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# AN ORDERED SUPRABARRELLED SPACE

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**Abstract**

A locally convex space $E$ is said to be ordered suprabarrelled if given any increasing sequence of subspaces of $E$ covering $E$ there is one of them which is suprabarrelled. In this paper we show that the space $m_0(X, \Sigma)$, where $X$ is any set and $\Sigma$ is a $\sigma$-algebra on $X$, is ordered suprabarrelled, given an affirmative answer to a previously raised question. We also include two applications of this result to the theory of vector measures.

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We denote by $\Sigma$ a $\sigma$-algebra on a set $X$ and for every $A$ of $\Sigma$ we set $e(A)$ to denote the characteristic function of $A$. Let $m_0(X, \Sigma)$ be the linear space over the field $K$ of the real or complex numbers generated by $\{e(A): A \in \Sigma\}$ endowed with the topology defined by the norm $\|x\| = \sup\{|x(t)|, t \in X\}$. Given a member $A$ of $\Sigma$, we denote by $m_0(A, \Sigma)$ the subspace of $m_0(X, \Sigma)$ generated by the functions $e(B)$ with $B \in \Sigma$ and $B \subset A$, and given a continuous linear form $u$ over $m_0(X, \Sigma)$, $u(A)$ stands for the restriction of $u$ to $m_0(A, \Sigma)$ and $\|u(A)\|$ denotes the norm of $u(A)$. On the other hand, we set $\Gamma$ to denote the family of all the finite dimensional subspaces of $m_0(X, \Sigma)$. If $X$ coincides with the set $\mathbb{N}$ of the positive integers and $\Sigma$ denotes the $\sigma$-algebra $2^N$ of all the subsets of $X$, we write $l_0^\infty$ instead of $m_0(\mathbb{N}, 2^N)$. In the sequel, by "space" we mean "locally convex Hausdorff space over the field of the real or complex numbers." A space $E$ is Baire-like





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[7] if given an increasing sequence of closed absolutely convex sets covering $E$ there is one of them which is a neighbourhood of the origin. We call $E$ suprabarrelled [9] if given an increasing sequence of subspaces of $E$ covering $E$, one of them is Baire-like; and $E$ is called ordered suprabarrelled [4] if given any increasing sequence of subspaces of $E$ covering $E$ there is one of them which is suprabarrelled. It is known that every barrelled dense subspace of a Baire-like space is Baire-like.

In [8], Valdivia shows that $m_0(X, \Sigma)$ is suprabarrelled and in [4] we asked if $l_0^\infty$ was an ordered suprabarrelled space. In fact, a positive answer to this question was already claimed in [6], but without giving any explicit proof. In this paper we actually show that the space $m_0(X, \Sigma)$ is ordered suprabarrelled. Our methods are in part based on those given in [8]. We have also included two applications of this result to the theory of vector measures.

LEMMA 1. *Let $E$ be a linear subspace of $m_0(X, \Sigma)$ and let $A$ be an element of $\Sigma$ such that $m_0(A, \Sigma) \not\subset E + F$ for every $F \in \Gamma$. If $\{P, Q\}$ is a partition of $A$, with $P, Q \in \Sigma$, then either $m_0(P, \Sigma) \not\subset E + F$ for every $F \in \Gamma$ or $m_0(Q, \Sigma) \not\subset E + F$ for every $F \in \Gamma$.*

PROOF. If $m_0(P, \Sigma) \not\subset E+F$ for every $F \in \Gamma$, then we are done. If this is not the case, then there exists an $F_1 \in \Gamma$ such that $m_0(P, \Sigma)$ is contained in $E + F_1$. This proves that $m_0(Q, \Sigma) \not\subset E + F$ for every $F \in \Gamma$ since if there exists an $F_2 \in \Gamma$ such that $m_0(Q, \Sigma)$ is contained in $E + F_2$ and, *a fortiori*, in $E + F_1 + F_2$, then given that $m_0(A, \Sigma)$ is the direct sum of $m_0(P, \Sigma)$ and $m_0(Q, \Sigma)$ it follows that $E + F_1 + F_2$ does no contain $m_0(P, \Sigma)$, a contradiction.

