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WHEN X^* IS A P'-SPACE

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ABSTRACT. In [7,3.1] the authors show that if a space X is realcompact and locally compact, then X^* is a P'-space. In this paper we show that the hypothesis of realcompactness can be weakened. We also look at other conditions on X that are sufficient to guarantee that X^* is a P'-space.

0. **Introduction.** A space X is a P'-space if every zero-set of X has non-empty interior in X or, equivalently, if every zero-set is regular closed in X. In [7,3.1] Fine and Gillman show that if a space X is locally compact and realcompact, then $\beta X - X$ is a P'-space. We show that the hypothesis that X is realcompact can be reduced. We introduce the class of p-realcompact spaces, a class that lies strictly between the c-realcompact spaces and the nearly realcompact spaces. We then show that locally compact p-realcompact spaces have growths that are P'-spaces.

Along the way to producing this result, we introduce some local conditions on a zero-set of a locally compact nearly realcompact space X that are necessary and sufficient to guarantee that the zero-set of its extension will trace to a set with empty interior in X^*.

1. **Definitions and Preliminaries.** There are a number of lines along which realcompactness has been generalized in the literature. By a "line" we mean a sequence of properties of a space, each weaker than its predecessor. One of these lines is an "isocompactness" line that characterizes spaces by which of their closed subsets are compact. The line we are mainly interested in concerns the structure of $\beta X - X$. The properties in this line generalize the characterization of realcompactness described in Proposition 1.0 below. It is in this line that we find the property that guarantees (in the presence of local compactness) that $\beta X - X$ is a P'-space. We will also be concerned with the "isocompactness line" and will discuss where the two lines intersect. Thus we need the following definitions and preliminary results.

Since we are concerned with the structure of the growth of a space X, that is, with $X^*(= \beta X - X)$, we will naturally assume that all our spaces are Tychonoff. As usual, we denote the collection of all real-valued continuous (resp. bounded continuous) functions on a space X by $C(X)$ (resp. $C^*(X)$). A zero-set Z of X is a set of the form $Z(f) = f^{\leftarrow}\{0\}$ where $f \in C(X)$. A cozero-set is the complement of a zero-set. A *z-ultrafilter* on X is a

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maximal filter in the collection of zero-sets of X. We denote the natural numbers by $\mathbb{N}$ and the non-negative integers by ω.

We assume that the reader is familiar with the basic theory of both the Stone-Čech compactification βX and the Hewitt realcompactification υX of a Tychonoff space X as described in [8]. In particular we think of points of βX as z-ultrafilters on X and points of υX as z-ultrafilters on X with the countable intersection property (and thus a space X is realcompact if and only if every z-ultrafilter with the countable intersection property converges). We will often use the following well known characterization of points in $\beta X - \upsilon X$.

PROPOSITION 1.0. *A point p in βX is in $\beta X - \upsilon X$ if and only if there is $g \in C(\beta X)$ with $g(p) = 0$ and $g > 0$ on X.*

Subsets A and B of X are *completely separated* in X if there is $f \in C(X)$ with $f^{\rightarrow}(A) \subset \{0\}$ and $f^{\rightarrow}(B) \subset \{1\}$. Completely separated subsets of X have disjoint closures in βX. A subset $A \subset X$ is *well embedded* in X if A is completely separated in X from every disjoint zero-set of X. It is well known that zero-sets of X are well embedded in X. A set $A \subset X$ is *relatively pseudocompact* (resp. *strongly relatively pseudocompact*) in X if every function in $C(X)$ (resp. for every cozero-set neighborhood P of A, every function in $C(P)$) is bounded on A. A space X is *isocompact* (resp. *strongly isocompact*) (resp. *hyperisocompact*) if every countably compact (resp. every strongly relatively pseudocompact) (resp. every relatively pseudocompact) closed subset of X is compact. Clearly hyperisocompact implies strongly isocompact implies isocompact.

Unfortunately there is some confusion in the literature about these terms. The term "strongly isocompact" was used in [1] to mean "hyperisocompact". The terminology used here follows [3].

2. ***p*-realcompact spaces.** We begin with the definition of some of the properties in the "structure of $\beta X - X$" line.

A space X is *c-realcompact* (see [9]) if for every $p \in X^*$, there is a decreasing sequence $\langle A_n : n \in \omega \rangle$ of regular closed subsets of βX with $p \in \bigcap_{n\in\omega} A_n$ while $\bigcap_{n\in\omega}(A_n \cap X) = \emptyset$. X is *nearly realcompact* (see [2]) if $\beta X - \upsilon X$ is dense in $\beta X - X$. Finally we define X to be *p-realcompact* if every zero-set of βX that meets X^* meets $\beta X - \upsilon X$.

