SOME RESTRICTED PARTITION FUNCTIONS: CONGRUENCES MODULO 5

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1. Introduction

Ramanujan was the first mathematician to discover some of the arithmetical properties of $p(n)$, the number of unrestricted partitions of n. His congruence,

$$p(5n+4) \equiv 0 \pmod 5, \tag{1}$$

for example, is famous [2; 3]. Some progress has been made since then; it is known that the congruence,

$$p(n) \equiv r \pmod 5, \tag{2}$$

has an infinitude of solutions for any arbitrary value of r [4]. This is a somewhat weak relation, and one would have liked to obtain, if possible, stronger results of the type,

$$\tau(n) \equiv 0 \pmod 5, \tag{3}$$

for 'almost all' values of n, which in its turn is derivable from another stronger relation, viz.,

$$\tau(n) \equiv n\sigma(n) \pmod 5, \tag{4}$$

also established by Ramanujan [2], where $\tau(n)$ is Ramanujan's function defined by

$$x\left[\prod_1^\infty(1-x^n)\right]^{24} = \sum_1^\infty \tau(n)x^n, \qquad |x| < 1. \tag{5}$$

What is done in this paper is to modify the unrestricted character of the partition function, $p(n)$, by the introduction of some simple restrictions, so that not only relations of the type (1) but also others including those of types (3) and (4) may be found.

2. The main results

It is obviously necessary to explain the nature of the above restrictions. The restriction merely imposes the condition that no number of the forms

$75n$, $75n \pm r$, (where r is a specified number), shall be a part (necessarily positive integral) of the relevant partitions. In other words, in order to determine the value of

$$ {}^{75}_{r}p(n), $$

the partition function restricted in this way, one should count all the unrestricted partitions of n excepting those which contain a number of any of the above forms as a part. We shall assume ${}^{75}_{r}p(n)$ and $p(n)$ to be unity when $n = 0$, and vanishing when n is negative. In this paper we shall deal mainly with the five restricted partition functions corresponding to $r = 5\rho$, $\rho = 1, 2, 4, 5$ and 7.

The restricted partition function ${}^{75}_{25}p(n)$ corresponding to $r = 25$, (or $\rho = 5$), has a somewhat simpler interpretation. It is easily seen that this particular function requires the count of all the unrestricted partitions of n excepting those which contain 25 or any multiple thereof as a part. We have, in this case, considered it desirable to use the simpler notation,

$$ {}^{25}p(n) = {}^{25}_{0}p(n) = {}^{75}_{25}p(n), \tag{6} $$

in order to emphasize the simpler interpretation.

We can now state our main results.

THEOREM 1. *For almost all values of n,*

$$ \begin{aligned} {}^{25}p(n) &\equiv 0 && \pmod{5}, \\ {}^{75}_{20}p(n) &\equiv {}^{75}_{5}p(n-5) && \pmod{5}, \\ {}^{75}_{35}p(n) &\equiv -{}^{75}_{10}p(n-5) && \pmod{5}. \end{aligned} $$

THEOREM 2. *For all values of n,*

$$ {}^{25}p(5n+4) \equiv 0 \pmod{5}; $$

and more generally for $\rho = 0, 1, 2, 3, 4, 5, 6, 7$,

$$ {}^{75}_{5\rho}p(5n+4) \equiv 0 \pmod{5}. $$

THEOREM 3. *For all values of n,*

$$ \begin{aligned} {}^{75}_{20}p(25n+23) - {}^{75}_{20}p(5n+3) &\equiv {}^{75}_{5}p(25n+18) - {}^{75}_{5}p(5n-2) && \pmod{5}, \\ {}^{75}_{35}p(25n) - {}^{75}_{35}p(5n) &\equiv -{}^{75}_{10}p(25n-5) + {}^{75}_{10}p(5n-5) && \pmod{5}. \end{aligned} $$

THEOREM 4. *The following linear expressions in the restricted partition functions are multiplicative modulo* 5 ($n > 0$),

$$ \begin{gathered} {}^{25}p(n-1); \\ 2 \cdot {}^{75}_{20}p(n-2) - 2 \cdot {}^{75}_{5}p(n-7) + {}^{25}p(n-1); \\ 2 \cdot {}^{75}_{35}p(n) + 2 \cdot {}^{75}_{10}p(n-5) - {}^{25}p(n-1). \end{gathered} $$

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3. Definitions and notations

We shall use m to denote an integer positive, zero or negative, but n is reserved to denote a positive or a non-negative integer only.

