EQUATIONS OF MOTION FOR ISOLATED BODIES WITH RELATIVISTIC CORRECTIONS INCLUDING THE RADIATION REACTION FORCE.

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ABSTRACT. We have derived in an explicit form the equations of motion for two spherically-symmetric non rotating bodies in the slow motion approximation. The equations include relativistic corrections of order $(v/c)^2$, $(v/c)^4$ and $(v/c)^5$ to the newtonian equations of motion. It is shown that the equations depend on the only parameter characterizing each body, namely on its relativistic mass, regardless of its internal structure and degree of compactness. This means that the equations can also be applied to bodies with a strong internal gravity, such as neutron stars and black holes. It is shown that in the $(v/c)^2$ and $(v/c)^4$ approximations the equations can be derived from a Lagrangian. The Lagrangian is given in an exact form. The integration of the equations of motion is performed by the method of osculating elements. The formulae for secular change of the semi-major axis and eccentricity coincide precisely with the standard ones whose derivation is based on a calculation of the energy flux in the outgoing gravitational waves.

1. INTRODUCTION

The detailed description of the motion of the gravitating bodies with all relativistic corrections taken into account, including radiation reaction force, has now acquired not only a theoretical, but also a practical meaning. This is accounted for by a persistent raising of precision of astronomical observations and discovery of such systems as the double pulsar PSR 1913+16 (Hulse and Taylor, 1975). The present techniques allow to measure the periodic and secular corrections to the Newton equations of motion of the order of $(v/c)^2$ (see Will, 1981 ; Kislik et al., 1980 and Weisberg and Taylor, 1984) as well as the secular radiation corrections to the order $(v/c)^5$ (Weisberg and Taylor, 1984) where v is a characteristic velocity of the bodies. It seems possible that the effect caused by terms of order $(v/c)^4$ may be measured in a near future (Shapiro, 1979). The progress of astronomical observations stimulates the development of rigorous relativistic theories of motion of celestial bodies.

An isolated pair of masses is a simple model for which the relativistic theory of motion up to terms of order $(v/c)^5$ inclusively can be

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developed. On the other hand this model provides an adequate approximation to most of the real double star systems.

The motion of a pair of masses with the radiative reaction force included has been a subject of a large number of works. The early papers on the gravitational radiative damping (Peters and Mathews, 1963 and Peters, 1964) have been based on the application of the Einstein formula for the gravity wave flux. The method of continuation of a radiative solution to the close-by zone has also been applied (Burke, 1971 and Misner et al., 1973). However, both these methods are not capable of describing the motion in full details, since they ignore the conservative relativistic corrections (such as a "perihelion shift") and they give information only about secular terms in the radiative approximation. Moreover, for some authors these methods seem doubtful (Ehlers et al., 1976).

The other approach to the problem of radiative damping is based on a solution of the relativistic hydrodynamic equations by successive approximations (see Chandrasekhar et al., 1965, 1969, 1970 ; Kerlik, 1980 ; Futamase, 1983 ; Anderson and Decanio, 1975). This approach gives relativistic corrections of order $(v/c)^2$, $(v/c)^4$ and $(v/c)^5$ to the equations of motion of a fluid element. However, these equations describe the motion of an element of a fluid and not a body as a whole.

One can distinguish three different ways in obtaining the equations of motion (eq.m.) for a body as a whole. These are :

- The Einstein-Infeld-Hoffman method (EIH) : Einstein et al., 1938.
- The method of asymptotic expansions (AE) : Demianski and Grishchuk, 1974 ; D'Eath, 1975a and b ; Kates, 1980 ; Thorne and Hartle, 1984 ; Zhang, 1984.
- The method of the post-Newtonian approximations (PNA) : Fock, 1959.

One should also mention the works by Dixon (1979), Ehlers and Rudolph (1977) and Schattner (1979). However, it is not yet clear how to apply the latter method to concrete astronomical objects.

According to the EIH method one treats the bodies as singularities of the gravitational field and derives their eq.m. from the vacuum Einstein equations. In order to simplify the calculations one can introduce the energy-momentum tensor containing delta-functions (Infeld, 1954). This leads to a necessity of regularizing the integrals, which diverge at the world lines of the sources. In the early papers the regularizing procedure has not been well defined. However, some information on the relativistic corrections was extracted in $(v/c)^4$ (Carmeli, 1965 ; Ohta et al., 1974) and $(v/c)^5$ (Carmeli, 1965, Infeld and Michalska-Trautman, 1969) approximations. Later, the regularizing procedure was substantially improved by Damour et al. (1981) and Damour (1983). Unfortunately, there is no proof that the different regularizing procedures lead to the same eq.m. in $(v/c)^4$ and $(v/c)^5$ approximations.