LEMMA 2. *For any positive integer $p > 1$, elements $x_1, x_2, \dots, x_r$ of $m_0(X, \Sigma)$ and a linear subspace $E$ of $m_0(X, \Sigma)$, if $A \in \Sigma$ is such that $m_0(A, \Sigma) \not\subset E+F$ for every $F \in \Gamma$, then there are $p$ elements $Q_1, Q_2, \dots, Q_p$ of $\Sigma$, which are a partition of $A$, such that $e(Q_i) \notin \langle E \cup \{x_1, x_2, \dots, x_r\}\rangle$ for $i = 1, 2, \dots, p$.*

PROOF. If $m_0(A, \Sigma) \not\subset E + F$ for every $F \in \Gamma$, there is a $P_1$ of $\Sigma$ contained in $A$ such that $e(P_1) \notin \langle E \cup \{e(A), x_1, \dots, x_r\}\rangle$ and therefore $e(A - P_1) \notin \langle E \cup \{e(A), x_1, \dots, x_r\}\rangle$ as well. Since $\{P_1, A - P_1\}$ is a partition of $A$, applying Lemma 1 we have that either $m_0(P_1, \Sigma) \not\subset E + F$ for every $F \in \Gamma$ or $m_0(A - P_1, \Sigma) \not\subset E + F$ for every $F \in \Gamma$. In the first case we set $Q_1 := A - P_1$ and $B_1 := P_1$, and in the second $Q_1 := P_1$ and $B_1 := A - P_1$. So we have

$$e(Q_1), e(B_1) \notin \langle E \cup \{x_1, \dots, x_r\}\rangle$$

and

$$m_0(B_1, \Sigma) \not\subset E + F \quad \text{for every } F \in \Gamma.$$

Replacing $A$ by $B_1$ in the former argument we obtain a partition of $B_1$ in two elements $\{Q_2, B_2\}$ of $\Sigma$, such that

$$e(Q_2), e(B_2) \notin \langle E \cup \{x_1, \dots, x_r\}\rangle$$

and

$$m_0(B_2, \Sigma) \not\subset E + F \quad \text{for every } F \in \Gamma.$$

Continuing in this way, we obtain a partition of $B_{p-2}$ in two elements $\{Q_{p-1}, B_{p-1}\}$ of $\Sigma$, such that

$$e(Q_{p-1}), e(B_{p-1}) \notin \langle E \cup \{x_1, \dots, x_r\}\rangle.$$

We set finally $Q_p := B_{p-1}$.

For the next result we suppose as given a family $\{E_{nm}, n, m = 1, 2, \dots\}$ of linear subspaces of $m_0(X, \Sigma)$, an element $A$ which belongs to $\Sigma$, vectors $x_1, x_2, \dots, x_r$ of $m_0(X, \Sigma)$, $p$ positive integers $n(1) < n(2) < \cdots < n(p)$ and, for each $i \in \{1, 2, \dots, p\}$, $q(i)$ positive integers $m(i, 1) < m(i, 2) < \cdots < m(i, q(i))$.

LEMMA 3. *Suppose first that $m_0(A, \Sigma) \not\subset E_{nm} + F$ for every $F \in \Gamma$ when $(n, m)$ takes the values $(n(i), m(i, j))$ with $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q(i)$. We also suppose that for each $i \in \{1, 2, \dots, p\}$ there are an infinity of positive integers $m > m(i, q(i))$ such that $m_0(A, \Sigma) \not\subset E_{n(i)m} + F$ for every $F \in \Gamma$. Finally we suppose there are an infinity of positive integers $n > n(p)$ such that for each one of them, there are an infinity of natural values of $m$ with $m_0(A, \Sigma) \not\subset E_{nm} + F$ for every $F \in \Gamma$. Under these conditions there exist $q(1) + q(2) + \cdots + q(p)$ pairwise disjoint elements $\{M_{ij}, i = 1, 2, \dots, p, \ j = 1, 2, \dots, q(i)\}$ of $\Sigma$ contained in $A$ such that*

$$e(M_{ij}) \notin \langle E_{n(i)m(i,j)} \cup \{x_1, x_2, \dots, x_r\}\rangle$$

*for $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q(i)$. In addition we have that*

$$m_0\left(A - \bigcup\{M_{ij}, i = 1, 2, \dots, p, \ j = 1, 2, \dots, q(i)\}, \Sigma\right) \not\subset E_{nm} + F$$

*for every $F \in \Gamma$ when $(n, m) = (n(i), m(i, j))$, $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q(i)$, fixed $i$ for every $F \in \Gamma$ when $(n, m)$ coincides with an infinity of pairs $(n(i), m)$ of natural numbers with $m > m(i, q(i))$, $i = 1, 2, \dots, p$ and for every $F \in \Gamma$ when $n$ takes infinitely many natural values greater than $n(p)$ and, for each one of those values of $n$, the second coordinate $m$ takes in turn an infinity of natural values.*