We define the function $u_r(x)$, or simply u_r, by

$$u_r = \sum_{n=0}^{\infty} n^r a_n x^n \cdot \sum_{n=0}^{\infty} p(n) x^n, \tag{7}$$

where a_n is defined by the well-known 'pentagonal number' theorem of Euler,

$$f(x) = \prod_{n=1}^{\infty}(1-x^n) = \sum_{-\infty}^{+\infty}(-1)^m x^{\frac{1}{2}m(3m+1)} = \sum_{n=0}^{\infty} a_n x^n, \tag{8}$$

and $p(n)$ is the number of unrestricted partitions of n given by the expansion,

$$[f(x)]^{-1} = \left[\prod_{n=1}^{\infty}(1-x^n)\right]^{-1} = \sum_{n=0}^{\infty} p(n) x^n. \tag{9}$$

When $n = r = 0$, n^r is to be interpreted as unity, so that

$$u_0 = \sum_{n=0}^{\infty} a_n x^n \cdot \sum_{n=0}^{\infty} p(n) x^n = 1. \tag{10}$$

We shall use v to denote the pentagonal numbers,

$$v = \tfrac{1}{2}m(3m+1),\ m = 0, \pm 1, \pm 2, \cdots; \tag{11}$$

and with each v there corresponds an 'associated' sign, viz., $(-1)^m$. We shall come across sums of the type

$$\sum_v [\mp V(v)],$$

where it is understood that the sign to be prefixed is the 'associated' one, which would thus be (a) negative if v is 1, 2, 12, 15, 35, $\cdots$, that is, when it is of the form $(2m+1)(3m+1)$, and (b) positive if it is 0, 5, 7, 22, 26, $\cdots$, that is, when it is of the form $m(6m+1)$. It is clear that with the above notation,

$$u_r = \sum_v (\mp v^r x^v)/f(x), \tag{12}$$

$$\sum_v (\mp x^v)/f(x) = 1. \tag{13}$$

We shall also require the functions U_i, $i = 0, 1, 2$ which are certain linear functions of u_r's, $r = 0, 1, 2$, as given below

$$U_0 = -2u_2 + u_1 + u_0, \tag{14}$$

$$U_1 = -u_2 + 2u_1, \tag{15}$$

$$U_2 = -2u_2 + 2u_1. \tag{16}$$

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We also need certain polynomials $P_i(v)$ in v, $i = 0, 1, 2$, which are obtained by replacing U_i by $P_i(v)$, and u_r by v^r in the above relations (14) etc. Thus

$$P_0(v) = -2v^2+v+1, \tag{17}$$

$$P_1(v) = -v^2+2v \quad , \tag{18}$$

$$P_2(v) = -2v^2+2v \quad . \tag{19}$$

4. Lemmas

Remembering that the pentagonal numbers fall only in the residue classes 0, 1, 2 modulo 5, the truth of the following lemma can be easily verified.

LEMMA 1.

$$\begin{aligned} P_i(v) &\equiv 1 \pmod 5, \quad \text{if} \quad v \equiv i \pmod 5, \\ &\equiv 0 \pmod 5, \quad \text{if} \quad v \not\equiv i \pmod 5. \end{aligned}$$

If we replace the u_r's appearing in the expressions for U_i by

$$\sum_v (\mp v^r x^v)/f(x),$$

which is justified according to (12), we obtain

$$U_i = \sum_v [\mp P_i(v) x^v]/f(x);$$

and then the use of Lemma 1 leads to Lemma 2 given below.

LEMMA 2.

$$U_i \equiv \sum_{v \equiv i} (\mp x^v)/f(x) \pmod 5,$$

the summation being extended over all pentagonal numbers $v \equiv i \pmod 5$.

The truth of the following lemma can be verified without much difficulty by writing $5m+j$ with $j = 0, -2; -1; 1, 2$ respectively in place of m in the expression $\frac{1}{2}m(3m+1)$ for the pentagonal numbers, and in $(-1)^m$ its associated sign. It is also to be remembered (when j is negative, say, $-j'$) that $\frac{1}{2}(5m-j')(15m-3j'+1)$ and $\frac{1}{2}(5m+j')(15m+3j'-1)$ represent the same set of numbers.

