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A GEOMETRICAL CHARACTERIZATION OF A CLASS OF HOLOMORPHIC VECTOR BUNDLES OVER A COMPLEX TORUS

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This note is to be a supplement of the preceeding paper in the journal by Matsushima, settling a question raised by him. In his paper he associates a holomorphic vector bundle over a complex torus to a holomorphic representation of what he calls Heisenberg group. We shall show that a simple holomorphic vector bundle is determined in this manner if and only if the associated projective bundle admits an integrable holomorphic connection. A theorem by Morikawa ([3], Theorem 1) is the motivation of this problem and is somewhat strengthened by our result.

Let V be a complex vector space of dimension n and let L be a lattice in V. The quotient group $V/L = E$ is a complex torus. It is known ([2], § 3) that a holomorphic vector bundle F of rank m over E is determined by a $GL(m, \boldsymbol{C})$-valued theta factor J, namely by a holomorphic map

$$J : L \times V \to GL(m, \boldsymbol{C})$$

satisfying the following equality:

$$(1) \qquad J(\alpha + \beta, u) = J(\alpha, \beta + u)J(\beta, u) \qquad \text{for } \alpha, \beta \in L \text{ and } u \in V .$$

We denote by F_J the holomorphic vector bundle over E determined by a theta factor J.

A résumé of Matsushima's construction of holomorphic vector bundles over E is in order. Let H be a hermitian form on $V \times V$. Let G_H be a nilpotent Lie group whose underlying manifold is $V \times \boldsymbol{C}^*$ and whose multiplication is defined by

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$$(u, c) \cdot (v, d) = \left(u + v, e\left[\frac{1}{2i} H(u, v)\right] cd\right) \qquad \text{for } (u, c), (v, d) \in V \times \boldsymbol{C}^* ,$$

where $e[x] = \exp 2\pi i x$.

We denote by $G_H(L)$ the subgroup $L \times \boldsymbol{C}^*$ in G_H, which is a complex Lie group. The right action of the complex Lie group $G_H(L)$ on the complex manifold $V \times \boldsymbol{C}^*$ is holomorphic. Thus $V \times \boldsymbol{C}^*$ is a holomorphic principal bundle over E with structure group $L \times \boldsymbol{C}^* = G_H(L)$.

If a holomorphic representation $\rho: G_H(L) \to GL(m, \boldsymbol{C})$ is given, a holomorphic vector bundle F_{H}, ρ is determined as the quotient space $V \times \boldsymbol{C}^* \times C^m / \{G_H(L), \rho\}$. Lemma 3.1 in [2] shows that a theta factor J_ρ associated to the holomorphic vector bundle F_H, ρ is given by

$$(2) \quad J_\rho(\alpha, u) = \rho\left(-\alpha, e\left[\frac{1}{2i}(H(u, \alpha) + H(\alpha, \alpha))\right]\right) \qquad \text{for } \alpha \in L,\ u \in V .$$

THEOREM. *Suppose that the associated projective bundle of a holomorphic vector bundle F over a complex torus E admits an integrable holomorphic connection, or equivalently admits a system of transition functions which are constant. Then, we can choose a hermitian form H_1 on $V \times V$ whose imaginary part assumes rational values on $L \times L$, and a holomorphic representation ρ of $G_{H_1}(L)$ so that F is isomorphic to F_{H_1}, ρ.*

Proof. (a) It is well known (Atiyah [1], Proposition 14) that the associated projective bundle $P(F)$ of a holomorphic vector bundle F admits an integrable holomorphic connection if and only if $P(F)$ arises from a homomorphism h of the fundamental group L of torus E into $PGL(m, \boldsymbol{C})$. A necessary and sufficient condition for the projective bundle $P(F)$ to have an integrable holomorphic connection is that one can choose a theta factor J of F such that

$$(3) \qquad J(\alpha, u) = J(\alpha, 0)\mu(\alpha, u)$$

with scalar function $\mu(\alpha, u)$ for each $\alpha \in L$.

