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DISCRETE COACTIONS ON C^*-ALGEBRAS

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Abstract

We will consider coactions of discrete groups on C^*-algebras and imitate some of the results about compact group actions on C^*-algebras. In particular, the crossed product of a reduced coaction ϵ of a discrete amenable group G on A is liminal (respectively, postliminal) if and only if the fixed point algebra of ϵ is. Moreover, we will also consider ergodic coactions on C^*-algebras.

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1. Introduction and notation

The concept of coactions is defined so that it is the dual notion of actions of groups on C^*-algebras. The most interesting and important results about coactions are the duality theorems (see [6, 7, 11]). Recently, Raeburn gave a systematic treatment of this subject [12]; the terminology we use here mainly follows this and [8]. In this paper, we will examine some properties of discrete reduced coactions on C^*-algebras. In the second section, we will give some elementary results of discrete coactions. When the group is abelian, discrete coactions correspond to compact actions. It is believed that many results in compact actions can be translated to the case of discrete coactions with more direct and elementary proofs (since the representation theory of $C_0(G)$ for G discrete is much simpler than that of $C^*(G)$ for G compact). Sections 3 and 4 are mainly used to demonstrate this. In the third section, we will translate results of Gootman and Lazar [3] which state that the crossed product of a C^*-dynamical system is liminal (respectively, postliminal) if and only if the fixed point algebra is. As a corollary, we show that if the fixed point algebra is postliminal, then the original algebra is nuclear. In Section 4, we try to translate some results in the paper [5]

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which include: If a compact group acts ergodically on a C^*-algebra A, then A has a tracial state which is, in fact, the unique G-invariant state of A. However, the referee has pointed out that there is some overlap between Section 4 and Theorem 3.7 of an unpublished paper of Quigg [10].

For the definition of coactions and reduced coactions and their crossed products, we refer the reader to [12, §2]. If A is a C^*-algebra and δ is a (respectively, reduced) coaction of a locally compact group G on A, then we call the triple (A, G, δ) a (respectively, reduced) codynamical system.

In this paper, unless otherwise specified, G is a discrete group with unit e, A is a C^*-algebra, all the coactions and crossed products are reduced, and all the tensor products considered are spatial ones. By [1, 7.15] reduced coactions of discrete groups are automatically non-degenerate.

2. Discrete coactions

Let ϵ be a reduced coaction of G on A. Since $C_r^*(G)$ is unital, $\epsilon(A) \subseteq \tilde{M}(A \otimes C_r^*(G)) = A \otimes C_r^*(G)$. Denote by λ_t the canonical image of $t \in G$ in $C_r^*(G)$. Then $\{\lambda_t : t \in G\}$ form a generating set of $C_r^*(G)$. Let $\psi_t \in C_r^*(G)^*$ be defined by $\psi_t(\lambda_s) = 1$ (if $s = t$) and $\psi_t(\lambda_s) = 0$ (if $s \neq t$). Note that ψ_e is the natural tracial state of $C_r^*(G)$ and $\psi_t = \lambda_t^{-1} \cdot \psi_e$ (that is, $\psi_t(x) = \psi_e(\lambda_t^{-1}x)$) and hence is continuous. Let S_t be the slice map corresponding to ψ_t ($t \in G$). Note that in this case, we have $S_t(a \otimes z) = z(t) \cdot a$ and $\delta_G(z) = \sum_{s\in G} z(s) \cdot \lambda_s \otimes \lambda_s$ for all $z \in \mathcal{K}(G)$, the space of continuous functions on G with compact, hence finite, support; $a \in A$; and where δ_G is the co-multiplication on $C^*(G)$. (For its definition we refer the reader to [12, §1].) Denote by M_f the canonical representation of $f \in C_0(G)$ in $l^2(G)$ by multiplication. In the following we will write M_t instead of M_{δ_t}, where $\delta_t \in C_0(G)$ is the map defined by $\delta_t(s) = 1$ if $s = t$ and $\delta_t(s) = 0$ if $s \neq t$.

The following definition is an analogue of the case of finite coactions (see the introduction to [8]) and the t-components correspond to the spectral subspaces in the case of compact actions.

DEFINITION 2.1. Let $A_t = \{a \in A : \epsilon(a) = a \otimes \lambda_t\}$, $(t \in G)$. We will call this the *t-component of* ϵ. The e-component of ϵ is called the *fixed point algebra* of ϵ and will also be denoted by A^ϵ.

