy to see that if $F$ satisfies the inequality

$$\left| \frac{z}{\eta_{\triangle^*}}\left(\frac{F''}{F'} - \frac{g''}{g'}\right)\right| \leq \inf_{z\in\triangle^*} \frac{k|z||\partial_{\overline{z}}(g\tau)| - |z\partial_z(g\tau)|}{|g(1-\tau)\eta_{\triangle^*}|}, \tag{5}$$

then $F$ must satisfy inequality (3), which implies that $F$ is univalent. Because

$$\|T_{F\circ g^{-1}}\|_{D^*}^* = \|T_F - T_g\|_{\triangle^*}^* = \sup_{z\in\triangle^*}\left| \frac{z}{\eta_{\triangle^*}}\left(\frac{F''}{F'} - \frac{g''}{g'}\right)\right|, \tag{6}$$

we can see, from (4) and (5), that if $F$ satisfies

$$\|T_{F\circ g^{-1}}\|^*_{D^*} \le \inf_{z\in\Delta^*} \frac{k|z||\partial_{\bar z}(g\tau)| - |z\partial_z(g\tau)|}{|g(1-\tau)\eta_{\Delta^*}|},$$

then $F$ is univalent in $\Delta^*$ and $F\circ g^{-1}$ is univalent in $D^*$. So

$$\begin{aligned}\sigma_I^*(D^*) &\ge \sup_{0\le k<1}\left\{\inf_{z\in\Delta^*} \frac{k|z||\partial_{\bar z}(g\tau)| - |z\partial_z(g\tau)|}{|g(1-\tau)\eta_{\Delta^*}|}\right\}\\ &= \inf_{z\in\Delta^*} \frac{|z||\partial_{\bar z}(g\tau)| - |z\partial_z(g\tau)|}{|g(1-\tau)\eta_{\Delta^*}|}\\ &= \inf_{w\in D^*} \frac{(|\lambda_{\bar w}(w)| - |\lambda_w(w)|)|g^{-1}(w)|}{|w-\lambda(w)|\eta_{D^*}(w)}.\end{aligned}$$ □

By using Theorem 2, we can get a lower bound of $\sigma_I(D^*)$, where $D$ is an ellipse domain.

THEOREM 3. *Let $D$ be an ellipse domain where the ratio of the minor axis to the major axis is $q$, so then*

$$\sigma_I^*(D^*) \ge \frac{2q}{q+1},$$

*where $D^* = \overline{\mathbb{C}} \setminus \overline{D}$.*

PROOF OF THEOREM 3. We may assume without loss of generality that $D^*$ is bounded by the ellipse $\{e^{i\theta}r + (e^{-i\theta})/r \mid 0 \le \theta < 2\pi\}$, where $r > 1$ is fixed, so then

$$q = \frac{r-(1/r)}{r+(1/r)} \quad \text{and} \quad r^2 = (1+q)/(1-q).$$

The function $w = g(z) = rz + 1/(rz)$ maps $\Delta^*$ onto $D^*$, $g(z)$ is conformal in $\Delta^* \setminus \{\infty\}$ and $g(\infty) = \infty$. The function

$$\phi(w) = \tfrac{1}{2}\left(w - \sqrt{w^2-4}\right) \quad (\phi(\infty) = 0),$$

maps $D^*$ onto the disk $|z| < 1/r$. It follows that

$$\lambda = \phi + r^2\overline{\phi},$$

is a quasiconformal reflection across $\partial D^*$.

