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THE STRICT TOPOLOGY ON A SPACE OF VECTOR-VALUED FUNCTIONS

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1. Introduction

Let X be a topological space, E a real or complex topological vector space, and $C(X, E)$ the vector space of all bounded continuous E-valued functions on X. The notion of the strict topology on $C(X, E)$ was first introduced by Buck (1) in 1958 in the case of X locally compact and E a locally convex space. In recent years a large number of papers have appeared in the literature concerned with extending the results contained in Buck's paper (1); see, for example, (14), (15), (3), (4), (12), (2), and (6). Most of these investigations have been concerned with generalising the space X and taking E to be the scalar field or a locally convex space.

In this paper we define the strict topology β on $C(X, E)$, where X is now any Hausdorff topological space and E an arbitrary Hausdorff topological vector space. In Section 3 we consider the properties of $(C(X, E), \beta)$ as a topological vector space and show that it has almost all the properties of the 'strict topology' studied by the above authors. In Section 4 we establish an analogue of the Stone–Weierstrass theorem in the β-topology setting.

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2. Notation and terminology

Throughout this paper we shall assume, unless stated otherwise, that X is a Hausdorff topological space, and E a non-trivial Hausdorff topological vector space and we let $\mathcal{W}$ denote a base of closed balanced neighbourhoods of 0 in E.

Let $B(X, E)$ be the vector space of all bounded E-valued functions on X and $B_0(X, E)$ (resp. $B_{00}(X, E)$) the subspace of $B(X, E)$ consisting of those functions which vanish at infinity (have compact support). The subspaces consisting of continuous functions in $B(X, E)(B_0(X, E), B_{00}(X, E))$ will be denoted by $C(X, E)(C_0(X, E), C_{00}(X, E))$. When E is the real or complex field, these spaces will be denoted by $B(X)$, $B_0(X)$, $B_{00}(X)$, $C(X)$, $C_0(X)$, and $C_{00}(X)$. We shall denote by $B(X) \otimes E$ the vector space spanned by the set of all functions of the form $\phi \otimes a$, where $\phi \in B(X)$, $a \in E$, and $(\phi \otimes a)(x) = \phi(x)a (x \in X)$.

Let $\phi, \phi_1 \in B(X)$. Then ϕ_1 is said to *dominate* ϕ if there exists a $\lambda > 0$ such that $|\phi(x)| \leq \lambda |\phi_1(x)|$ for all $x \in X$.

3. The strict topology on $C(X, E)$

We begin by describing a general method of defining linear topologies on $C(X, E)$.

Definition 3.1. Let S be any subset of $B(X)$. We define the *S-topology* on $C(X, E)$ to be the linear topology which has a sub-base of neighbourhoods of 0 consisting of all sets of the form

$$U(\phi, W) = \{f \in C(X, E) : \phi(x)f(x) \in W \quad \text{for all } x \in X\},$$

where $\phi \in S$ and $W \in \mathcal{W}$.

Lemma 3.2. (cf. (3), Lemma 2.1). *Let S and S_1 be subsets of $B(X)$. If each element of S is dominated by an element of S_1, then the S-topology on $C(X, E)$ is weaker than the S_1-topology.*

Proof. Let U_1 be any S-neighbourhood of 0 in $C(X, E)$, and suppose $U_1 \supseteq \bigcap_{i=1}^{n} U(\phi_i, W_i)$, where $\phi_1, \ldots, \phi_n \in S$ and $W_1, \ldots, W_n \in \mathcal{W}$. For each ϕ_i $(i = 1, \ldots, n)$, choose a $\lambda_i > 0$ and a $\psi_i \in S_1$ such that $|\phi_i(x)| \leq \lambda_i |\psi_i(x)|$ for all $x \in X$. Let $U_2 = \bigcap_{i=1}^{n} U(\psi_i, (1/\lambda)W_i)$, where $\lambda = \max\{\lambda_1, \ldots, \lambda_n\}$. Then U_2 is an S_1-neighbourhood of 0 in $C(X, E)$ and $U_2 \subseteq U_1$, as required.

Using the notion of an S-topology, we now introduce the strict topology and other related topologies on $C(X, E)$, as follows.