REMARK 2.2. Note that the fixed point algebra can be defined without G being discrete (since $\lambda_e = 1$) but in the case when G is discrete and $A \neq (0)$, we have $A^\epsilon \neq (0)$. Suppose $A^\epsilon = (0)$. Let $\{u_\tau\}$ be an approximate unit of A. By the non-degeneracy of ϵ (as a *-homomorphism), $\{\epsilon(u_\tau)\}$ converges strictly to the unit of $M(A \otimes C_r^*(G))$. Since S_e is strictly continuous (see [8, 1.5]), $S_e(\epsilon(u_\tau))$ converges

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strictly to the unit of $M(A)$. However 2.3(ii) implies that $S_e(\epsilon(u_\tau)) \in A^\epsilon = (0)$, which gives a contradiction.

LEMMA 2.3. *Let ϵ be a reduced coaction of G on A and S_t as above.*

(i) $A_t^* = A_{t^{-1}}$ *and* $A_t \cdot A_s \subseteq A_{ts}$;

(ii) $S_t(\epsilon(A)) = A_t$ *for all* $t \in G$.

PROOF. (i) is clear.

(ii) For any $a \in A$, $\epsilon(S_t(\epsilon(a)) = S_t[(\epsilon \otimes i)(\epsilon(a))] = S_t[(i \otimes \delta_G)(\epsilon(a))] = S_t(\epsilon(a)) \otimes \lambda_t$. (Here S_t in the first and fourth terms means the slice map of ψ_t on $A \otimes C_r^*(G)$, while S_t in the other terms means the slice map of ψ_t on $A \otimes C_r^*(G) \otimes C_r^*(G)$.)

PROPOSITION 2.4. *The set $\{(a_r \otimes \lambda_r)(1 \otimes M_t) : r, t \in G,\ a_r \in A_r\}$ is a generating set for the reduced crossed product $A \times_\epsilon G$. Moreover, for any $r, s, t, u \in G$, $a_r \in A_r \setminus (0)$ and $b_s \in A_s \setminus (0)$, we have $(a_r \otimes \lambda_r)(1 \otimes M_t) = (b_s \otimes \lambda_s)(1 \otimes M_u)$ if and only if $r = s$, $t = u$ and $a_r = b_s$.*

PROOF. For simplicity, we will assume A acts faithfully on a Hilbert space H and $A \times_\epsilon G$ is a subalgebra of $B(H \otimes L^2(G))$ (see [8, 2.6(3)]). By an easy calculation, we have $\lambda_s M_t = M_{st}\lambda_s$ and $M_t \lambda_r M_s = \delta_{rs,t} \cdot \lambda_r M_s$, where δ is the Kronecker delta. Now let $x \in A \otimes C_r^*(G)$ be an elementary tensor $b \otimes z$, where $z \in \mathcal{K}(G)$. Then

$$\begin{aligned}(1 \otimes M_t)x(1 \otimes M_s) &= b \otimes z(ts^{-1})\lambda_{ts^{-1}} M_s \\ &= [S_{ts^{-1}}(b \otimes z)] \otimes \lambda_{ts^{-1}} M_s \\ &= (S_{ts^{-1}}(x) \otimes \lambda_{ts^{-1}})\,(1 \otimes M_s).\end{aligned}$$

In fact, the above equality holds for any $x \in A \otimes C_r^*(G)$ by linearity and continuity. Moreover, by [8, 2.5], the set $\{(1 \otimes M_t)\epsilon(a)(1 \otimes M_s) : r, s \in G,\ a \in A\}$ spans a dense subspace of $A \times_\epsilon G$. Thus, using the above calculation and Lemma 2.3(ii), part one is proved.

Let $a \in A_r \setminus (0)$ and $b \in A_s \setminus (0)$ be such that $(a \otimes \lambda_r)(1 \otimes M_t) = (b \otimes \lambda_s)(1 \otimes M_u)$. Then by multiplying $(1 \otimes M_e)$ on the left of both sides and using the calculations above, we have $S_{t^{-1}}(a \otimes \lambda_r) \otimes M_e \lambda_{t^{-1}} = S_{u^{-1}}(b \otimes \lambda_s) \otimes M_e \lambda_{u^{-1}}$ and thus $t = u$. Similarly, by multiplying $(1 \otimes M_{rt})$ on both sides, we have $(a \otimes \lambda_r)(1 \otimes M_t) = (S_r(b \otimes \lambda_s) \otimes \lambda_r)(1 \otimes M_t)$. Hence $r = s$ and $a = b$ as required.