Proof. As $m_0(A, \Sigma) \not\subset E_{n(1)m(1,1)} + F$ for every $F \in \Gamma$, setting $s := q(1) + q(2) + \cdots + q(p)$ we apply Lemma 2 to find a partition of $A$ in $p+s+2$ elements $\{Q_1, Q_2, \ldots, Q_{p+s+2}\}$ of $\Sigma$ such that

$$(1) \qquad e(Q_i) \notin \langle E_{n(1)m(1,1)} \cup \{x_1, \ldots, x_r\}\rangle$$

for $i = 1, 2, \ldots, p+s+2$. On the other hand, as a consequence of Lemma 1, there is some $i(1) \in \{1, 2, \ldots, p+s+2\}$ such that $m_0(Q_{i(1)}, \Sigma) \not\subset E_{n(1)m} + F$ for every $F \in \Gamma$ when $m$ takes an infinity of values greater than $m(1, q(1))$. Similarly, there is some $i(2) \in \{1, 2, \ldots, p+s+2\}$ with $m_0(Q_{i(2)}, \Sigma) \not\subset E_{n(2)m} + F$ for every $F \in \Gamma$ when $m$ takes an infinity of values greater than $m(2, q(2))$. We continue in this way until we find some $i(p) \in \{1, 2, \ldots, p+s+2\}$ such that $m_0(Q_{i(p)}, \Sigma) \not\subset E_{n(p)m} + F$ for every $F \in \Gamma$ when $m$ takes an infinity of values greater than $m(p, q(p))$. Now, if $n'$ is the first natural number greater than $n(p)$ such that $m_0(A, \Sigma) \not\subset E_{n'm} + F$ for every $F \in \Gamma$ when $m$ takes an infinity of values, again because of Lemma 1 there exists some $j \in \{1, 2, \ldots, p+s+2\}$ with $m_0(Q_j, \Sigma) \not\subset E_{n'm} + F$ for every $F \in \Gamma$ when $m$ takes an infinity more of values. Since there are infinitely many values of $n > n(p)$ having the property above, applying repeatedly Lemma 1 we obtain that there exists some $i(p+1) \in \{1, 2, \ldots, p+s+2\}$ such that $m_0(Q_{i(p+1)}, \Sigma) \not\subset E_{nm} + F$ for every $F \in \Gamma$ when $n$ takes an infinity of values greater than $n(p)$ and, given each one of those values of $n$, the second coordinate of the pair $(n, m)$ takes in turn an infinity of natural number values. Thus, if we set $Q_0 := \bigcup\{Q_{i(k)},\ k = 1, 2, \ldots, p+1\}$, we have proved that $m_0(Q_0, \Sigma) \not\subset E_{nm} + F$ for every $F \in \Gamma$ when, for fixed $i$, $(n, m)$ is equal to an infinity of pairs $(n(i), m)$ with $m > m(i, q(i))$, $i = 1, 2, \ldots, p$, and for every $F \in \Gamma$ when $n$ takes an infinity of values greater than $n(p)$ and, for each one of those values of $n$, $m$ takes infinitely many values.

Reindexing the remainders $Q_i$ we have that $\{Q_0, Q_1, \ldots, Q_{s+1}\}$ is a partition of $A$ and so, using Lemma 1 again, we have that there is some $r(1) \in \{0, 1, 2, \ldots, s+1\}$ such that $m_0(Q_{r(1)}, \Sigma) \not\subset E_{n(1)m(1,1)} + F$ for every $F \in \Gamma$, some $r(2) \in \{0, 1, 2, \ldots, s+1\}$ such that $m_0(Q_{r(2)}, \Sigma) \not\subset E_{n(1)m(1,2)} + F$ for every $F \in \Gamma, \ldots$, and some $r(s) \in \{0, 1, \ldots, s+1\}$ with $m_0(Q_{r(s)}, \Sigma) \not\subset E_{n(p)m(p,q(p))}$ for every $F \in \Gamma$.

Clearly, there are two elements of the set $\{0, 1, 2 \ldots, s+1\}$ which are not contained in the set $\{r(1), r(2), \ldots, r(s)\}$ and at least one of them, say $h$, is different from $0$. Since $m_0(Q_{r(1)} \cup Q_{r(2)} \cup \cdots \cup Q_{r(s)}, \Sigma) \not\subset E_{nm} + F$ for every $F \in \Gamma$ and every $(n, m) = (n(i), m(i, j))$ with $i = 1, 2, \ldots, p$. and $j = 1, 2, \ldots, q(i)$, we conclude that $m_0(A - Q_h, \Sigma) \not\subset E_{nm} + F$ for every $F \in \Gamma$ when $(n, m)$ coincides with each one of all the aforementioned pairs. Furthermore, since $h \neq 0$, then $Q_0$ is contained in $A - Q_h$ and hence

$m_0(A - Q_h, \Sigma) \not\subset E_{nm} + F$ for every $F \in \Gamma$ when $(n, m)$ coincides with all the pairs considered before the index rearrangement.