We have the following implications already available. The first implication is an easy consequence of Proposition 1.0 and the second is due to Blair and van Douwen and is proved in [11,14.3].

PROPOSITION 2.0. *Realcompact spaces are c-realcompact, and c-realcompact spaces are nearly realcompact.*

We show next where p-realcompactness fits into this scheme, but first we need the following facts from elsewhere.

PROPOSITION 2.1 [2,1.4]. *If G is open in X, then the following are equivalent.*

(1) G is relatively pseudocompact in X.

(2) If $\langle F_n : n \in \omega \rangle$ is a decreasing sequence of regular closed subsets of X with $F_n \cap G \neq \emptyset$ for all $n \in \omega$, then $\bigcap_{n\in\omega} F_n \neq \emptyset$.

We use the following two propositions often. The proof of the first is trivial and the proof of the second can be found in [3].

PROPOSITION 2.2. *For all $f \in C(X)$, if $Z(f^\beta) \subset \upsilon X$, then $\mathrm{cl}_{\beta X} Z(f) = Z(f^\beta)$.*

PROPOSITION 2.3 [3,2.6]. *If A is a subspace of the space X, then the following are equivalent.*

(1) A is relatively pseudocompact in X.

(2) $\mathrm{cl}_{\beta X} A \subset \upsilon X$.

PROPOSITION 2.4. *c-realcompact spaces are p-realcompact and p-realcompact spaces are nearly realcompact.*

PROOF. The second implication is trivial. We prove the first. Let X be a c-realcompact space and let $f: \beta X \longrightarrow [0,1]$ with $p \in Z(f) \cap X^*$. There is a decreasing sequence $\langle A_n : n \in \omega \rangle$ of regular closed sets in βX with $p \in \bigcap_{n\in\omega} A_n$ and $\bigcap_{n\in\omega}(A_n \cap X) = \emptyset$. Assume that $Z(f) \subset \upsilon X$. Then by Proposition 2.2, $\mathrm{cl}_{\beta X}\big(Z(f) \cap X\big) = Z(f)$.

For all $n \in \mathbb{N}$, let

$$B_n = (\mathrm{int}_{\beta X} A_n) \cap f^{\leftarrow}(0, 1/n) \cap X.$$

We will show that $B_n \neq \emptyset$ for all $n \in \mathbb{N}$. Suppose, on the contrary, that there is $N \in \mathbb{N}$ with $B_N = \emptyset$. Now for all $n \geq N$, $f^{\leftarrow}[0, 1/n) \cap \mathrm{int}_X(A_n \cap X) \neq \emptyset$, and so by our assumption, $\emptyset \neq \mathrm{int}_X(A_n \cap X) \cap f^{\leftarrow}[0, 1/n) \subset Z(f)$ for all $n \geq N$. Thus we have, for all $n \geq N$, $A_n \cap \mathrm{int}_X\big(Z(f) \cap X\big) \neq \emptyset$. Now by Proposition 2.3, $\mathrm{int}_X\big(Z(f) \cap X\big)$ is relatively pseudocompact in X and so by Proposition 2.1, $\emptyset \neq \bigcap_{n\geq N}(A_n \cap X) \subset \bigcap_{n\in\mathbb{N}}(A_n \cap X)$, a contradiction. We conclude that $B_n \neq \emptyset$ for all $n \in \mathbb{N}$.

We define, recursively, sequences $\langle Y_n : n \in \mathbb{N} \rangle$ of zero-sets of X, $\langle P_n : n \in \mathbb{N} \rangle$ of cozero-sets of X, $\langle q_n : n \in \mathbb{N} \rangle \subset X$, $\langle r_n : n \in \mathbb{N} \rangle$ and $\langle s_n : n \in \mathbb{N} \rangle \subset \mathbb{R}$, satisfying:

(1) $0 < s_{n+1} < r_n < s_n < \frac{1}{n+1}$ for all $n \in \mathbb{N}$; and

(2) $q_n \in Y_n \subset P_n \subset f^{\leftarrow}(r_n, s_n) \cap \mathrm{int}_X A_n$ for all $n \in \mathbb{N}$.