Indeed, this condition is sufficient. Suppose that $P(E)$ admits an integrable holomorphic connection. Then $P(E)$ arises from a homomorphism $h: L \to PGL(m, \boldsymbol{C})$. Let us denote by $\tilde{J}(\alpha, u)$ the image of a theta factor $J(\alpha, u)$ under the natural homomorphism of $GL(m, \boldsymbol{C})$ onto $PGL(m, \boldsymbol{C})$. Since the $PGL(m, \boldsymbol{C})$-valued factor $\tilde{J}$ and the homomorphism h define the same bundle $P(F)$,

$$h(\alpha) = \tilde{\varphi}(u + \alpha)\tilde{J}(\alpha, u)\tilde{\varphi}(u)^{-1}$$

with a $PGL(m, \boldsymbol{C})$-valued holomorphic function $\tilde{\varphi}$ on V. Since V is simply connected, we can lift $\tilde{\varphi}$ to a holomorphic map $\varphi: V \to SL(m, \boldsymbol{C})$ so that $\varphi(u)$ is lying above $\tilde{\varphi}(u)$. Then, $J'(\alpha, u) = \varphi(u + \alpha)J(\alpha, u)\varphi(u)^{-1}$ is a theta factor with required property.

(b) Let us assume that a holomorphic vector bundle F satisfies the condition in the theorem and that a theta factor J of F is chosen so that the condition (3) is satisfied. From the condition (1) on J, it follows that the scalar function μ determined by (3) satisfies the following equalities:

(i) $\mu(\alpha, \beta)\mu(\alpha + \beta, u) = \mu(\alpha, \beta + u)\mu(\beta, u)$, for $\alpha, \beta \in L$, $u \in V$;

(ii) $\mu(\alpha, 0) = \mu(0, u) = 1$, for $\alpha \in L$, $u \in V$;

(iii) $\mu(\alpha, -\alpha) = \mu(-\alpha, \alpha), \alpha \in L$.

We define a multiplication $\times$ on $L \times \boldsymbol{C}^*$ in terms of μ and make $L \times \boldsymbol{C}^*$ a complex Lie group $G_\mu(L)$:

$$(\alpha, c) \times (\beta, d) = (\alpha + \beta, \mu(\beta, \alpha)cd) \qquad \text{for } (\alpha, c), (\beta, d) \in L \times \boldsymbol{C}^* .$$

The associative law is verified by (i). The identity is $(0, 1)$, because of (ii) and the inverse of (α, c) is $(-\alpha, \mu(-\alpha, \alpha)^{-1}c)$.

Define a map

$$f: G_\mu(L) \to GL(m, \boldsymbol{C})$$

by $f(\alpha, c) = J(\alpha, 0)^{-1}c$. Then, f is a holomorphic representation. In fact,

$$f((\alpha, c) \times (\beta, d)) = J(\alpha + \beta, 0)^{-1}\mu(\beta, \alpha)cd .$$

Since $J(\alpha + \beta, 0) = J(\beta, \alpha)J(\alpha, 0) = J(\beta, 0)J(\alpha, 0)\mu(\beta, \alpha)$ by (1) and (3),

$$\begin{aligned} f((\alpha, c) \times (\beta, d)) &= J(\alpha, 0)^{-1}J(\beta, 0)^{-1}cd \\ &= f(\alpha, c)f(\beta, d) . \end{aligned}$$

(c) The map $L \times V \to \boldsymbol{C}^*$ given by $(\alpha, u) \to \det J(\alpha, u)$ is a $\boldsymbol{C}^*$-valued theta factor corresponding to the line bundle $\det F$, which is equivalent to a normalized theta factor ([4], p. 111). We choose a $\boldsymbol{C}^*$-valued holomorphic function φ on V, a hermitian form H on $V \times V$ whose imaginary part assumes integral values on $L \times L$ and a semi-character $\chi: L \to \boldsymbol{C}^*$ such that

$$\det J(\alpha, u) = \varphi(u + \alpha)\chi(\alpha)e\left[\frac{1}{2i}H(u, \alpha) + \frac{1}{4i}H(\alpha, \alpha)\right]\varphi(u)^{-1} .$$

On the other hand from (3),

$$\det J(\alpha, u) = \det J(\alpha, 0)\mu^m(\alpha, u) .$$

Thus,

$$\det J(\alpha, 0) = \varphi(\alpha)\varphi(0)^{-1}\chi(\alpha)e\left[\frac{1}{4i}H(\alpha, \alpha)\right]$$

and

$$\mu^m(\alpha, u) = \varphi(u + \alpha)\varphi(\alpha)^{-1}e\left[\frac{1}{2i}H(u, \alpha)\right]\varphi(u)^{-1}\varphi(0) .$$