We put now $M_{11} := Q_h$ and so by relation (1),

$$e(M_{11}) \notin \langle E_{n(1)m(1,1)} \cup \{x_1, x_2, \dots, x_r\}\rangle$$

and $m_0(A - M_{11}, \Sigma) \not\subset E_{nm} + F$ for every $F \in \Gamma$ when $(n, m) = (n(i), m(i, j))$ with $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q(i)$, for each $i \in \{1, 2, \dots, p\}$ for every $F \in \Gamma$ when $(n, m)$ coincides with an infinity of pairs $(n(i), m)$ with $m > m(i, q(i))$, and for every $F \in \Gamma$ when $n$ takes infinitely many values greater than $n(p)$ and, for each one of them, $m$ takes an infinity of values.

We repeat again the previous argument taking $A - M_{11}$ instead of $A$. In fact, since $m_0(A - M_{11}, \Sigma) \not\subset E_{n(1)m(1,2)} + F$ for every $F \in \Gamma$, we can use Lemma 2 to obtain a partition of $A - M_{11}$ in $p + s + 2$ elements of $\Sigma$ whose characteristic functions are not contained in $\langle E_{n(1)m(1,2)} \cup \{x_1, x_2, \dots, x_r\}\rangle$. Using repeatedly Lemma 1, we can choose some element of this partition, which we denote by $M_{12}$, such that

$$e(M_{12}) \notin \langle E_{n(1)m(1,2)} \cup \{x_1, x_2, \dots, x_r\}\rangle$$

and moreover $m_0(A - M_{11} \cup M_{12}, \Sigma) \not\subset E_{nm} + F$ for every $F \in \Gamma$ when $(n, m) = (n(i), m(i, j))$ for $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q(i)$, for $i \in \{1, 2, \dots, p\}$ for every $F \in \Gamma$ when $(n, m)$ coincides with an infinity of pairs $(n(i), m)$ with $m > m(i, q(i))$, and for every $F \in \Gamma$ when $n$ takes an infinity of values greater than $n(p)$ and, for each one of them, $m$ takes an infinity of values. We continue in this way until we find a last $M_{n(p)n(p,q(p))} \in \Sigma$ which establishes the lemma.

Lemma 4. *Let* $\{E_{nm}, n, m = 1, 2, \dots\}$ *be a sequence of linear subspaces of* $m_0(X, E)$ *with* $m_0(X, E) \neq E_{nm} + F$ *for every* $F \in \Gamma$ *and for* $n, m = 1, 2 \dots$. *Then, there exists a sequence* $\{M_{ijk}, i, j, k = 1, 2, \dots\}$ *of pairwise disjoint members of* $\Sigma$, *a strictly increasing sequence* $\{(n(i), i = 1, 2, \dots\}$ *of positive integers and, for each* $i \in \mathbb{N}$, *a strictly increasing sequence* $\{m(i, j), j = 1, 2 \dots\}$ *of positive integers, such that*

$$e(M_{ijk}) \notin \langle E_{n(i)m(i,j)} \cup \{e(M_{rst}), \ r, s, t \in \mathbb{N}, \ r + s + t < i + j + k\}\rangle$$

*for* $i, j, k = 1, 2, \dots$.

Proof. Let $n(1) = m(1, 1) = 1$. We are supposing that $m_0(X, \Sigma) \not\subset E_{n(1)m(1,1)} + F$ for every $F \in \Gamma$, $m_0(X, \Sigma) \not\subset E_{n(1)m} + F$ for every $F \in \Gamma$ when $m > m(1, 1)$ and, given each $n > n(1)$, $m_0(X, \Sigma) \not\subset E_{nm} + F$ for every $F \in \Gamma$ when $m$ takes any value of $\mathbb{N}$. Then, by Lemma 3, there exists some $M_{111}$ in $\Sigma$ with $e(M_{111}) \notin E_{n(1)m(1,1)}$ and furthermore we have that $m_0(X - M_{111}, \Sigma) \not\subset E_{nm} + F$ for every $F \in \Gamma$ when $(n, m) = (n(1), m(1, 1))$, for every $F \in \Gamma$ when $(n, m)$ coincides with an infinity of pairs $(n, m)$ with $n = n(1)$ and $m > m(1, 1)$ and for every $F \in \Gamma$ when $n$ takes an infinity of values greater than $n(1)$ and, for each one of them, $m$ takes infinitely many values. Let $n(2)$ be the first of those values of $n$, let $m(1, 2)$ be the first of the infinity of values of $m$ greater than $m(1, 1)$ such that $m_0(X - M_{111}, \Sigma) \not\subset E_{n(1)m} + F$ for every $F$, and let finally $m(2, 1)$ be the first of the infinity of natural values of $m$ such that $m_0(X - M_{111}, \Sigma) \not\subset E_{n(2)m} + F$ for every $F \in \Gamma$.