LEMMA 3. *With respect to the modulus* 5 *the pentagonal numbers* v *fall in the three residue classes* $i = 0, 1, 2$; *and the solutions of*

$$v \equiv i \pmod 5$$

are given below together with the associated signs.

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i	solutions (1st set):	sign	solutions (2nd set):	sign
0	$\frac{1}{2}(75m^2+5m)$:	$(-1)^m$	$\frac{1}{2}(75m^2+55m)+5$:	$(-1)^m$
1	$\frac{1}{2}(75m^2+25m)+1$:	$(-1)^{m+1}$		
2	$\frac{1}{2}(75m^2+35m)+2$:	$(-1)^{m+1}$	$\frac{1}{2}(75m^2+65m)+7$:	$(-1)^m$

The identities given in the next lemma are simple applications of a special case of a famous identity of Jacobi [**3**, p. 283], viz.,

$$\prod_{n=0}^{\infty}[(1-x^{2kn+k-l})(1-x^{2kn+k+l})(1-x^{2kn+2k})] = \sum_{-\infty}^{+\infty}(-1)^m x^{km^2+lm}. \tag{20}$$

In establishing this lemma k and l are given values which are in conformity with the quadratic expressions in m given in Lemma 3. As an illustration, we have the initial steps for $\sum_{v\equiv 2}(\mp x^v)$ as

$$\begin{aligned}\sum_{v\equiv 2}(\mp x^v) &= \sum_{-\infty}^{+\infty}(-1)^{m+1}x^{\frac{1}{2}(75m^2+35m)+2}+\sum_{-\infty}^{+\infty}(-1)^m x^{\frac{1}{2}(75m^2+65m)+7} \\ &= -x^2\sum_{-\infty}^{+\infty}(-1)^m x^{\frac{1}{2}(75m^2+35m)}+x^7\sum_{-\infty}^{+\infty}(-1)^m x^{\frac{1}{2}(75m^2+65m)}.\end{aligned}$$

LEMMA 4. *Writing $v \equiv i$ simply for $v \equiv i$* (mod 5),

$$\begin{aligned}\sum_{v\equiv 0}(\mp x^v) &= \prod_{n=0}^{\infty}[(1-x^{75n+35})(1-x^{75n+40})(1-x^{75n+75})] \\ &\quad +x^5\prod_{n=0}^{\infty}[(1-x^{75n+10})(1-x^{75n+65})(1-x^{75n+75})], \\ \sum_{v\equiv 1}(\mp x^v) &= -x\prod_{n=0}^{\infty}(1-x^{25n+25}), \\ \sum_{v\equiv 2}(\mp x^v) &= -x^2\prod_{n=0}^{\infty}[(1-x^{75n+20})(1-x^{75n+55})(1-x^{75n+75})] \\ &\quad +x^7\prod_{n=0}^{\infty}[(1-x^{75n+5})(1-x^{75n+70})(1-x^{75n+75})].\end{aligned}$$

The next lemma is derived from Lemma 2 after the substitution in it of the product expressions for $\sum_{v\equiv i}(\mp x^v)$ as given in Lemma 4. The following fact is to be used in addition.

$$\begin{aligned}&\prod_{n=0}^{\infty}[(1-x^{75n+r})(1-x^{75n+75-r})(1-x^{75n+75})]/f(x) \\ &= \prod_{n=0}^{\infty}[(1-x^{75n+r})(1-x^{75n+75-r})(1-x^{75n+75})]/[(1-x)(1-x^2)(1-x^3)\cdots] \\ &= \sum_{n=0}^{\infty}{}_{r}^{75}p(n)x^n.\end{aligned} \tag{21}$$

LEMMA 5.

$$U_0 \equiv \sum_{n=0}^{\infty} {}^{75}_{35}p(n)x^n + \sum_{n=0}^{\infty} {}^{75}_{10}p(n-5)x^n \quad (\text{mod } 5),$$

$$U_1 \equiv -\sum_{n=0}^{\infty} {}^{25}p(n-1)x^n \quad (\text{mod } 5),$$

$$U_2 \equiv -\sum_{n=0}^{\infty} {}^{75}_{20}p(n-2)x^n + \sum_{n=0}^{\infty} {}^{75}_{5}p(n-7)x^n \quad (\text{mod } 5).$$