REMARK 2.5. (1) It is easily seen from the calculation in 2.4 that for any $r, s, t, u \in G$, we have $(A_r \otimes \lambda_r)(1 \otimes M_t) \cdot (A_s \otimes \lambda_s)(1 \otimes M_u) \subseteq (A_{rs} \otimes \lambda_{rs})(1 \otimes M_u)$ and also $(A_r \otimes \lambda_r)(1 \otimes M_t)(A_s \otimes \lambda_s)(1 \otimes M_u) = (0)$ if $t \neq su$.

(2) Let α be the dual action on $B = A \times_\epsilon G$ (see for example [8, proof of 4.8]). If we define p_t to be the projection $1 \otimes M_t \in M(B)$, then p_t satisfies the following:

(i) $\alpha_t(p_e) = p_{t^{-1}}$;
(ii) $p_t \cdot p_s = \delta_{t,s} p_s$ (where δ means the Kronecker delta);
(iii) Let $\mathscr{F}(G)$ be the collection of all finite subsets of G and let $p_E = \sum_{t\in E} p_t$. Then p_E converge to 1 strictly in $M(B)$.

In fact, by a result of Quigg (see [9, 4.3]), it is easily seen that given a dynamical system (B, α), the existence of the projection p_e satisfying (i)-(iii) is equivalent to (B, α) being a dual system.

PROPOSITION 2.6. *Let A_t be the t-component of ϵ. Then $A = \overline{\bigoplus_{t\in G} A_t}$. Moreover, if $A = \overline{\bigoplus_{t\in G} B_t}$ then $B_r \cdot B_s \subseteq B_{rs}$, $B_r^* = B_{r^{-1}}$ and $\|\sum_{t\in G} b_t\| = \|\sum_{t\in G} b_t \otimes \lambda_t\|$ (for a finite number of non-zero $b_t \in B_t$) if and only if there exists a reduced coaction δ of G on A such that $A_t = B_t$ for all $t \in G$.*

PROOF. Note that $S_e(\epsilon(a)(1 \otimes \mu\lambda_t)) = a_{t^{-1}}\mu$ for any $a \in A$ and $\mu \in \mathbb{C}$. By [1, 7.15], $\epsilon(A)(1 \otimes C_r^*(G))$ is dense in $A \otimes C_r^*(G)$. Now for any $a \in A$, $a \otimes 1 = \lim_n \sum_{n,t} \epsilon(a_n)(1 \otimes \mu_n^t\lambda_t)$ (where only finitely many μ_n^t are non-zero). Hence $a = S_e(a \otimes 1) = \lim_n \sum_{n,t} (a_n)_t \mu_n^t \in \overline{\bigoplus_{t\in G} A_t}$. Thus, the first part is proved. Suppose now that $A = \overline{\bigoplus_{t\in G} B_t}$ such that B_t satisfies the prescribed properties. Define δ on $\bigoplus_{t\in G} B_t$ by $\delta(b_t) = b_t \otimes \lambda_t$ for $b_t \in B_t$. Then δ is an isometric *-homomorphism from $\bigoplus_{t\in G} B_t$ to $A \otimes C_r^*(G)$. Hence δ extends to a reduced coaction on A.

EXAMPLE 2.7. Let (A, G, α) be a C^*-dynamical system and B be its reduced crossed product with dual coaction ϵ (where α is considered as in Proposition 2.6). Then $B = C^*\{\alpha(a)(1 \otimes \lambda_r) : a \in A,\ r \in G\}$. Let B_t be the t-component of ϵ. For any $x \in B_t$, there exist $x_n \in B$ of the form $\sum_{s\in F_n} \alpha(a_{s,n})(1 \otimes \lambda_s)$ (where $\{s \in G : a_{s,n} \neq 0\}$ is finite for all such n) such that x_n converges to x in norm. Hence, $\alpha(a_{t,n})(1 \otimes \lambda_t) = S_t \circ \epsilon(x_n)$ converges to $S_t \circ \epsilon(x) = x$. Since α is a monomorphism, $a_{t,n}$ converges to some element $a \in A$. Thus $x = \alpha(a)(1 \otimes \lambda_t)$ and $B_t = \{\alpha(a)(1 \otimes \lambda_t) : a \in A\}$. Moreover, $B = \overline{\bigoplus_{t\in G} B_t}$ and $A \otimes \mathscr{K}(l^2(G)) \cong (A \times_{\alpha,r} G) \times_{\epsilon,r} G \cong C^*\{[\alpha(a)(1 \otimes \lambda_t)] \otimes \lambda_t M_t : r, t \in G, a \in A\}$.