Taking $X - M_{111}$ instead of $A$ in Lemma 3, $x_1 = e(M_{111})$, $p = 2$, $q(1) = 2$ and $q(2) = 1$, we have $q(1) + q(2) = 3$ pairwise disjoint members $\{M_{112}, M_{121}, M_{211}\}$ of $\Sigma$, each one of them contained in $X - M_{111}$, such that $e(M_{112}) \notin \langle E_{n(1)m(1,1)} \bigcup \{e(M_{111})\}\rangle$, $e(M_{121}) \notin \langle E_{n(1)m(1,2)} \bigcup \{e(M_{111})\}\rangle$, $e(M_{211}) \notin \langle E_{n(2)m(2,1)} \bigcup \{e(M_{111})\}\rangle$ and

$$m_0\left(X - \bigcup\{M_{rst},\ r, s, t \in \mathbb{N},\ r + s + t \le 4\}, \Sigma\right) \not\subset E_{nm} + F$$

for every $F \in \Gamma$ when $(n, m) = (n(i), m(i, j))$ with $i = 1, 2$ and $j \le q(i)$, for every $F \in \Gamma$ when $(n, m)$ coincides with an infinity of pairs $(n(1), m)$ with $m > m(1, 2)$, for every $F \in \Gamma$ when $(n, m)$ coincides with an infinity of pairs $(n(2), m)$ with $m > m(2, 1)$, and for every $F \in \Gamma$ when $n$ takes an infinity of values greater than $n(2)$ and, for each one of them, $m$ takes infinitely many values. We proceed now by recurrence, supposing we have obtained $p$ positive integers $n(1) < n(2) < \cdots < n(p)$, $p - i + 1$ positive integers $m(1, 1) < m(i, 2) < \cdots < m(i, p - i + 1)$ for $i = 1, 2, \ldots, p$ and a family $\{M_{ijk}, i + j + k \le p + 2\}$ of pairwise disjoint elements of $\Sigma$ such that

$$e(M_{ij}) \notin \left\langle E_{n(i)m(i,j)} \bigcup \{e(M_{rst}),\ r + s + t < i + j + k\}\right\rangle$$

for every $(i, j, k) \in \mathbb{N}^3$ with $i + j + k \le p + 2$. Moreover,

$$(2) \qquad m_0\left(X - \bigcup\{M_{ijk},\ i + j + k \le p + 2\}, \Sigma\right) \not\subset E_{nm} + F$$

for every $F \in \Gamma$ when $(n, m) = (n(i), m(i, j))$ with $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, p - i + 1$, for each $i \in \{1, 2, \ldots, p\}$ for every $F \in \Gamma$ when $(n, m)$ coincides with an infinity of pairs $(n(i), m)$ with $m > m(i, p-i+1)$, and for every $F \in \Gamma$ when $n$ takes an infinity of values greater than $n(p)$ and, given each one of them, $m$ takes an infinity of values. Now let $n(p+1)$ be the first of those values of $n > n(p)$ and let $m(p+1, 1)$ be the first of the corresponding values of $m$ of that pair. We take for each $i \in \{1, 2, \ldots, p\}$ as $m(i, p - i + 2)$ the first value of $m$ which satisfies relation (2) with $n = n(i)$. We apply Lemma 3 with $X - \bigcup\{M_{ijk},\ i + j + k \le p + 2\}$ instead

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of $A$, $p+1$ instead of $p$, $x_1 = e(M_{111})$, $x_2 = e(M_{112}), \dots, x_r = e(M_{p11})$, with $r = \sum_{i=1}^{p} i(i+1)/2$, and $q(i) = p - i + 2$, $i = 1, 2, \dots, p+1$. This ensures the existence of $q(1)+q(2)+\cdots+q(p+1) = (p+1)(p+2)/2$ pairwise disjoint elements of the $\sigma$-algebra $\Sigma$ contained in the set

$$X - \bigcup\{M_{ijk}, i+j+k \leq p+2\},$$

and indexed by the solutions in $\mathbb{N}$ of the equation $i+j+k = p+3$, which satisfy the requested conditions.

THEOREM 1. *$m_0(X, \Sigma)$ is ordered suprabarrelled.*

PROOF. We shall prove that given any increasing sequence of subspaces of $m_0(X, \Sigma)$ covering $m_0(X, \Sigma)$ there is one of them which is suprabarrelled.

Suppose this is not true. There exists an increasing sequence $\{F_n, n = 1, 2, \dots\}$ of subspaces of $m_0(X, \Sigma)$ covering $m_0(X, \Sigma)$ such that for every positive integer $n$ there is an increasing sequence $\{F_{nm}, m = 1, 2, \dots\}$ of non Baire-like subspaces of $F_n$ covering $F_n$. Hence, in each $F_{nm}$, $n, m = 1, 2, \dots$ there is some increasing sequence $\{S_{nmr}, r = 1, 2, \dots\}$ of closed absolutely convex sets covering $F_{nm}$ such that no $S_{nmr}$, $r = 1, 2, \dots$, is a neighbourhood of the origin in $F_{nm}$. Let $R_{nmr}$ be the closure of $S_{nmr}$ in $m_0(X, \Sigma)$ for $n, m, r = 1, 2, \dots$ and put $E_{nm} := \bigcup\{R_{nmr}, r = 1, 2, \dots\}$.