Let $s_0 = \frac{1}{2}$. Assume now that $n \in \mathbb{N}$ and that for all $j \leq n$ the sequences have been constructed and that s_n has been defined. There is $k \in \mathbb{N}$ with $\frac{1}{k} < \min\{s_n, \frac{1}{n+1}\}$. Pick $q_n \in B_k$. Then $0 < f(q_n) < \min\{s_n, \frac{1}{n+1}\}$. Pick r_n with $0 < r_n < f(q_n)$. Finally pick a zero-set neighborhood Y_n and a cozero-set neighborhood P_n of q_n such that

$$Y_n \subset P_n \subset f^{\leftarrow}(r_n, s_n) \cap \mathrm{int}_X A_n.$$

Now pick $s_{n+1} < \min\{r_n, \frac{1}{n+2}\}$ with $0 < s_{n+1}$. This completes the recursion.

Let $Z_n = \bigcup_{j>n} Y_j$ for all $n \in \mathbb{N}$ and note that Z_n is a zero-set of X. Let q be a z-ultrafilter on X containing $\{Z_n : n \in \mathbb{N}\}$. Since $\bigcap_{n\in\mathbb{N}} Z_n \subset \bigcap_{n\in\omega}(A_n \cap X) = \emptyset$, $q \in \beta X - \upsilon X$. Clearly $f(q) = 0$. This contradiction completes the proof.

We denote the extension of the function $f \in C^*(X)$ to βX by f^β.

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PROPOSITION 2.5 [3,2.7]. *If A is a subspace of X, then the following are equivalent.*

(1) A is strongly relatively pseudocompact in X.

(2) A is relatively pseudocompact and well embedded in X.

Obviously every relatively pseudocompact zero-set is strongly relatively pseudocompact. We denote the extension of the function $f \in C^*(X)$ to βX by f^β.

PROPOSITION 2.6. *If every relatively pseudocompact zero-set of X is compact, then X is p-realcompact. Hence strongly isocompact spaces are p-realcompact.*

PROOF. Let $Z(f^\beta)$ be a zero-set of βX and suppose that $Z(f^\beta) \subset \upsilon X$. By Proposition 2.2, $\mathrm{cl}_{\beta X} Z(f) = Z(f^\beta)$ and so by Proposition 2.3 $Z(f)$ is relatively pseudocompact and hence compact. Therefore $Z(f^\beta) \cap X^* = \emptyset$.

COROLLARY 2.7. *Topologically complete spaces are p-realcompact.*

PROOF. Topologically complete spaces are hyperisocompact (see [5,3.1]).

EXAMPLES 2.8. The following examples show that p-realcompact spaces need not be c-realcompact and that nearly realcompact spaces need not be p-realcompact.

(1) A nearly realcompact space that is not p-realcompact. Let $X = \mathbb{Q} \times \omega_1$ where $\mathbb{Q}$ is the set of rationals. By [2,1.11], X is nearly realcompact. X is not p-realcompact since the set $\{0\} \times \omega_1$ ($= Z(f)$ where $f(\langle x, \alpha\rangle) = |x|$) is a non-compact zero-set of X but $Z(f^\beta) = \{0\} \times [0, \omega_1] \subset \upsilon X$.

(2) A p-realcompact space that is neither c-realcompact nor isocompact.

Let T be the Tychonoff Plank $([0, \omega_1] \times [0, \omega]) - \{\langle \omega_1, \omega\rangle\}$ and let $A = \{x_\alpha^\beta : \alpha, \beta \in \omega_1\}$. Let $Y = T \cup A$. We topologize Y by isolating all points of A, giving all points of T that do not lie on the top edge of T their usual neighborhoods in T and letting points of the top edge of T have their usual neighborhoods in T together with sequences from A that converge to each point of the neighborhood. That is, a basic neighborhood of $\langle \alpha, \omega\rangle$ will be of the form $U \cup \{x_\delta^\gamma : \gamma > \beta$ and $\langle \delta, \omega\rangle \in U\}$ where U is a usual neighborhood in T and $\beta \in \omega_1$.

Note that if Z is a non-compact zero-set of Y, then $|Z \cap A| \geq \omega$. Every countable subset of A is clopen and so no infinite subset of A is relatively pseudocompact in Y. Thus if Z is a non-compact zero-set of Y then Z is not relatively pseudocompact. We conclude, by Proposition 2.6, that Y is p-realcompact.

To see that Y is not c-realcompact, we note first that T is C-embedded in Y and so $p = \langle \omega_1, \omega\rangle \in \upsilon T \subset \upsilon Y$. But if $p \in \bigcap_{n \in \omega} A_n$ where each A_n is regular closed in βY, then $\bigcap_{n \in \omega}(A_n \cap Y) \neq \emptyset$.