After the above preparation, next we compute

$$\inf_{w\in D^*} \frac{(|\lambda_{\bar w}(w)| - |\lambda_w(w)|)|g^{-1}(w)|}{|w-\lambda(w)|\eta_{D^*}(w)}.$$

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We know that

$$g^{-1}(w)=\frac{2}{r\big(w-\sqrt{w^2-4}\big)},$$
$$\eta_{D^*}(w)=\frac{r/2|1-w(w^2-4)^{-(1/2)}|}{1-\big|(r/2)\big(w-\sqrt{w^2-4}\big)\big|^2},$$
$$\lambda_{\overline{w}}=\phi_{\overline{w}}+r^2\overline{\phi}_{\overline{w}}=\phi_{\overline{w}}+r^2\overline{\phi_w},$$
$$\lambda_w=\phi_w+r^2\overline{\phi}_w=\phi_w+r^2\overline{\phi_{\overline{w}}},$$

where

$$\phi_w=\tfrac{1}{2}[1-w(w^2-4)^{-1/2}]\quad\text{and}\quad\phi_{\overline{w}}=0.$$

So

$$\lambda_{\overline{w}}=r^2\overline{\phi_w}=r^2\tfrac{1}{2}\Big[1-\overline{w(w^2-4)^{-1/2}}\Big],$$
$$\lambda_w=\phi_w=1/2[1-w(w^2-4)^{-1/2}],$$
$$|\lambda_{\overline{w}}|-|\lambda_w|=\tfrac{1}{2}(r^2-1)|1-w(w^2-4)^{-1/2}|,$$
$$|w-\lambda(w)|=|w-\phi-r^2\overline{\phi}|=\Big|w-\tfrac{1}{2}\Big(w-\sqrt{w^2-4}\Big)-r^2\tfrac{1}{2}\Big(\overline{w}-\overline{\sqrt{w^2-4}}\Big)\Big|.$$

Hence

$$\begin{aligned}\inf_{w\in D^*}&\frac{(|\lambda_{\overline{w}}(w)|-|\lambda_w(w)|)|g^{-1}(w)|}{|w-\lambda(w)|\eta_{D^*}(w)}\\&=\inf_{w\in D^*}\frac{(r^2-1)/2|1-w(w^2-4)^{-1/2}|2/\big(r\big|w-\sqrt{w^2-4}\big|\big)}{\big|w-\big(w-\sqrt{w^2-4}\big)/2-r^2\big(\overline{w}-\overline{\sqrt{w^2-4}}\big)/2\big|}\\&\quad\times\frac{1-\big|(r/2)\big(w-\sqrt{w^2-4}\big)\big|^2}{(r/2)|1-w(w^2-4)^{-1/2}|}\\&=\inf_{w\in D^*}\frac{(r^2-1)/r1/\big|w-\sqrt{w^2-4}\big|}{\big|w-\big(w-\sqrt{w^2-4}\big)/2-r^2\big(\overline{w}-\overline{\sqrt{w^2-4}}\big)/2\big|}\\&\quad\times\frac{1-\big|r/2\big(w-\sqrt{w^2-4}\big)\big|^2}{r/2}.\end{aligned}$$

Let $\zeta=w-\sqrt{w^2-4}$, then $|\zeta|<2/r$ and $w=(\zeta/2)+(2/\zeta)$,

$$\begin{aligned}\inf_{w\in D^*}&\frac{(|\lambda_{\overline{w}}(w)|-|\lambda_w(w)|)|g^{-1}(w)|}{|w-\lambda(w)|\eta_{D^*}(w)}\\&=\inf_{|\zeta|<(2/r)}\frac{(r^2-1)/r}{|\zeta/2+2/\zeta-\zeta/2-r^2/2\overline{\zeta}|}\frac{1-r^2/4|\zeta|^2}{r/2}=\frac{r^2-1}{r^2}=\frac{2q}{q+1}.\end{aligned}$$

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So

$$\sigma_I^*(D^*) \geq \frac{2q}{1+q}. \qquad \square$$

REMARK. Because $q$ is the ratio of the minor axis to the major axis of the ellipse and $q < 1$, we find that

$$\frac{2q}{1+q} < 1 \quad \text{and} \quad \frac{2q}{1+q} \to 1 \quad (q \to 1),$$

which is compatible with $\sigma_I^*(\triangle^*) = 1$.

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