It is obvious that if G is a locally compact group and G_d is its discrete group, then any action of G on A is an action of G_d on A. We now show the dual property in the case of coactions.

PROPOSITION 2.8. *Let G be a locally compact group and G_d be its discrete group. Then any (respectively, reduced) coaction δ of G_d on A induces a (respectively, reduced) coaction δ' of G on A.*

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PROOF. Here, we show the case of reduced coaction only. Let μ be the canonical map from $C_r^*(G_d)$ to $M(C_r^*(G))$ and let $u_t = \mu(\lambda_t)$ (that is, u_t is the canonical image of $t \in G$ in $C_r^*(G)$). Define $\delta' = (i \otimes \mu) \circ \delta$ from A to $M(A \otimes C_r^*(G))$. The non-degeneracy of δ' (as a *-homomorphism) follows from that of $i \otimes \mu$ and δ. Firstly we show that $\delta'(A) \subseteq \tilde{M}(A \otimes C_r^*(G))$. Let $a \in A$. Since $\delta(a) \in A \otimes C_r^*(G_d)$, there is $x_n = \sum_k a_{k,n} \otimes z_{k,n}$ ($a_{k,n} \in A$, $z_{k,n} \in C_r^*(G_d)$) such that x_n converges to $\delta(a)$. Hence, for any $z \in C_r^*(G)$, we have $a_{k,n} \otimes \mu(z_{k,n})z \in A \otimes C_r^*(G)$ (as $\mu(z_{k,n}) \in M(C_r^*(G))$) and so $\delta'(a)(1 \otimes z) = \lim \sum a_{k,n} \otimes \mu(z_{k,n})z \in A \otimes C_r^*(G)$ (since $i \otimes \mu$ is a homomorphism, the convergence above is in the norm limit). Finally, we check the coaction identity. For any $z \in \mathscr{K}(G_d)$, $z = \sum_{t \in F} \nu_t \lambda_t$ for some $F \in \mathscr{F}(G)$ and $\nu_t \in \mathbb{C}$. Thus, $(\mu \otimes \mu)\delta_{G_d}(z) = (\mu \otimes \mu)(\sum_{t \in F} \nu_t \lambda_t \otimes \lambda_t) = \sum_{t \in F} \nu_t u_t \otimes u_t$. On the other hand, $\delta_G(\mu(z)) = \delta_G(\sum_{t \in F} \nu_t u_t) = \sum_{t \in F} \nu_t u_t \otimes u_t = (\mu \otimes \mu)\delta_{G_d}(z)$. Now, for any $a \in A$, $(\delta' \otimes i)\delta'(a) = (i \otimes \mu \otimes \mu)(\delta \otimes i)\delta(a) = (i \otimes (\mu \otimes \mu) \circ \delta_{G_d})\delta(a) = (i \otimes \delta_G \circ \mu)\delta(a) = (i \otimes \delta_G)\delta'(a)$ as required.

3. Liminality of crossed products by discrete coactions

The ideas of the following results come from [3]. Lemmas 3.1 to 3.3 are well-known results and are true for general locally compact groups.

LEMMA 3.1. *Let α be an action of G on A. Then the largest liminal ideal I of A is α-invariant.*

LEMMA 3.2. *Let α be an action of G on A, I be an α-invariant ideal of A, q be the quotient map from A to A/I and β be the induced action of α on A/I. If J is a β-invariant ideal of A/I, then $q^{-1}(J)$ is an α-invariant ideal of A.*

LEMMA 3.3. *Let α be an action of G on A. The largest postliminal ideal I of A is α-invariant.*

LEMMA 3.4. *Let G be a discrete group and ϵ be a reduced coaction of G on A. If J is a non-trivial ϵ-invariant ideal of A, then we have $A^\epsilon/(A^\epsilon \cap J) \cong (A/J)^\delta \neq (0)$ (here δ is the coaction induced by ϵ on A/J).*

PROOF. Let q be the quotient map from A to A/J. Then by 2.3(ii), $q(A^\epsilon) = q\,[S_e(\epsilon(A))] = S_e\,[(q \otimes i)(\epsilon(A))] = S_e[\delta(q(A))] = S_e[\delta(A/J)] = (A/J)^\delta$.