Finally, the top edge of T witnesses that Y is not isocompact.

(3) A c-realcompact space X that is not isocompact.

Let X be the Tychonoff plank T with a copy of $(0, 1) = G_n$ inserted between $\langle \omega_1, n\rangle$ and $\langle \omega_1, n+1\rangle$ for each $n \in \omega$. That is, we identify the 0 of the n-th copy of $[0, 1]$ with $\langle \omega_1, n\rangle$ and the 1 of $[0, 1]$ with $\langle \omega_1, n+1\rangle$. Every point of X^* is in $\bigcap_{n \in \omega} \mathrm{cl}_{\beta X}(\bigcup_{j > n} G_j)$ and so X is c-realcompact. Clearly X is not isocompact. (This example appears in [2,1.15].)

We remark that p-realcompactness does not have nice hereditary properties. In Example (3) T is a regular closed C-embedded zero-set of a p-realcompact space that is not p-realcompact.

We give next a couple of sufficient conditions for a space to be p-realcompact. We use the notation

$$\mathrm{Ex}(U) = \beta X - \mathrm{cl}_{\beta x}(X - U)$$

for open subsets U of X.

PROPOSITION 2.9. *If every point in $\upsilon X - X$ has neighborhoods of the form $\{\mathrm{Ex}(\bigcup_{j>n} P_j) : n \in \omega\}$ where $\{P_n : n \in \omega\}$ is a discrete sequence of non-empty open sets of a space X, then X is p-realcompact.*

PROOF. Let $Z = Z(f^\beta)$ be a zero-set of βX with $p \in X^* \cap Z$. We may assume that $p \in \upsilon X - X$ and so p has neighborhoods as in the hypothesis. Recursively pick $x_{j_n} \in P_{j_n} \cap f^{\leftarrow}[0, 1/n)$ for each $n \in \mathbb{N}$. Now D=$\{x_{j_n} : n \in \mathbb{N}\}$ is a closed discrete C-embedded subset of X and so D is not relatively pseudocompact in X. By Proposition 2.3 $\mathrm{cl}_{\beta X} D \not\subseteq \upsilon X$, but any point in $\mathrm{cl}_{\beta X} D - \upsilon X$ is in $Z - \upsilon X$.

A map $f: X \longrightarrow Y$ is *hyper-real* if $f^{\beta\rightarrow}(\beta X - \upsilon X) \subset \beta Y - \upsilon Y$. Hyper-real maps were introduced by Blair and are maps that carry realcompactness forward and pseudocompactness backward. Maps that are fiber countably compact and that carry zero-sets to closed sets (*e.g.* perfect maps) are hyper-real (see [13, 17.17, 17.19]). Not surprisingly hyper-real maps preserve p-realcompactness.

PROPOSITION 2.10. *If X is p-realcompact and $f: X \longrightarrow Y$ is a hyper-real surjective map, then Y is p-realcompact.*

PROOF. Let Z be a zero-set of βY that meets Y^*. Then $W = f^{\beta\leftarrow}(Z)$ is a zero-set of βX that meets X^*, and so there is a point $p \in W - \upsilon X$. Since f is hyper-real, then, $f^\beta(p) \in f^{\beta\rightarrow}(W) - \upsilon X \subset Z - \upsilon X$.

The next proposition shows that p-realcompact non-compact spaces have growths with rich structure.

PROPOSITION 2.11. *If X is p-realcompact and if Z is a zero-set of βX that meets X^*, then $X^* \cap Z$ contains a copy of $\beta\mathbb{N} - \mathbb{N}$ and hence has cardinality $\geq 2^c$.*

PROOF. This essentially follows from [8,9.5].

The question of whether the set-theoretic hypothesis MA + $\neg$ CH implies that every perfectly normal space is realcompact is still open as far as the authors know. It was a long-time conjecture of Blair that the question has an affirmative answer. (See [1] for a discussion of this question.) Of course, the question easily reduces to whether MA+$\neg$ CH implies that perfectly normal spaces are c-realcompact and so the following proposition is of some interest.

PROPOSITION 2.12 (MA $+\neg$ CH). *Every perfectly normal space is hyperisocompact, hence p-realcompact.*

PROOF. This an easy consequence of Weiss's theorem in [14] that MA $+\neg$ CH implies that every perfect countably compact space is compact.

3. **P'-spaces.** Before showing that locally compact p-realcompact spaces have growths that are P'-spaces, we need the following lemma. The proof given here is essentially that of [7,3.1].