The $B_0(X)$-topology on $C(X, E)$ is called the *strict topology* and is denoted by β. The $B(X)$-topology is called the *uniform topology* and is denoted by υ. It easily follows from Lemma 3.2 that the υ-topology is the same as the $\{1\}$-topology, where $1 \in B(X)$ is the function identically 1 on X. The $B_{00}(X)$-topology is called the *compact-open topology* and is denoted by κ. It is evident that the κ-topology is the linear topology which has a sub-base of neighbourhoods of 0 consisting of all sets of the form $U(\chi_\kappa, W)$, where χ_κ is the characteristic function of any compact set K in X and $W \in \mathcal{W}$. Let $B_\rho(X)$ be the subspace of $B(X)$ consisting of functions with finite support. Then the $B_\rho(X)$-topology is called the *point-wise topology* and is denoted by ρ. It is easily seen that $\rho \leq \kappa \leq \upsilon$; if X is compact, then κ and υ coincide, and if X is discrete, then ρ and κ coincide.

The following lemma gives us a convenient form for the base of neighbourhoods of 0 in $C(X, E)$ for each of the topologies defined above.

Lemma 3.3. *Let S denote any one of the sets $B(X)$, $B_0(X)$, $B_{00}(X)$, or $B_\rho(X)$. Then the S-topology on $C(X, E)$ has a base of neighbourhoods of 0 consisting of all sets of the form $U(\phi, W)$, where $\phi \in S$ with $0 \leq \phi \leq 1$ and $W \in \mathcal{W}$.*

Proof. Let U_1 be any S-neighbourhood of 0, and suppose $U_1 \supseteq \bigcap_{i=1}^{m} U(\phi_i, W_i)$, where $\phi_1, \ldots, \phi_m \in S$ and $W_1, \ldots, W_m \in \mathcal{W}$. Let $\lambda = \max_{1 \leq i \leq m}\{\|\phi_i\|\}$. If $\lambda > 0$, choose a $W \in \mathcal{W}$ with $\lambda W \subseteq \bigcap_{i=1}^{m} W_i$. Define

$$\phi(x) = \max_{1 \le i \le m} \left\{ \frac{|\phi_i(x)|}{\lambda} \right\} \quad (x \in X).$$

Then $\phi \in S$, $0 \le \phi \le 1$, and it is easy to show that $U(\phi, W) \subseteq U_1$. If $\lambda = 0$, then $U_1 = C(X, E)$, and so, if we take $\phi_0 = 0$, we have $U_1 \supseteq U(\phi_0, W)$ for any W in $\mathcal{W}$. Thus the S-neighbourhoods of 0 have a base of the required form.

The properties of $(C(X, E), \beta)$ are given in the following two theorems which extend results proved by Buck ((1), Theorem 1), Giles ((3), Theorem 2.4), and other authors ((4), (5), (12)).

Theorem 3.4. (i) $\rho \le \kappa \le \beta \le \upsilon$.

(ii) *If X is completely regular, then*

(a) *υ and β coincide if and only if X is compact;*

(b) *β and κ coincide if and only if every σ-compact subset of X is relatively compact.*

(iii) *υ and β have the same bounded sets in $C(X, E)$.*

(iv) *β and κ coincide on υ-bounded subsets of $C(X, E)$.*

(v) *A sequence $\{f_n\}$ in $C(X, E)$ is β-convergent if and only if it is υ-bounded and κ-convergent.*

Proof. (i) This follows immediately from Lemma 3.2.

(ii) (a) Suppose $\upsilon \le \beta$. Then, by Lemma 3.3, for any $W \in \mathcal{W}$, there exist a $\phi \in B_0(X)$ with $0 \le \phi \le 1$ and a $V \in \mathcal{W}$ such that $U(\phi, V) \subseteq U(1, W)$. If $E \backslash W \ne \phi$, let $c \in E \backslash W$ and choose $\lambda > 0$ such that $c \in \lambda V$. If X is not compact, then $X \backslash F \ne \phi$ for every compact set F in X. Since $\phi \in B_0(X)$, the set $\{x \in X : \phi(x) \ge 1/\lambda\}$ has a compact closure, K say, in X. Let $x_0 \in X \backslash K$, and choose a $\psi \in C(X)$ such that $0 \le \psi \le 1$, $\psi(x_0) = 1$, and $\psi(K) = 0$. Let $g = \psi \otimes c$. Then $g \in U(\phi, V)$ but $g \notin U(1, W)$, which is a contradiction. If $W = E$, choose a W_0 in $\mathcal{W}$ such that $W_0 \subset E$ and then argue as above with W_0 replacing W. On the other hand, if X is compact, then $\kappa = \upsilon$ and so, from (i), $\beta = \upsilon$, as required.