The following is actually a direct consequence of strong Morita equivalence, but we give an elementary proof here.

LEMMA 3.5. *Let B be a hereditary subalgebra of A such that there is no non-trivial ideal in A containing B. If B is liminal (respectively, postliminal), then so is A.*

PROOF. For any $(\pi, H) \in \hat{A}$, let (ϕ, K) be the restriction of (π, H) to B. Then (ϕ, K) is an irreducible representation of B. If $x \in B$ is such that $\phi(x) \in \mathscr{K}(K)$, then $\pi(x) \in \mathscr{K}(H)$. Suppose B is postliminal. Then, since A is the only ideal in A containing B, we have $\phi \neq 0$. Thus, there exists $x \in B$ such that $\phi(x) \in \mathscr{K}(K) \setminus (0)$. Hence $\pi(x) \in \mathscr{K}(H) \setminus (0)$ and A is postliminal. Moreover, if B is liminal, then $\phi(x) \in \mathscr{K}(K)$ for all $x \in B$. Hence B is contained in the largest liminal ideal I of A. Thus $I = A$, by hypothesis.

THEOREM 3.6. *Let ϵ be a reduced coaction of a discrete amenable group G on A. Then A^ϵ is liminal (respectively, postliminal) if and only if $A \times_\epsilon G$ is liminal (respectively, postliminal).*

PROOF. Necessity: By Proposition 2.4, $M = \{(a \otimes \lambda_e)(1 \otimes M_e) : a \in A_e\} = \{a \otimes M_e : a \in A^\epsilon\}$ is a subalgebra of $A \times_\epsilon G$ which is isomorphic to A^ϵ. Hence the liminal or postliminal property of $A \times_\epsilon G$ implies that of A^ϵ.

Sufficieny: Let $p = 1 \otimes M_e \in M(A \times_\epsilon G)$. Then p is a projection and by the proof of Proposition 2.4, we have $p(A \times_\epsilon G)p = M$ (where M is as in the first part of the proof). Hence M is a liminal (respectively, postliminal) hereditary subalgebra of $A \times_\epsilon G$. Therefore, M is contained in the largest liminal (respectively, postliminal) ideal I of $A \times_\epsilon G$ by 3.5. Now we claim that $I = A \times_\epsilon G$. Suppose not: Then since I is $\hat{\epsilon}$-invariant for the dual action $\hat{\epsilon}$ of ϵ, there is an ϵ-invariant ideal J of A such that $I = J \times_\epsilon G$ (by [2, 3.7]). By Lemma 3.4, there exists $x \in A^\epsilon$ such that $x \notin J$. But since $x \otimes M_e \in M \subseteq I = J \times_\epsilon G$, $x \in J$ (see Proposition 2.4) which gives a contradiction.

REMARK. Note that [2, 3.7] is the only place where we need the amenability of G and so the proposition can be improved if [2, 3.7] is true for any reduced coaction of discrete group.

COROLLARY 3.7. *Let G be a finite group and ϵ a reduced coaction of G on A. Then the following are equivalent:*

(i) *A^ϵ is liminal (respectively, postliminal);*
(ii) *$A \times_\epsilon G$ is liminal (respectively, postliminal);*
(iii) *A is liminal (respectively, postliminal).*

For G not finite, we have a weaker result as follows:

COROLLARY 3.8. *Let G be a discrete amenable group and ϵ a reduced coaction of G on A. If A^ϵ is postliminal, then A is nuclear.*

PROOF. By 3.6, $A \times_\epsilon G$ is postliminal and hence nuclear. Now, by Katayama's duality theorem [7, 8] and the fact that nuclearity is preserved under crossed products of amenable groups (see [4, 14]), $A \otimes \mathcal{K}(L^2(G))$ is nuclear and so is A.