We require another set of congruences. These are obtained from the classical result, — due to Catalan [1, p. 290], —

(22) $$p(n-1)+2p(n-2)-5p(n-5)-7p(n-7)+\cdots = \sigma(n),$$

and the fairly old result, — due to Glaisher [1, p. 312], —

(23) $$p(n-1)+2^2p(n-2)-5^2p(n-5)-7^2p(n-7)+\cdots = -\tfrac{1}{12}[5\sigma_3(n)-(18n-1)\sigma(n)].$$

These results can be re-written according to our notation as

(24) $$\sum_v [\mp vp(n-v)] = -\sigma(n),$$

(25) $$12\sum_v [\mp v^2p(n-v)] = 5\sigma_3(n)-(18n-1)\sigma(n).$$

Now from (12) we have

(26) $$\begin{aligned} u_r &= \sum_v (\mp v^r x^v)/f(x) \\ &= \sum_v (\mp v^r x^v)\cdot\sum_{n=0}^{\infty} p(n)x^n \\ &= \sum_{n=1}^{\infty}\{\sum_v [\mp v^r p(n-v)]\}x^n, \quad r>0. \end{aligned}$$

It is now easy to establish the validity of the following lemma from the above three relations (24), (25) and (26).

LEMMA 6.

$$u_1 = -\sum_{n=1}^{\infty}\sigma(n)x^n,$$

$$u_2 \equiv -2\sum_{n=1}^{\infty}(2n+1)\sigma(n)x^n \quad (\text{mod } 5).$$

The following lemma can be easily obtained by the application of Lemma 6 to the expressions (14) etc. for U_i's in terms of u_r's.

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LEMMA 7.

$$U_0 - 1 \equiv -2 \sum_{n=1}^{\infty} (n+1)\sigma(n)x^n \pmod 5,$$

$$U_1 \equiv - \sum_{n=1}^{\infty} n\sigma(n)x^n \pmod 5,$$

$$U_2 \equiv -2 \sum_{n=1}^{\infty} (n-1)\sigma(n)x^n \pmod 5.$$

It is of special interest to note that the five restricted partition functions appearing in our theorems are connected by the identical relation given in Lemma 8.

LEMMA 8.

$${}^{75}_{35}p(n+1) - {}^{75}_{20}p(n-1) + {}^{75}_{10}p(n-4) + {}^{75}_{5}p(n-6) = {}^{25}p(n).$$

This lemma can be easily derived from (13) which can be put in the form

$$\sum_{i=0}^{2} [\sum_{v \equiv i} (\mp x^v)/f(x)] = 1. \tag{27}$$

All that we need do now is to apply Lemma 4 and the relation (21), before equating the coefficients of x^n, $n > 0$, on both sides of (27).

5. Proof of the Theorems

By comparing the coefficients of like powers of x in the two (right hand) expressions for U_i modulo 5 given in Lemmas 5 and 7 we obtain the following congruences for $n > 0$.

$${}^{75}_{35}p(n) \quad + {}^{75}_{10}p(n-5) \equiv -2(n+1)\sigma(n) \pmod 5, \tag{28}$$

$${}^{25}p(n-1) \equiv n\sigma(n) \pmod 5, \tag{29}$$

$$-{}^{75}_{20}p(n-2) + {}^{75}_{5}p(n-7) \equiv -2(n-1)\sigma(n) \pmod 5. \tag{30}$$

These are the basic relations on which our final conclusions are drawn.

Remembering the well-known congruence [5; 2 p. 167],

$$\sigma_k(n) \equiv 0 \quad \pmod M \quad \text{for almost all } n \tag{31}$$

for arbitrarily fixed M and odd k, it is a straightforward matter to deduce Theorem 1 from the above basic relations.