LEMMA 3.0. *If X is locally compact, then every non-empty zero-set of βX that misses X has non-empty interior in X^*.*

PROOF. Let $Z = Z(f^\beta)$ be a non-empty zero-set of βX that misses X. Assume $f > 0$ on X. We can construct, recursively, sequences $\langle x_n : n \in \omega\rangle$ and $\langle f^{\leftarrow}(r_n, s_n) : n \in \omega\rangle$ such that $x_n \in f^{\leftarrow}(r_n, s_n) \subset f^{\leftarrow}(0, 1/n)$ and $f(x_n) \searrow 0$. Then for all n, we can pick a zero-set Z_n, a cozero-set P_n, and a compact set K_n with

$$x_n \subset Z_n \subset P_n \subset K_n \subset f^{\leftarrow}(r_n, s_n).$$

Let $P = \bigcup_{n\in\omega} P_n$. P is a cozero-set of X. We claim that $\emptyset \neq \mathrm{Ex}(P) \cap X^* \subset Z(f^\beta)$.

To show that $\emptyset \neq \mathrm{Ex}(P) \cap X^*$, let q be a limit point of $\{x_n : n \in \omega\}$ in βX. Since $q \in \mathrm{cl}_{\beta X} \bigcup_{n\in\omega} Z_n$, and $\bigcup_{n\in\omega} Z_n$ is a zero-set of X disjoint from $X - P$, $q \in \mathrm{Ex}(P) \cap X^*$. Now let $p \in \mathrm{Ex}(P) \cap X^*$ and suppose $f^\beta(p) = \epsilon > 0$. There is $k \in \omega$ with $s_k < \epsilon$. Then $p \notin \mathrm{cl}_{\beta X} \bigcup_{n\geq k} f^{\beta\leftarrow}(r_n, s_n)$ and so $p \notin \mathrm{cl}_{\beta X} \bigcup_{n\geq k} K_n$. But $\bigcup_{n<k} K_n$ is compact and so $p \notin \mathrm{cl}_{\beta X} \bigcup_{n\in\omega} K_n$, contradicting that $p \in \mathrm{Ex}(P)$. We conclude that $p \in Z(f^\beta)$.

To see that local compactness is necessary in the hypothesis of Lemma 3.0 we note (as did Fine and Gillman) that every zero-set of $\beta\mathbb{Q}$ that misses $\mathbb{Q}$ has empty interior in $\mathbb{Q}^*$

COROLLARY 3.1. *If X is locally compact and if Z is a zero-set of βX with $Z \not\subset \upsilon X$, then* $\mathrm{int}_{X^*}(X^* \cap Z) \neq \emptyset$.

PROOF. Let $p \in Z - \upsilon X$. There is $f \in C(\beta X)$ with $f(p) = 0$ and $f > 0$ on X. Then $\emptyset \neq Z(f) \cap Z \cap X^*$ while $Z(f) \cap Z \cap X = \emptyset$. By Lemma 3.0 $\emptyset \neq \mathrm{int}_{X^*}\big(Z(f) \cap Z\big) \subset \mathrm{int}_{X^*}(X^* \cap Z)$.

The following theorem is an immediate consequence of Corollary 3.1.

THEOREM 3.2. *If X is locally compact and p-realcompact, then X^* is a P'-space.*

PROPOSITION 3.3. *Let X be locally compact and nearly realcompact and let Z be a zero-set of βX. Then $Z \subset \upsilon X$ if and only if* $\mathrm{int}_{X^*}(Z - X) = \emptyset$.

PROOF. One direction is immediate from the definition of nearly realcompactness and the other is a consequence of Corollary 3.1.

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COROLLARY 3.4. *If X is locally compact and nearly realcompact, then X^* is a P'-space if and only if X is p-realcompact.*

We give examples now to show that locally compact nearly realcompact spaces need not have P' remainders, that is, need not be p-realcompact.

EXAMPLE 3.5. This example of a locally compact, nearly realcompact space that is not p-realcompact is essentially due to Mrówka in [10]. Before describing it we give a bit of terminology and background.

Two infinite sets X and Y are said to be *almost disjoint* if $X \cap Y$ is finite. A family A of infinite sets is *almost disjoint* if each pair from A is almost disjoint. If A is maximal with respect to the property of being almost disjoint we say that A is a mad family. If A is an almost disjoint family of infinite subsets of ω, then there is a natural topology on the set $\Psi(A) = A \cup \omega$: Points of ω are isolated and if $a \in A$, then $\{\{a\} \cup (a - F) : F$ is a finite subset of $A\}$ is a neighborhood base at a. We call any such space a Ψ-space. Obviously Ψ-spaces are locally compact, and it is a fact that $\Psi(A)$ is pseudocompact if and only if A is maximal. See [8,5I] for more about Ψ-spaces.