By replacing 3.6, [7, 8] and [4, 14] by [3, 3.2], [11, 7] and [12, 4.6] in the proof of 3.8, we get the following result:

PROPOSITION 3.9. *Let G be a compact group and α an action of G on A. If A^α is postliminal, then A is nuclear.*

REMARK 3.10. Note that by Corollary 3.8, if there exists an ergodic coaction (see 4.1) of a discrete amenable group on A, then A is nuclear. In fact, in the next section, we can show that A has a continuous faithful trace (see 4.6).

4. Ergodic coactions

DEFINITION 4.1. A reduced coaction ϵ of a discrete group G on a unital C^*-algebra A is said to be *ergodic* if A^ϵ $(= A_e)$ consists of scalar multiples of the unit only.

LEMMA 4.2. *Let (A, G, ϵ) be a discrete reduced codynamical system. If A is unital and ϵ is ergodic, then for any $t \in G$, all elements in A_t are scalar multiples of unitaries. Moreover, A_t is one dimensional if $A_t \neq 0$.*

PROOF. For any $x \in A_t$, $u = x^*x$ and $v = xx^*$ belong to $A_e = \mathbb{C}$ by 2.3. Hence x is a scalar multiple of an isometry and a co-isometry and thus x is a scalar multiple of a unitary. Now, suppose $A_t \neq 0$ and let x and y be two unitaries in A_t. Then x^*y is a unitary in A_e and so $x^*y = e^{i\theta} \cdot 1$ for some $\theta \in \mathbb{R}$. Hence, x is a scalar multiple of y and thus A_t is one dimensional.

If ϵ is an ergodic coaction of G on A $(1 \in A)$, then the map ω defined by $\omega(x)1 = S_e \circ \epsilon(x)$ is a state on A. Moreover, ω is averagely ϵ-invariant in the following sense:

DEFINITION 4.3. Let (A, G, ϵ) be a discrete (reduced) codynamical system. Then $\phi \in A^*$ is said to be *averagely ϵ-invariant* if $\phi = (\phi \otimes \psi_e) \circ \epsilon$.

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REMARK 4.4. (a) The above definition corresponds to $\phi(x) = \phi(\int \alpha_s(x)\, ds)$ in the case of actions. This is why the term 'averagely' is used. For simplicity, we will drop the term 'averagely' in the following.

(b) ω is ϵ-invariant since $(\omega \otimes \psi_e) \circ \epsilon = [(i \otimes \psi_e) \circ \epsilon \otimes \psi_e] \circ \epsilon = (i \otimes \psi_e \otimes \psi_e) \circ (i \otimes \delta_G) \circ \epsilon = (i \otimes \psi_e) \circ \epsilon = \omega$.

(c) For ergodic coactions, the ϵ-invariant state is unique. In fact, if ω' is an ϵ-invariant state, then $\omega'(x) = \omega'((i \otimes \psi_e)\epsilon(x)) = \omega'(\omega(x)1) = \omega(x)$.

(d) If ϕ is ϵ-invariant, then ϕ vanishes on A_t (for $t \neq e$). In fact, if $x \in A_t$, then $\phi(x) = (\phi \otimes \psi_e)(x \otimes \lambda_t) = 0$.

PROPOSITION 4.5. *The state ω defined above is faithful.*

PROOF. Since ϵ is injective, it suffices to show that $(i \otimes \psi_e)(y^* y) = 0$ implies that $y = 0$ for $y \in A \otimes C_r^*(G)$. Note that ψ_e is a faithful state on $C_r^*(G)$. Now for any state f of A, $\psi_e[(f \otimes i)(y^* y)] = f \otimes \psi_e(y^* y) = 0$ which implies that $(f \otimes i)(y^* y) = 0$. Hence $y = 0$ as required.

Corresponding to the results of Hoegh-Krohn, Landstad and Stormer (see [5]), we have the following results.

PROPOSITION 4.6. *If ϵ is an ergodic coaction as above, then the unique ϵ-invariant state is a tracial state.*

PROOF. Since ϵ is a homomorphism, we have to show that $S_e(y^* y) = S_e(yy^*)$ for any $y \in \epsilon(A)$. By 2.6, it is required to show that this holds for $y \in \epsilon(\bigoplus_{t\in G} A_t)$. Now for all $y \in \epsilon(\bigoplus_{t\in G} A_t)$, $y = \sum_{t\in F} a_t \otimes \lambda_t$ for some $a_t \in A_t$ and $F \in \mathscr{F}(G)$. So $S_e(y^* y) = S_e(\sum_{r,s\in F} a_r^* a_s \otimes \lambda_{r^{-1}s}) = \sum_{r\in F} a_r^* a_r = \sum_{r\in F} a_r a_r^* = S_e(yy^*)$ by 4.2 and hence the proposition is proved.