(b) If every σ-compact subset of X is relatively compact, then it is easy to show that $\beta \le \kappa$. Conversely, let $\beta \le \kappa$, and suppose that there is a set $G = \bigcup_{n=1}^{\infty} K_n$ (K_n compact in X) which is not relatively compact. Then, for each compact set F in X, $G \backslash F \ne \phi$. Let $\phi = \sum_{n=1}^{\infty} 2^{-n} \chi_{K_n}$. Then $\phi \in B_0(X)$ and $\phi = 0$ outside of G. For any $W \in \mathcal{W}$, there exist a compact set K in X and a $V \in \mathcal{W}$ such that $U(\chi_K, V) \subseteq U(\phi, W)$. If $E \backslash W \ne \phi$, let $d \in E \backslash W$, and $y_0 \in G \backslash K$. Choose a $\psi_1 \in C(X)$ with $0 \le \psi_1 \le (1/\phi(y_0))$, $\psi_1(y_0) = 1/\phi(y_0)$, and $\psi_1(K) = 0$. Let $h = \psi_1 \otimes d$. Then $h \in U(\chi_K, V)$ but $h \notin U(\phi, W)$, a contradiction. If $W = E$, choose a W_0 *in* $\mathcal{W}$ such that $W_0 \subset E$ and then argue as above with W_0 replacing W.

(iii) Suppose there is a set $A \subseteq C(X, E)$ which is β-bounded but not υ-bounded. Then there exist sequences $\{f_n\} \subseteq A$, $\{x_n\} \subseteq X$, and a $W \in \mathcal{W}$ such that $f_n(x_n) \notin n^2 W$. Let $\phi(x) = 1/n$ if $x = x_n$, and $\phi(x) = 0$ if $x \ne x_n$ $(n = 1, 2, \ldots)$. Then $\phi \in B_0(X)$ but $\phi(x_n) f_n(x_n) \notin nW$; that is, $\{f_n\}$, and hence A, is not β-bounded. This contradiction proves the result.

(iv) The proof follows from standard arguments (see (3, Theorem 2.4(iv)) and is omitted.

(v) This follows immediately from (iii) and (iv).

Theorem 3.5. (i) *$C_{00}(X, E)$ is β-dense in $C(X, E)$ if and only if X is locally compact.*

(ii) *If X is a k-space and E is complete, then $C(X, E)$ is β-complete.*

(iii) *If $(C(X, E), \beta)$ is metrizable, then β and υ coincide.*

Proof. (i) Suppose X is locally compact, and let $f \in C(X, E)$. Let $\phi \in B_0(X)$, $0 \leq \phi \leq 1$, and $W \in \mathcal{W}$.

Let $K \subseteq X$ be a compact set such that $\phi(x)f(x) \in W$ for $x \notin K$. Choose a $\psi \in C_{00}(X)$ such that $0 \leq \psi \leq 1$ and $\psi(K) = 1$. Let $g = \psi f$. Then $g \in C_{00}(X, E)$ and

$$\phi(x)(g(x) - f(x)) = \phi(x)(\psi(x) - 1)f(x) \begin{cases} = 0 & \text{if } x \in K, \\ \in (\psi(x) - 1)W \subseteq W & \text{if } x \notin K \end{cases}$$

(since W is balanced).

Thus $g - f \in U(\phi, W)$, and so f belongs to the β-closure of $C_{00}(X, E)$; that is, $C_{00}(X, E)$ is β-dense in $C(X, E)$, as required.

Conversely, suppose $C_{00}(X, E)$ is β-dense in $C(X, E)$ but that X is not locally compact. Then there exists a $y \in X$ which has no compact neighbourhood. Consequently $f(y) = 0$ for all $f \in C_{00}(X, E)$. It follows that, if h is any non-zero constant function in $C(X, E)$, then h does not belong to the ρ-closure, and hence to the β-closure of $C_{00}(X, E)$; that is, $C_{00}(X, E)$ is not β-dense in $C(X, E)$.

(ii) The proof may be carried out by using an argument similar to the one used in (**3**, Theorem 2.4(v)).

(iii) Suppose $(C(X, E), \beta)$ is metrizable. By Theorem 3.4(iii), the identity mapping $i: (C(X, E), \beta) \to (C(X, E), \upsilon)$ takes bounded sets into bounded sets. Hence, by (**11**, Theorem 1.32), i is continuous; that is, $\upsilon \leq \beta$.