The barrelledness of $m_0(X, \Sigma)$ implies that $m_0(X, \Sigma) \neq E_{nm} + F$ for every $F \in \Gamma$ and for every pair $(n, m)$ of positive integers, since otherwise there would exist some $E_{pq}$ of finite codimension which would be Baire-like and therefore some $R_{pqr}$ would be a neighbourhood of the origin in $E_{pq}$, a contradiction.

By Lemma 4, there exist a sequence $\{M_{ijk}, i, j, k = 1, 2, \dots\}$ of pairwise disjoint members of $\Sigma$, a strictly increasing sequence $\{n(i), i = 1, 2, \dots\}$ of positive integers and, for each $i \in \mathbb{N}$, an increasing sequence $\{m(i, j), j = 1, 2, \dots\}$ of positive integers, such that

$$e(M_{ijk}) \notin \langle E_{n(i)m(i,j)} \cup \{e(M_{rst}), r+s+t < i+j+k\}\rangle$$

for $i, j, k = 1, 2, \dots$.

In this way, with $T_{ijk} := R_{n(i)m(i,j)k}$, it is clear that

$$e(M_{ijk}) \notin 3(T_{ijk} + \delta(i+j+k)\Gamma\{e(M_{rst}), r+s+t < i+j+k\})$$

where

$$\delta(i+j+k+) \geq \operatorname{card}\{e(M_{rst}), r+s+t < i+j+k\} = \sum\left\{\binom{p}{2}, 2 \leq p \leq i+j+k-2\right\}.$$

By the Hahn-Banach theorem, for each set $(i, j, k)$ of natural numbers, there is some continuous linear form $u_{ijk}$ on $m_0(X, \Sigma)$ such that

(3)
$$|\langle e(M_{ijk}), u_{ijk}\rangle| > 3, \quad \sum\{|\langle e(M_{rst}), u_{ijk}\rangle|, \; r+s+t < i+j+k\} \leq 1 \quad \text{and } |\langle z, u_{ijk}\rangle| \leq 1$$

for every $z \in T_{ijk}$

If we endow $\mathbb{N}^3$ with the diagonal ordering $((i_1, i_2, i_3) < (j_1, j_2, j_3)$ if either $i_1 + i_2 + i_3 < j_1 + i_2 + i_3$ or if $i_1 + i_2 + i_3 = j_1 + j_2 + j_3$ and there is some index $1 \leq r \leq 3$ such that $i_r < j_r$ with $i_k = j_k$ for $1 \leq k < r)$ and $\{\alpha(n), \; n = 1, 2, \dots\}$ denotes the sequence of the ordered elements of $\mathbb{N}^3$, we are going to find by recurrence a decreasing sequence $\{N^{(\alpha(n))}, \; n = 1, 2, \dots\}$ of subsets of $\mathbb{N}^3$ such that given any pair $(p, q)$ of positive integers, there are infinitely many elements in each $N^{(ijk)}$ whose two first coordinates are $(p, q)$, and verifying the relations

(4)
$$\|u_{ijk}\left(\bigcup\{M_{rst}, \; (r, s, t) \in N^{(ijk)}\}\right)\| < 12$$

for $i, j, k = 1, 2, \dots$.

Let $G := \bigcup\{M_{ijk}, \; i, j, k = 1, 2, \dots\}$ and let $m$ be a positive integer such that $\|u_{111}(G)\| < m$. We make a partition of $\mathbb{N}^3$ in $m$ parts $P_r$, $1 \leq r \leq m$, so that, in each one of them, given any pair $(p, q)$ of positive integers, there are infinitely many elements whose two first components coincide with $(p, q)$. Now it is easy to note [8] that

$$\Sigma\{\|u_{111}\left(\bigcup\{M_{ijk}, \; (i, j, k) \in P_r\}\right)\|, \; r = 1, 2, \dots, m\} \leq \|u_{111}(G)\| < m$$

and hence there is some $s$, $1 \leq s \leq m$, such that

$$\|u_{111}\left(\bigcup\{M_{ijk}, \; (i, j, k) \in P_s\}\right)\| < 1$$

Then we set $N^{(111)} := P_s$.