DEFINITION 4.7. Let (A, G, ϵ) be a reduced codynamical system and ϕ an ϵ-invariant state of A. Then ϵ is said to be

(a) *cyclic* with respect to ϕ if there exists an $x \in A$ such that for any $y \in A$, $(\phi \otimes \psi_t)[(y \otimes 1)\epsilon(x)] = 0$ (for all $t \in G$) implies $y = 0$.
(b) *weakly cyclic* with respect to ϕ if there is a family $\{z_t \in A_t : t \in G\}$ such that for any $y \in A$, $\phi(yz_t) = 0$ (for all $t \in G$) implies $y = 0$.

LEMMA 4.8. *Let (A, G, ϵ) be a reduced codynamical system and ϕ an ϵ-invariant state of A. If ϵ is cyclic with respect to ϕ, then it is weakly cyclic with respect to ϕ. The converse holds if G is countable.*

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PROOF. For any $x \in A$, $x = \lim_n \sum_{t\in G} x_{n,t}$ by 2.6 (where for a fixed n, only finitely many $x_{n,t} \in A_t$ are non-zero) and $x_{n,t}$ will converge to $x_t = S_t(\epsilon(x))$. Now we have $(\phi \otimes \psi_t)[(y \otimes 1)\epsilon(x)] = \lim_n \phi(yx_{n,t}) = \phi(yx_t)$. If ϵ is cyclic with respect to ϕ, then it is clear that it is also weakly cyclic with respect to ϕ. Now suppose that G is countable and let $G = \{t_n : n \in \mathbb{N}\}$. If ϵ is weakly cyclic with respect to ϕ, let $\{z_t\}_{t\in G}$ be the collection of elements as in 4.7(b). Let $x_t = z_t/\|z_t\|$ if $z_t \neq 0$ and $x_t = 0$ if $z_t = 0$. Define $x = \sum_{n\in\mathbb{N}} 2^{-n} x_{t_n}$. Then $(\phi \otimes \psi_t)[(y \otimes 1)\epsilon(x)] = 0$ if and only if $\phi(yz_t) = 0$, and thus the converse is proved.

PROPOSITION 4.9. *Let (A, G, ϵ) be a discrete reduced codynamical system with A unital. Then ϵ is ergodic if and only if there exists an ϵ-invariant state ω such that ϵ is weakly cyclic with respect to ω.*

PROOF. Necessity: Define $\phi \in A'$ by $\phi(y) = \omega(yz_e)$ (where $\{z_t\}$ are the elements in 4.7(b)). For any $y \in A^\epsilon$, $\omega(yz_t) = (\omega \otimes \psi_e)(\epsilon(yz_t)) = \delta_{t,e}\phi(y)$ (as ω is ϵ-invariant). Hence $\phi(y) = 0$ if and only if $\omega(yz_t) = 0$ for all $t \in G$, if and only if $y = 0$. Now ϕ is an injection from A^ϵ to $\mathbb{C}$ and the necessity is shown.

Sufficiency: Since ϵ is ergodic, $\omega(x)1 = S_e(\epsilon(x))$ is the unique ϵ-invariant state by 4.4(c). Now for any $t \in G$, take any $x_t \in A_t$ with $\|x_t\| = 1$ if $A_t \neq (0)$ and $x_t = 0$ otherwise. By 4.2, the set $\{x_t : t \in G\}$ generates $\bigoplus_{t\in G} A_t$. Consider the GNS-representation (π, H) for ω and let $\varphi : A \to H$ be the canonical map. Then $\{\varphi(x_t)\}$ generates H (since $\varphi(\bigoplus_{t\in G} A_t)$ is dense in H). Hence, for any $y \in A$, if $\langle\varphi(y^*), \varphi(x_s)\rangle = \omega(yx_s) = 0$ for all $s \in G$, then $\varphi(y^*) = 0$ in H and thus $y = 0$. (Note that φ is injective since ω is faithful.)

REMARK. As a corollary of 4.8 and 4.9 we obtain the translated version of [5, 4.4 and 5, 4.5].

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