Normal structure of *p*-groups

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It is suggested by a result of [2] that the Frattini subgroup of a
finite *p*-group is quite restricted in its structure, but it is not known
what this structure must be. This thesis provides some information on this
matter by considering normal subgroups of a *p*-group embedded in its
Frattini subgroup.

In the first chapter it is shown that frequently in such a normal
subgroup, its derived subgroup must be contained in the subgroup generated
by the *p*-th powers of its elements or by their 4-th powers if $p = 2$.
This must be the case, for example, if the embedded normal subgroup has at
most *p* generators and is contained in the group generated by the *p*-th
powers of elements in the embedding group.

The structure of groups defined in this way is dealt with. It is
shown that, while they are not necessarily regular, they share many of the
properties of regular groups. A particular case of the groups studied is
that of groups which are products of cyclic *p*-groups which permute in
pairs (*p* odd).

The second chapter considers the structure of those 2-generator
p-groups which have metacyclic factor groups of large order. Arising out
of this are new necessary and sufficient conditions for a *p*-group to be
metacyclic. Some of these results extend to 2-groups theorems known
previously only for odd primes *p* , and in addition provide tests for
whether a metacyclic 2-group is ordinary (that is, possesses a
presentation like that of the odd-prime groups, [3]) or is exceptional. In
particular, it is shown that it is sufficient to test whether the second
Frattini factor group is metacyclic and if so, whether it is an ordinary

metacyclic group.

Finally, three applications are made. The first section of Chapter 3 proves that the only 2-generator groups of the kind discussed in Chapter 1 are the ordinary metacyclic groups. Furthermore, no subgroup of an ordinary metacyclic group is exceptional. The second application is the determination of precisely which 2-generator p-groups can be normal subgroups of a p-group contained in its Frattini subgroup.

The problems studied in the thesis originated in attempts to extend a problem of Blackburn [1] on the embedding of 2-generator groups in p-groups. The remaining application in Chapter 3 is to a special case of Blackburn's problem, for odd primes p .

References

[1] Norman Blackburn, "Generalizations of certain elementary theorems on p-groups", *Proc. London Math. Soc.* (3) 11 (1961), 1-22.

[2] Wolfgang Gaschütz, "Über die Φ-Untergruppe endlicher Gruppen", *Math. Z.* 58 (1953), 160-170.

[3] Bruce W. King, "Presentations of metacyclic groups", *Bull. Austral. Math. Soc.* 8 (1973), 103-131.