Suppose we have determined $N^{(ijk)}$ and that $(r, s, t)$ is the element following $(i, j, k)$ in the ordering of $\mathbb{N}^3$. If $q \in \mathbb{N}$ is such that $\|u_{rst}(G)\| < q$ then we make a partition of the set $N^{(ijk)}$ in $q$ parts $Q_g$, $1 \leq g \leq q$, so that, in each one of them, given any pair $(p, q)$ of $\mathbb{N}^2$, there are infinitely many elements whose two first components coincide with $(p, q)$.

Given that

$$\sum\left\{\left\|u_{rst}\left(\bigcup\{M_{ijk}, \; (i, j, k) \in Q_g\}\right)\right\|, \; g = 1, 2, \dots, q\right\} \leq \|u_{rst}(G)\| < q$$

there is some $h$ with $1 \le h \le q$, such that

$$\left\| u_{rst}\left(\bigcup\{M_{ijk},\ (i,j,k) \in Q_h\}\right)\right\| < 1.$$

Then we set $N^{(rst)} := Q_h$.

Next we determine a sequence $S = \{(i(n), j(n), k(n)),\ n = 1, 2, \ldots\}$ in $\mathbb{N}^3$ whose terms verify the following conditions.

(A) $(i(n+1), j(n+1), k(n+1)) \in N^{(i(n)j(n)k(n))}$.

(B) $i(n) + j(n) + k(n) < i(n+1) + j(n+1) + k(n+1)$.

(C) $\{T_{i(n)j(n)k(n)},\ n = 1, 2, \ldots\}$ covers the whole space $m_0(X, \Sigma)$.

We start by setting $(i(1), j(1), k(1)) = (1, 1, 1)$ and having determined the $n-1$ first terms we take $(i(n), j(n), k(n)) \in N^{(i(n-1)j(n-1)k(n-1))}$ such that $(i(n), j(n))$ is equal to the two first coordinates of the $n$th element, $\alpha(n)$, of $\mathbb{N}^3$ and $k(n)$ is such that $i(n-1) + j(n-1) + k(n-1) < i(n) + j(n) + k(n)$. This choice is always possible because of the properties of the sets $N^{(ijk)}$.

Setting $Q := \bigcup\{M_{rst},\ (r,s,t) \in S\}$, as a consequence of the property (C) of the sequence $S$, there is some $(i,j,k) \in S$ such that $e(Q) \in T_{ijk}$. Using then the last relation of (3), this implies that $|\langle e(Q), u_{ijk}\rangle| \le 1$.

On the other hand, as $S$ satisfies condition (B) we have that

$$\begin{aligned}\langle e(Q), u_{ijk}\rangle = \langle e(M_{ijk}), u_{ijk}\rangle &\\ + \left\langle e\left(\bigcup\{M_{rst},\ r+s+t < i+j+k,\ (r,s,t) \in S\}\right), u_{ijk}\right\rangle &\\ + \left\langle \left(\bigcup\{M_{rst},\ r+s+t > i+j+k,\ (r,s,t) \in S\}\right), u_{ijk}\right\rangle &.\end{aligned}$$

Therefore, using now property (A) of $S$, we have

$$\begin{aligned}&|\langle e(Q), u_{ijk}\rangle| \\ &\quad \ge |\langle e(M_{ijk}), u_{ijk}\rangle| - \sum\{|\langle e(M_{rst}), u_{ijk}\rangle|,\ r+s+t < i+j+k\} \\ &\quad\quad - \left\| e\left(\bigcup\{M_{rst},\ (r,s,t) \in N^{(ijk)}\}\right)\right\|.\end{aligned}$$

From this, according to (3) and (4), it follows that $|\langle e(Q), u_{ijk}\rangle| > 1$, a contradiction.

Definition. A double sequence $\{F_{ij},\ i,j = 1, 2, \ldots\}$ of subspaces of a space $F$ will be called doubly increasing if it satisfies the two following properties:

(1) for each $i \in \mathbb{N}$, the sequence $\{F_{ij}, j = 1, 2, \ldots\}$ is increasing;

(2) the sequence $\{\bigcup\{F_{ij},\ j = 1, 2, \ldots\},\ i = 1, 2, \ldots\}$ is increasing.