In [10,3.11], Mrówka proves the existence of an Ψ-space $Y = \Psi(A)$ such that $\beta Y = Y \cup \{\infty\}$ is the one point compactification of Y. It is this space that we modify slightly for our example.

Let E be the set of even integers and let O be the set of odd integers. Let A_0 be a mad family on $\mathcal{P}(E)$ such that the Stone-Čech compactification of $\Psi(A_0)$ is the one point compactification of $\Psi(A_0)$. Let A_1 be a copy of A_0 on the odd integers and fix a countable $B \subset A_0$. Fix a bijection $f: (A_0 - B) \longrightarrow A_1$ and let X be the quotient space on $\Psi(A_0 - B) \cup \Psi(A_1)$ obtained by identifying each $a \in A_0 - B$ with $f(a)$. Let i_0 and i_1 be the inclusion maps from $\Psi(A_0 - B)$ and $\Psi(A_1)$ to X. We claim that X is nearly realcompact but not p-realcompact.

To see that X is nearly realcompact, fix any $p \in \beta X$ and fix any zero-set $Z \subseteq X$ in p. Let $Z_1 = A_1 \cap i_1^{\leftarrow}(Z)$. Now, Z_1 is a zero-set in $\Psi(A_1)$ and therefore is either finite (in which case Z is finite and $p \in X$), or co-countable. So suppose that it is co-countable, and fix an open set U containing Z. We claim that there is a $b \in B$ such that $U \cap b$ is infinite. To see this, let $Z_0 = A_0 \cap i_0^{\leftarrow}(Z)$. Z_0 is a co-countable subset of A_0 disjoint from B. Since A_0 is a mad family, $A_0 - Z_0$ is a mad family on the set $(\bigcup B) - U$. But no countable almost disjoint family can be maximal. Therefore there is a $b \in B$ such that $U \cap b$ is infinite. Since it is also clopen and discrete in X, we have that $\mathrm{Ex}(U) - \upsilon X \neq \emptyset$.

To see that X is not p-realcompact, we first define a continuous function witnessing that $A = i_1^{\rightarrow}(A_1)$ is a zero-set in X. Let $B = \{b_k : k < \omega\}$ be an enumeration of B. Let $g(a) = 0$ for $a \in A$, $g(n) = 1/n$ for $n \in O$ and $g(n) = 1/k$ for $n \in B_k$. Note that the co-countable filter on A is a z-ultrafilter that converges to some $p \in \upsilon X$. The fact that A_0 and A_1 are mad families implies that $Z(g^\beta) = A \cup \{p\}$ and therefore $A \cup \{p\}$ is a zero set in βX witnessing that X is not p-realcompact.

Therefore this is an example of a locally compact and nearly realcompact space for which $\beta X - X$ is not P'. While this example is completely regular, it is not normal. Assuming CH, we can construct a normal example.

EXAMPLE 3.6. (CH) implies that there is a separable, normal, locally compact, nearly realcompact space that is not p-realcompact.

PROOF. Fix $A = \{a_\alpha : \alpha < \omega_1\}$, a copy of ω_1 disjoint from ω, and let $X = A \cup \omega$. Fix a partition of $\omega = \bigcup_{n<\omega} B_n$ such that each B_n is infinite. We define a topology τ on X such that

(1) (X, τ) is locally compact, locally countable, first countable and 0-dimensional.
(2) ω consists of isolated points.
(3) $A \cup \bigcup_{k>n} B_k$ is open for each n, and hence if $f\colon X \longrightarrow [0,1]$ is defined by $f^{\leftarrow}\{0\} = A$ and $f^{\leftarrow}\{1/n\} = B_n$ for each n, then f is continuous.
(4) A is homeomorphic to ω_1 with the order topology.
(5) For each infinite $b \subseteq \omega$, if $b \cap B_n \neq \emptyset$ for infinitely many $n \in \omega$, then b has an accumulation point in A.
(6) For each open U containing a co-countable subset $A' \subseteq A$, there is an open set V such that $A' \subseteq V \subseteq U$ and $U - V$ is infinite.

The local compactness and (4) imply that any such space is normal. Also it is nearly realcompact but not p-realcompact: The fact that the co-countable filter on A is a z-ultrafilter follows from (3) and (4). If $p \in \upsilon X$ is its limit, then (5) implies that $A \cup \{p\}$ is a zero-set witnessing that the space is not p-realcompact. Normality along with (6) imply nearly realcompactness. The proofs are analogous to the same results for the first example.