A subset A of $C(X, E)$ is said to be *equicontinuous* at $x \in X$ if, for each $W \in \mathcal{W}$, there exists a neighbourhood $N(x)$ of x such that $f(y) - f(x) \in W$ for all $y \in N(x)$ and $f \in A$. A is said to be equicontinuous on X if it is equicontinuous at each point of X.

We now give an analogue of the Arzelà–Ascoli theorem.

Theorem 3.6. *Let X be a k-space and E a topological vector space. Then a subset A of $C(X, E)$ is β-compact if and only if the following conditions hold*:

(i) *A is β-closed*;

(ii) *A is β-bounded*;

(iii) *$A(x) = \{f(x) : f \in A\}$ is relatively compact in E for each $x \in X$*;

(iv) *A is equicontinuous on each compact subset of X.*

Proof. Suppose A is β-compact in $C(X, E)$. Then conditions (i) and (ii) hold trivially. Since $\kappa \leq \beta$, A is κ-compact and so (iii) and (iv) follow from (**7**, p. 81, Exercise H(d)).

Conversely, suppose that a subset A of $C(X, E)$ satisfies conditions (i)–(iv). Since A, being β-bounded, is υ-bounded, the topologies β and κ coincide on A (Theorem 3.4(iv)). Thus, to show that A is β-compact, it is only necessary to show that A is κ-compact. Now, by using the same argument as the one used to prove Theorem 3.4(iv), we can show that the β and κ closures of A are the same. Consequently, A is κ-closed. This fact together with conditions (iii) and (iv) imply that A is κ-compact (see (**7**, p. 81, Exercise H(d)). This completes the proof.

Let $S_0(X)$ denote the set of all non-negative upper semi-continuous functions on X

which vanish at infinity. Then the $S_0(X)$-topology on $C(X, E)$ is called the *weighted topology* and is denoted by ω (cf. (**10**), p. 283).

Theorem 3.7. *The topologies ω and β coincide on $C(X, E)$.*

Proof. It is clear that $\omega \leq \beta$. Now, let $\phi \in B_0(X)$. By Lemma 3.2, it is sufficient to show that there exists a function in $S_0(X)$ which dominates ϕ. For each n, the set $\{x \in X : |\phi(x)| \geq 2^{-n}\}$ has compact closure, K_n say, in X. Let $\psi = \Sigma_{n=1}^{\infty} 2^{-n}\chi_{K_n}$. Then it is not difficult to show that $\psi \in S_0(X)$ and ψ dominates ϕ.

We conclude this section with an open problem. Let β' denote the finest linear topology on $C(X, E)$, which coincides with the κ-topology on υ-bounded sets. Clearly $\beta \leq \beta'$. Katsaras (**6**, Theorem 3.4) has shown that, if X is completely regular and E a normed space, then $\beta = \beta'$ (see also, Fontenot (**2**, p. 844)). However, we do not know whether or not $\beta = \beta'$ when X is completely regular and E a general topological vector space.

4. A Stone–Weierstrass theorem for $(C(X, E), \beta)$

The Stone–Weierstrass theorem for $(C(X, E), \beta)$ was first established by Buck (**1**) for X a locally compact metrizable space and E finite dimensional. This result was later extended to locally compact space X and locally convex space E by Todd (**14**) and Wells (**15**). In this section we establish a Stone–Weierstrass type theorem with E any topological vector space but introducing an additional condition on X which we define as follows.

Definition 4.1. (**9**, p. 9). Let $\mathcal{U}$ be a collection of subsets of a topological space X. For any $x \in X$, we define $\text{ord}_x\mathcal{U}$, the order of $\mathcal{U}$ at x, as the number of members of $\mathcal{U}$ which contain x, and we define $\text{ord}\,\mathcal{U} = \sup_{x\in X}\{\text{ord}_x\mathcal{U}\}$. The *covering dimension* of X is defined as the least positive integer n such that, for any finite open covering $\mathcal{U}$ of X, there exists an open covering $\mathcal{B}$ such that $\mathcal{B}$ is a refinement of $\mathcal{U}$ and $\text{ord}\,\mathcal{B} \leq n+1$. If no such finite n exists, then we say that X has an infinite covering dimension.