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PROPOSITION 1. *Suppose that $W$ is a doubly increasing sequence of subspaces of a space $F$ covering $F$ and let $f$ be a linear mapping from $m_0(X, \Sigma)$ into $F$ with closed graph. If each $L \in W$ has a locally convex topology $\tau_L$ stronger than the final one such that $L(\tau_L)$ is a $\Gamma_r$-space, then there is a $G \in W$ containing the range space of $f$ such that $f$, considered as a mapping from $m_0(X, \Sigma)$ into $G(\tau_G)$, is continuous.*

PROOF. By the previous theorem there is some $G \in W$ such that $E := f^{-1}(G)$ is dense in $m_0(X, \Sigma)$ and barrelled. Now if $g$ denotes the restriction of $f$ on $E$ and $x \in m_0(X, \Sigma) - E$, there exists a $G$-valued linear extension $h$ of $g$ over the subspace $L := \langle\{x\} \cup E\rangle$ with closed graph. As $E$ is dense and barrelled in $m_0(X, \Sigma)$, then $L$ is barrelled. Now the closed graph theorem of [10] establishes the continuity of $h$. If $\{x_n, n = 1, 2, \ldots\}$ is a sequence of points of $E$ which converges to $x$ under the norm topology of the space $m_0(X, \Sigma)$, we have that $h(x_n) \to h(x)$ in $G$ and, the graph of $f$ being closed, that $f(x) = h(x) \in G$. Thus $x \in E$, a contradiction. This shows that $f$ is $G$-valued. Furthermore, $f: m_0(X, \Sigma) \to G(\tau_G)$ is continuous.

THEOREM 2. *Let $\mu$ be a finitely additive measure on $\Sigma$ with values in a space $E$ and let $H$ be a $\sigma(E'E)$-total subset of $E'$. Suppose that $E$ contains a doubly increasing sequence $W$ of subspaces of $E$ covering $E$ such that in each $L \in W$ there exists some locally convex topology $\tau_L$, stronger than the final one, under which $L(\tau_L)$ is a $\Gamma_r$-space which does not contain a copy of $l^\infty$ If $u \circ \mu$ is a countably additive scalar measure for each $u \in H$, there exists a $G \in W$ such that $\mu$ is a $G$-valued countably additive vector measure.*

PROOF. Define $S: m_0(X, \Sigma) \to E$ such that $S(e(A)) = \mu(A)$ for every $A \in \Sigma$ and let $F$ denote the linear hull of $H$. If $\{z_i, i \in I, \geq\}$ is a net of points of $m_0(X, \Sigma)$ such that $z_i \to z$ in $m_0(X, \Sigma)$, then $\langle z_i, u \circ \mu\rangle \to \langle z, u \circ \mu\rangle$ for every $u \in F$. In fact, $u \circ \mu$ is a bounded finitely additive scalar measure when $u \in H$ and it can be identified with an element of the dual space of $m_0(X, \Sigma)$. Thus, $\langle S(z_i), u\rangle \to \langle S(z), u\rangle$ for every $u \in F$. This shows that $S: m_0(X, \Sigma) \to E(\sigma(E, F))$ is continuous. So, $S$ is a mapping from $m_0(X, \Sigma)$ into $E$ with closed graph. By Proposition 1, there is some $G \in W$ such that $S: m_0(X, \Sigma) \to G(\tau_G)$ is continuous. Now, by [10, Corollary 1.14] there is a $G$-valued continuous linear extension $T$ of $S$ over the completion $m(X, \Sigma)$ of $m_0(X, \Sigma)$. As $G(\tau_G)$ contains no copy of $l^\infty$, $T$ is weakly compact [3]. From this fact, taking into account that $u \circ \mu$

is a countably additive scalar measure for every $u \in H$, is easy to show that $u \circ \mu$ is countably additive for every $u \in E'$. Now the Orlicz-Pettis theorem for locally convex spaces [5, 9.4] applies.

THEOREM 3. *Let $\mu$ be a mapping from $\Sigma$ into a space $E$ and let $H$ be a $\sigma(E'E)$-total subset of $E'$. Suppose that $E$ has a doubly increasing sequence $W$ of subspaces of $E$ covering $E$ such that in each $L \in W$ there exists some locally convex topology $\tau_L$, stronger than the final one, under which $L(\tau_L)$ is a $\Gamma_r$-space. If $u \circ \mu$ is a bounded finitely additive scalar measure for each $u \in H$, then there is some $G \in W$ such that $\mu$ is a $G$-valued bounded vector measure.*

PROOF. The totality of $H$ implies the finite additivity of $\mu$. Now defining $S$ as in the previous theorem, the boundedness of $u \circ \mu$ for each $u \in H$ guarantees that $S: m_0(X, \Sigma) \to E$ has closed graph. By the proposition above there is some $G$ such that $S: m_0(X, \Sigma) \to G(\tau_G)$ is continuous. Therefore, $S(\{e(A), A \in \Sigma\})$ is a bounded subset of $G(\tau_G)$ and hence $\{\mu(A), A \in \Sigma\}$ is bounded in $G$.

REMARK. Theorem 2 generalizes the implication (i) $\Rightarrow$ (iii) of [1, Theorem 1.1] and Theorem 3 generalizes [2, Corollary I.3.3].

NOTE. After we sent this paper we have shown in [11], using different methods and giving different applications, that $m_0(X, \Sigma)$ has a stronger barrelledness property than that of being ordered suprabarrelled.

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