The AE method is based on matching the solution valid in near-field and far-field zones of the sources. By this method the eq.m. are derived in $(v/c)^2$ approximation. For higher approximations this method is not yet developed properly.

The PNA method deals explicitely with the energy-momentum tensor of bodies consisting of the hydrodynamic fluid. The iteration scheme for solving the Einstein equations is developed in Chandrasekhar (1965, 1966

1970), Anderson and Decanio (1975), Futamase and Shutz (1983) as well as Ehlers (1980). The eq.m. for a body as a whole are obtained by integrating the hydrodynamic equations over the whole body (Fock, 1959 ; Papapetrou, 1951 ; Brumberg, 1972 ; Petrova, 1949 ; Petrova and Sandina, 1974 ; Breuer and Rudolph, 1981a,b ; Papapetrou and Linet, 1981 ; Spyrou 1977, 1978 ; Caporali, 1980 ; Kopejkin, 1985). This method is technically simpler than the EIH and AE methods but it assumes that the motion of the bodies is slow and the gravitational field is weak everywhere outside and inside the bodies. It will be shown below that, in fact, the eq. m. do not depend on the condition of weakness of the field inside the bodies, i.e. the parameter which characterizes the compactness of the bodies does not enter the eq.m. . This fact implies the validity of the PNA method for a wide class of objects, from ordinary stars to black holes.

The aim of this paper is to derive and solve the relativistic eq.m. for the isolated bodies with the same degree of completeness which is accepted in ordinary celestial mechanics. We analyse the motion of a pair of spherically-symmetric non-rotating bodies. All relativistic corrections up to terms of order $(v/c)^5$ inclusively are taken into account. We are using the PNA method whose main features are described in § 2. The limitations on the dimensionless parameters which characterize the system are discussed in § 3. The eq.m. including the radiation reaction force are derived in § 4. In § 5 we show that the eq.m. in $(v/c)^2$ and $(v/c)^4$ approximations can be presented in the form of the Euler-Lagrange equations. The explicit expression for the Lagrangian is given and the quantities conserved in these approximations are obtained. The rate of change of these quantities in the next $(v/c)^5$ approximation is also derived. In § 6 we solve the equations of motion by using the method of the osculating elements. The exact form of the parameters of the Keplerian orbit is found including secular terms caused by the gravitational radiation. Thus, by applying the PNA method, we have obtained the full celestial mechanical description of a double system which can consist either of extended bodies or compact bodies such as neutron stars or black holes.

2. THE PNA METHOD

Let us assume that the bodies consist of a fluid with the energy-momentum tensor :

$$T^{\alpha\beta} = (\mu + p)\, u^{\alpha} u^{\beta} - p g^{\alpha\beta} \qquad , \qquad \mu = \rho\, (c^2 + \Pi)$$

(greek indices have the values 0,1,2,3 ; Latin indices, 1,2,3. A coma denotes an ordinary derivative and a semicolon - covariant derivative. Latin indices are moved with the help of the unity matrix δ_{ik}). Here ρ denotes the rest mass density in the comoving frame, p is the isotropic pressure connected with ρ by an equation of state $p = p(\rho)$, $u^{\alpha} = dx^{\alpha}/ds$ is the 4-velocity of a fluid element, Π is the internal energy density satisfying the thermodynamic equation $d\,\Pi + p\,d\,(1/\rho) = 0$. The gravitational field is decribed by quantities :

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$$\gamma^{\alpha\beta} = \sqrt{-g}\ g^{\alpha\beta} - \eta^{\alpha\beta}$$

where $g^{\alpha\beta}$ is the metric tensor and $\eta^{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$. The harmonic frame is chosen, i.d. :

$$\gamma^{\alpha\beta}{}_{,\beta} = 0$$

In this frame the full set of the Einstein equations takes the form (Anderson and Decanio, 1975) :

$$\Box\, \gamma^{\alpha\beta} = 16\pi\, G/c^4\ \Lambda^{\alpha\beta}, \Box = \eta^{\mu\nu}\ \partial_\mu \partial_\nu$$
$$\Lambda^{\alpha\beta} = (-g)(T^{\alpha\beta}+t^{\alpha\beta}) - c^4/16\pi\, G(\gamma^{\alpha\beta}\gamma^{\mu\nu} - \gamma^{\alpha\mu}\gamma^{\beta\nu})_{,\mu\nu} \tag{2.1}$$

where $t^{\alpha\beta}$ is the Landau-Lifshitz (1975) pseudotensor.
The formal solution of (2.1) is :

$$\gamma^{\alpha\beta} = 16\pi\, G/c^4\ \Box^{-1}\ \Lambda^{\alpha\beta} \tag{2.2}$$