Theorem 4.2. *Let X be a completely regular space of finite covering dimension and E a topological vector space. If A is a $C(X)$-submodule of $C(X, E)$ such that, for each $x \in X$, $A(x)$ is dense in E, then A is β-dense in $C(X, E)$.*

Proof. Suppose X has covering dimension of order n, and let $f \in C(X, E)$. Let $\phi \in B_0(X)$, $0 \leq \phi \leq 1$, and $W \in \mathcal{W}$. There exists a $V \in \mathcal{W}$ such that $V + V + \cdots + V((n+2)-\text{terms}) \subseteq W$. Let K be a compact subset of X such that $\phi(x)f(x) \in V$ for $x \notin K$. For each $x \in X$, choose a function g_x in A and an open neighbourhood $N(x)$ of x such that $g_x(y) - f(y) \in V$ for all $y \in N(x)$. The sets $\{N(x) : x \in K\}$ form an open covering of K, and so there exists a finite open covering, $\{N(x_j) : j = 1, \ldots, m\}$ say, of K. The sets $\mathcal{U} = \{X\backslash K, N(x_j)(j = 1, \ldots, m)\}$ form a finite open covering of X, and so, by hypothesis, there exists an open covering $\mathcal{B}$ of X such that $\mathcal{B}$ is a refinement of $\mathcal{U}$ and

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ord $\mathscr{B} \leqslant n+1$. Since K is compact, a finite number of members of $\mathscr{B}$, $N_1, \ldots, N_r$ say, will cover K. Moreover, since $\mathscr{B}$ is a refinement of $\mathscr{U}$, for each $1 \leqslant i \leqslant r$, there exists a j_i, $1 \leqslant j_i \leqslant m$, such that $N_i \subseteq N(x_{j_i})$. Let $\{\phi_i : i = 1, \ldots, r\}$ be a collection of functions in $C(X)$ such that $0 \leqslant \phi_i \leqslant 1$, $\phi_i = 0$ outside of N_i, $\Sigma_{i=1}^r \phi_i(x) = 1$ for $x \in K$, and $\Sigma_{i=1}^r \phi_i(x) \leqslant 1$ for $x \in X$ (**8**, p. 69, Lemma 2). We define an E-valued function g on X by

$$g(x) = \sum_{i=1}^{r} \phi_i(x) g_{x_{j_i}}(x),$$

where $g_{x_{j_i}}$ is the function in A chosen as indicated earlier. Then $g \in A$. Let y be any point in X. If $I_y = \{i : y \in N_i\}$, then I_y has at most $(n+1)$-members and $\phi_i(y) = 0$ if $i \notin I_y$. Consequently if $y \in K$, then

$$\begin{aligned}\phi(y)(g(y) - f(y)) &= \phi(y)\left\{\sum_{i=1}^{r} \phi_i(y)(g_{x_{j_i}}(y) - f(y))\right\}\\ &= \phi(y)\left\{\sum_{i \in I_y} \phi_i(y)(g_{x_{j_i}}(y) - f(y))\right\}\\ &\in V + V + \cdots + V \quad \text{(at most } (n+1)\text{-times)}\\ &\subseteq W.\end{aligned}$$

If $y \notin K$, we have

$$\begin{aligned}\phi(y)(g(y) - f(y)) &= \phi(y) \sum_{i=1}^{r} \phi_i(y)(g_{x_{j_i}}(y) - f(y)) + \left\{\sum_{i=1}^{r} \phi_i(y) - 1\right\}\phi(y) f(y)\\ &\in V + \cdots + V \quad \text{(at most } (n+1)\text{-times)} + V\\ &\subseteq W.\end{aligned}$$

Thus $g - f \in U(\phi, W)$, and so f belongs to the β-closure of A; that is, A is β-dense in $C(X, E)$, as required.

Corollary 4.3. *Let X and E be as in the theorem, and let A be a $C(X)$-submodule of $C(X, E)$ and $f \in C(X, E)$. Then f belongs to the β-closure of A if and only if, for each $x \in X$, $f(x)$ belongs to the closure of $A(x)$ in E.*

The following result is a generalisation of (**13**, Theorem 1).

Corollary 4.4. *Let X and E be as in the theorem. Then $C(X) \otimes E$ is β-dense in $C(X, E)$.*

If E is locally convex, then the proof of Theorem 4.2 can be modified slightly to give the following

Theorem 4.5. *Let X be completely regular and E a locally convex space. If A is a $C(X)$-submodule of $C(X, E)$ such that, for each $x \in X$, $A(x)$ is dense in E, then A is β-dense in $C(X, E)$.*

The above extends the results of Wells (**15**, Theorem 2) and Todd (**14**, Theorem 3). Consequently, Theorems 4 and 5 in (**14**), which characterise the β-closed maximal $C(X)$-submodules of $C(X, E)$, will be true for X completely regular.

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