Before we present the construction we need some notation. Let $A_\alpha = \{a_\beta : \beta \leq \alpha\}$. We will call a countable set $b \subseteq \omega$ a transversal sequence if b is infinite and $b \cap B_n \leq 1$ for each $n < \omega$. Using CH we enumerate all transversal sequences as $\{b_\alpha : \alpha < \omega_1\}$. The example is defined recursively by constructing topologies τ_α on $X_\alpha = A_\alpha \cup \omega$ satisfying the following inductive hypotheses:

(a) $\beta < \alpha$ implies that τ_α is a conservative extension of τ_β (*i.e.*, τ_β coincides with the subspace topology on X_β from τ_α, and X_β is open in X_α).
(b) (X_α, τ_α) is locally compact, 0-dimensional, locally countable and first countable.
(c) b_β has an accumulation point in A_α for each $\beta < \alpha$.
(d) ω consists of isolated points in X_α.
(e) $A_\alpha \cup \bigcup_{k>n} B_k$ is open in X_α.

At limit stages $\alpha < \omega_1$ let τ'_α be the topology on $X_\alpha - \{a_\alpha\}$ generated by taking $\bigcup_{\beta<\alpha} \tau_\beta$ as a base. Note that τ'_α is a conservative extension of each τ_β. To define τ_α it suffices to define a local neighborhood base at a_α. Fix an increasing sequence $(\alpha_n)_{n\in\omega}$ with supremum α. For each n, the set $(a_{\alpha_n}, a_{\alpha_{n+1}}]$ is compact with respect to τ'_α. Therefore we can fix a compact open neighborhood $V_n \supseteq (a_{\alpha_n}, a_{\alpha_{n+1}}]$, so that $V_n \cap \omega \subseteq \bigcup_{k>n} B_k$. We also demand that the V_n's are pairwise disjoint. Let $U_n = \{a_\alpha\} \cup \bigcup_{k>n} V_k$ for each $n \in \omega$. If we let τ_α be generated by $\tau'_\alpha \cup \{U_n : n \in \omega\}$, then the inductive hypotheses are preserved.

At successor stages $\alpha = \beta + 1$ we consider the transversal sequence b_β. If it already has an accumulation point in A_β then we leave a_α isolated. Otherwise we extend τ_β to τ_α by letting $U_F = \{a_\alpha\} \cup b_\beta \setminus F$ be open for each finite set $F \subseteq b_\beta$. This makes b_β converge

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to a_α in (X_α, τ_α). Clearly the inductive hypotheses are preserved, thus completing the construction.

We need to check that the example satisfies the requirements (1)–(6). The inductive hypotheses imply (1), (2), (3), and (5). Hypothesis (4) follows easily from the construction. To see that (6) holds, fix an open set U containing a co-countable subset C of A. And suppose that $U \cap B_n$ were finite for each $n \in \omega$. Let $(a_{\alpha_n})_{n\in\omega}$ be an enumeration of the complement of C. For each n, fix a compact neighborhood U_n of a_{α_n} such that $U_n \cap \omega \subseteq \bigcup_{k>n} B_k$. Note also that, by compactness, U_n meets each of the B_k in a finite set. Let $V = U \cup \bigcup_{n\in\omega} U_n$. Then V is an open set containing A such that $V \cap B_n$ is finite for each n. This contradicts requirement (5). Therefore there is an n such that $U \cap B_n$ is infinite. Then $V = U - \bigcup_{k\leq n} B_k$ is an open set, $C \subseteq V \subseteq U$ and $U - V$ is infinite. This completes the construction.

We do not know whether there is an example in ZFC of a normal locally compact nearly realcompact space that is not p-realcompact.

We turn now to the question of what local conditions on a zero-set $Z(f)$ of a locally compact nearly realcompact space X will guarantee that $\mathrm{int}_{X^*}\big(X^* \cap Z(f^\beta)\big) = \emptyset$. We say that $f \in C(X)$ is a *well separated function* in X if whenever $Z(f) \subset P$ where P is a cozero-set of X, there is $n \in \mathbb{N}$ with $f^{\leftarrow}(-1/n, 1/n) \subset P$. We will use the following result.

PROPOSITION 3.7 [12,4.1]. *Let X be a space and let $f \in C^*(X)$. Then $\mathrm{cl}_{\beta X} Z(f) = Z(f^\beta)$ if and only if f is well separated in X.*

PROPOSITION 3.8. *For any space X, the following are equivalent.*

(1) X is p-realcompact

(2) Every relatively pseudocompact zero-set of a well separated function is compact.