We shall now establish other relations which are stronger being true for all values of n. Theorem 2 gives one such set of congruence relations. The first relation of the theorem is directly obtained from (29) by writing $5n+5$ for n. One can also obtain with equal ease the relations,

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$$ {}^{75}_{35}p(5n+4) \equiv -{}^{75}_{10}p(5n-1) \pmod 5, $$
$$ {}^{75}_{20}p(5n+4) \equiv {}^{75}_{5}p(5n-1) \pmod 5, $$

directly from the other basic relations (28) and (30) by writing $5n+4$ and $5n+6$ respectively in place of n. But still stronger relations asserting the divisibility of each side of the above congruences by 5 hold. These divisibility properties along with others are covered by the general proposition enunciated in Theorem 2. This general result actually emanates from the first relation, viz.,

$$ (32) \qquad {}^{25}p(5n+4) \equiv 0 \pmod 5; $$

and the process of derivation has two stages. In the first one Ramanujan's congruence (1) is obtained from the first relation (32), and this derived relation is used in the second stage to establish the general proposition. It easily follows from (21), (20) and (9) that ${}^{75}_{5\rho}p(n)$ can be expressed in the (really finite) form

$$ (33) \qquad {}^{75}_{5\rho}p(n) = p(n) + \sum_{n'=1}^{\infty} \varepsilon(n')p(n-5n') $$

where $\varepsilon(n') = 0$ or ± 1. For the special case corresponding to $\rho = 5$ we have the fully specified expression,

$$ (34) \qquad {}^{25}p(n) = \sum_{v} [\mp p(n-25v)]. $$

Keeping in mind the first relation of Theorem 2, viz., (32), Ramanujan's congruence (1) is seen to be valid by putting successively $n = 4, 9, 14, 19, \cdots$ in (34). And to derive the general proposition we merely write $5n+4$ for n in (33) and make use of Ramanujan's congruence.

Theorem 3 involves two differently restricted partition functions in each of the congruences. As an illustration of the method used we shall establish the first congruence. Writing $5n$ in place of n in (30) and making use of the relation,

$$ (35) \qquad \sigma(5n) \equiv \sigma(n) \pmod 5, $$

which is easily established, we have

$$ (36) \qquad -{}^{75}_{20}p(5n-2) + {}^{75}_{5}p(5n-7) \equiv 2\sigma(n) \pmod 5. $$

Subtracting (36) from (30), and then writing $5n+5$ in place of n again, we have the first congruence of Theorem 3.

There are of course congruences for linear functions with constant coefficients involving more than two differently restricted partition functions. Two interesting ones are

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$$(37) \qquad {}^{75}_{35}p(5n+3)+{}^{75}_{10}p(5n-2) \equiv -{}^{25}p(5n+2) \pmod{5},$$

$$(38) \qquad {}^{75}_{20}p(5n)-{}^{75}_{5}p(5n-5) \equiv {}^{25}p(5n+1) \pmod{5}.$$

From (28) and (29), and also from (29) and (30) we get the following linear congruences with variable coefficients.

$$\begin{aligned} n \cdot {}^{75}_{35}p(n) \quad\; + n \cdot {}^{75}_{10}p(n-5) &\equiv -2(n+1) \cdot {}^{25}p(n-1) \pmod{5}, \\ -n \cdot {}^{75}_{20}p(n-2) + n \cdot {}^{75}_{5}p(n-7) &\equiv -2(n-1) \cdot {}^{25}p(n-1) \pmod{5}. \end{aligned}$$

In order to pass on to linear functions with constant coefficients it is only necessary to substitute $5n+i$, $i = 0, 1, 2, 3, 4$, for n in these congruences. Substitutions of $5n+3$ in the first one, and $5n+2$ in the other give (37) and (38) respectively. Use of other values of i in the substitution process does not lead essentially to any congruence not already covered. In this connection the congruence relation (40) given at the end of the paper is to be kept in mind.

Then there are congruences involving the restricted partition functions which are the result of exploiting the multiplicative property of $\sigma(n)$. The multiplicative property modulo 5 of the first expression of Theorem 4, as given below explicitly,

$$(39) \qquad {}^{25}p(n-1) \cdot {}^{25}p(n'-1) \equiv {}^{25}p(nn'-1) \pmod{5}, \qquad (n, n') = 1,$$

is a direct consequence of the multiplicative property of $\sigma(n)$ and the relation (29). Also the relations (28) and (30) give rise to the congruence

$$(40) \qquad {}^{75}_{35}p(n)+{}^{75}_{10}p(n-5)+{}^{75}_{20}p(n-2)-{}^{75}_{5}p(n-7) \equiv \sigma(n) \pmod{5}.$$

The four unrestricted partition functions appearing in (40) may be reduced to three with the use of Lemma 8; and the multiplicative property of $\sigma(n)$ again come to our aid in establishing the multiplicative property modulo 5 of the other two expressions given in Theorem 4.

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