Since φ is a nowhere vanishing holomorphic function on a simply connected space V, there is a nowhere vanishing holomorphic function ψ on V such that $\psi^m = \varphi$. For each α, an m^{th} root of unity ε_α is determined by

$$\mu(\alpha, u) = \varepsilon_\alpha\psi(u + \alpha)\psi(\alpha)^{-1}e\left[\frac{1}{2mi}H(u, \alpha)\right]\psi(u)^{-1}\psi(0) .$$

Putting $u = 0$, we see that $1 = \mu(\alpha, 0) = \varepsilon_\alpha$. Thus,

$$(5)\qquad \mu(\alpha, u) = \frac{\psi(u + \alpha)}{\psi(\alpha)}e\left[\frac{1}{2mi}H(u, \alpha)\right]\frac{\psi(0)}{\psi(u)} .$$

(d) The above relation enables us to establish an isomorphism of $G_{H/m}(L)$ and $G_\mu(L)$. Put

$$\lambda(\alpha) = \psi(\alpha)/\psi(0) .$$

Then from (5),

$$(6)\qquad \mu(\alpha, \beta) = \frac{\lambda(\alpha + \beta)}{\lambda(\alpha)\lambda(\beta)}e\left[\frac{1}{2mi}H(\beta, \alpha)\right] .$$

Making use of the $\lambda(\alpha)$'s, we define a map

$$(7)\qquad g: G_{H/m}(L) \to G_\mu(L)$$

by

$$g(\alpha, c) = (\alpha, \lambda(\alpha)c) , \qquad (\alpha, c) \in L \times \boldsymbol{C}^* .$$

We claim that g is an isomorphism. Obviously, g is 1 : 1 and onto.

$$\begin{aligned} g((\alpha, c)\cdot(\beta, d)) &= g\Big(\alpha + \beta, e\Big[\frac{1}{2mi}H(\alpha,\beta)\Big]cd\Big) \\ &= \Big(\alpha + \beta, \lambda(\alpha + \beta)e\Big[\frac{1}{2mi}H(\alpha,\beta)\Big]cd\Big) \\ &= (\alpha + \beta, \lambda(\alpha)\lambda(\beta)\mu(\beta,\alpha)cd) \\ &= g(\alpha, c) \times g(\beta, d)\ , \end{aligned}$$

on account of (6). Thus g is an isomorphism.

(e) From (4) and (7), $\rho = f \circ g$ is a holomorphic representation of $G_{H/m}(L)$ into $GL(m, \boldsymbol{C})$ given by

$$\rho(\alpha, c) = J(\alpha, 0)^{-1}\lambda(\alpha)c\ .$$

The theta factor J' associated to the representation ρ in the formula (2) is

$$\begin{aligned} J'(\alpha, u) &= \rho\Big(-\alpha, e\Big[\frac{1}{2mi}(H(u,\alpha) + H(\alpha,\alpha))\Big]\Big) \\ &= J(-\alpha, 0)^{-1}\lambda(-\alpha)e\Big[\frac{1}{2mi}(H(u,\alpha) + H(\alpha,\alpha))\Big]\ . \end{aligned}$$

Making use of the equalities

$$\begin{aligned} J(-\alpha, 0)^{-1} &= J(\alpha, 0)\mu(0, -\alpha)\ , \\ \mu(\alpha, -\alpha) &= \lambda(\alpha)^{-1}\lambda(-\alpha)^{-1}e\Big[-\frac{1}{2mi}H(\alpha,\alpha)\Big] \end{aligned}$$

and of the equality (5), we see that

$$J'(\alpha, u) = \psi(u + \alpha)^{-1}J(\alpha, u)\psi(u)\ .$$

Thus, we have seen that the theta factors J and J' are equivalent and hence $F \cong F_{H/m}, \rho$, finishing the proof.

Remark. In order to prove the converse of the theorem, we assume that a holomorphic vector bundle F_ρ over E associated to a holomorphic representation ρ of $G_H(L)$ is simple. Then, the image of the central subgroup $\{0\} \times \boldsymbol{C}^*$ is of scalar matrices and hence $\rho(0, c) = c^k$ for some integer k. The projective representation $\tilde{\rho}$ reduces to a projective representation of L. By Atiyah's proposition, $P(F_\rho)$, which arises from $\tilde{\rho}$ of L, admits an integrable holomorphic connection.

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