PROOF. (1) $\Rightarrow$ (2). Let $Z(f)$ be a relatively pseudocompact and let f be a well separated function in X. By Proposition 2.3 and 3.7, $Z(f^\beta) = \mathrm{cl}_{\beta X} Z(f) \subset \upsilon X$. Since X is p-realcompact, $Z(f^\beta) \cap X^* = \emptyset$ which implies that $Z(f)$ is compact.

(2) $\Rightarrow$ (1). Let (1) be false and let $\emptyset \neq Z(f^\beta) \cap X^* \subset \upsilon X$. By Proposition 2.2, $\mathrm{cl}_{\beta X} Z(f) = Z(f^\beta)$ and so by Proposition 3.7 and 2.3, $Z(f)$ is relatively pseudocompact and f is well separated. Then (2) is false.

PROPOSITION 3.9. *Let X be locally compact and let $Z(f)$ be a zero-set of X. Then (1) $\Rightarrow$ (2) below and if, in addition, X is nearly realcompact, then (1) and (2) are equivalent.*

(1) $\mathrm{int}_{X^*}\big(X^* \cap Z(f^\beta)\big) = \emptyset$.

(2) $Z(f)$ is relatively pseudocompact, and f is well separated in X.

PROOF. (1) $\Rightarrow$ (2). If $Z(f)$ is not relatively pseudocompact, then by Proposition 2.3 $\mathrm{cl}_{\beta X} Z(f) \not\subseteq \upsilon X$ and so by Corollary 3.1 $\mathrm{int}_{X^*}\big(X^* \cap Z(f^\beta)\big) \neq \emptyset$. Suppose next that f is not well separated. Then by Proposition 3.7, $\mathrm{cl}_{\beta X} Z(f) \neq Z(f^\beta)$ and so by Proposition 2.2, $Z(f^\beta) \not\subseteq \upsilon X$. Then by Corollary 3.1, (1) is false.

(2) $\Rightarrow$ (1). Assume now that X is nearly realcompact. Then this implication follows from Proposition 3.7, 2.2 and 3.3.

COROLLARY 3.10. *If X is locally compact, then (2) $\Rightarrow$ (1). If, in addition, X is nearly realcompact, then (1) and (2) are equivalent.*

(1) X^* is P'.

(2) Every relatively pseudocompact zero-set of a well separated function is compact.

We remark that locally compact spaces with P' growths need not have any weak realcompactness properties. Any almost compact non-compact space (*e.g.* ω_1) has a P' growth but has no weak realcompactness properties. Thus the hypothesis that X is nearly realcompact cannot be dropped in Proposition 3.9 or Corollary 3.10.

REFERENCES

1. R. L. Blair, *Spaces in which special sets are z-embedded*, Canad. J. Math. **28**(1976), 673–690.
2. R. L. Blair and E. K. van Douwen, *Nearly realcompact spaces*, Topology Appl. **47**(1992), 209–221.
3. R. L. Blair and M. A. Swardson, *Spaces with an Oz Stone-Čech compactification*, Topology Appl. **36**(1990), 73–92.
4. E. K. van Douwen, *Remote points*, Dissertationes Math. **188**(1980), PWN, Warsaw.
5. N. Dykes, *Mappings and realcompact spaces*, Pacific J. Math. **31**(1969), 347–358.
6. ———, *Generalizations of realcompact spaces*, Pacific J. Math. **33**(1970), 571–581.
7. N. J. Fine and L. Gillman, *Extension of continuous functions in* $\beta\mathbb{N}$, Bull. Amer. Math. Soc. **66**(1960), 376–381.
8. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, 1960.
9. K. Hardy and R. G. Woods, *On c-realcompact spaces and locally bounded normal functions*, Pac. J. Math. **43**(1972), 647–656.
10. S. Mrówka, *Some set-theoretic constructions in topology*, Fund. Math. **94**(1977), 83–92.
11. J. Schommer, *Nearly realcompact and nearly pseudocompact spaces*, PhD. dissertation, Ohio University, Athens.
12. M. A. Swardson, *The character of certain closed sets*, Canad. J. Math. **36**(1984), 38–57.
13. M. D. Weir, *Hewitt-Nachbin spaces*, North Holland, Amsterdam, 1975.
14. W. Weiss, *Countably compact spaces and Martin's axiom*, Canad. J. Math. **30**(1978), 243–249.

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