By expanding the retarded time $t' = t - (x-y)\, c^{-1}$, one reduces (2.2) to:

$$\gamma^{\alpha\beta}\left[\vec{x},t\right] = 4G/c^4 \sum_{k=0}^{\infty} (-1)^k/k!\ I_{k-1}(\partial\Lambda^{\alpha\beta}/\partial t^k) \tag{2.3}$$

where $I_k(f)\left[\vec{x},t\right] = \int_{R^3} d^3y\ f(\vec{y},t)(\vec{x} - \vec{y})^k$,

and R^3 means integration over the whole space. In (2.3) the time derivative is left under the sign of the integral according to the suggestion by Ehlers (1980). This prevents all the metric coefficients up to the $(v/c)^6$ approximation from the appearance of the divergent integrals (Kerlick, 1980 ; Futamase, 1983 ; Futamase and Schutz, 1983 ; Breuer and Rudolph, 1981 a,b).

Let us begin the calculations from $\gamma^{\alpha\beta} = 0$ in the r.h.s. of (2.3). By three successive iterations one obtains the metric components expanded in powers of 1/c (Anderson and Decanio, 1975) :

$$g_{oo} = 1 + c^{-2}\ \underset{2}{g}_{oo} + c^{-4}\underset{4}{g}_{oo} + c^{-5}\underset{5}{A}_{oo} + c^{-6}\underset{6}{g}_{oo} + c^{-7}\underset{7}{A}_{oo} + O(c^{-8})$$

$$g_{oi} = c^{-3}\underset{3}{g}_{oi} + c^{-5}\underset{5}{g}_{oi} + c^{-6}\underset{6}{A}_{oi} + O(c^{-7}) \tag{2.4}$$

$$g_{ik} = \eta_{ik} + c^{-2}\underset{2}{g}_{ik} + c^{-4}\underset{4}{g}_{ik} + c^{-5}\underset{5}{A}_{ik} + O(c^{-6})$$

where

$$\underset{2}{g}_{oo} = -\ 2U\ ,$$

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$$\underset{4}{g}_{oo} = 2U^2 - 4\Phi + \chi_{,tt},$$

$$\underset{3}{g}_{oi} = 4W_i$$

$$\underset{2}{g}_{ik} = -2U\delta_{ik}$$

$$U = GI_{-1}(\mu)$$

$$\Phi = GI_{-1}(\mu v^2 + \mu U + 3/2p)$$

$$\chi = 1/2\pi\ I_{-1}(U)$$

$$W_i = GI_{-1}(\mu v^i)$$

$$v^i = u^i/u^o$$

The remaining coefficients in the (2.4) are also presented as the integrals from the, as yet unknown, functions $\mu(\vec{x},t)$, $v^i(\vec{x},t)$ and the coefficients themselves. Their functional form can be found in Anderson and Decanio (1975) and in Breuer and Rudolph (1981,a,b).

The hydrodynamic equations of motion :

$$T^{\alpha\beta}{}_{;\beta} = T^{\alpha\beta}{}_{,\beta} + \Gamma^{\alpha}{}_{\mu\nu}\, T^{\mu\nu} + \Gamma^{\mu}{}_{\nu\mu}\, T^{\alpha\nu} = 0$$

after substitution, $\Gamma^{\alpha}{}_{\beta\gamma}$ calculated from (2.4) get the form (Breuer and Rudolph, 1981,a,b) :

$$\begin{aligned}
\mu_{,t} &= -(\mu v_k)_{,k} + c^{-2}k + c^{-4}q + c^{-5}s + O(c^{-6}) \\
(\mu v_i)_{,t} &= -\theta_{ik,k} + c^{-2}k_i + c^{-4}q_i + c^{-5}s_i + O(c^{-6}) \qquad (2.5) \\
\theta_{ik} &= \mu v_i v_k + p\delta_{ik} + 1/8\pi G\{2U_{,i}U_{,k} - \delta_{ik}(U_{,n})^2\}
\end{aligned}$$

where k,q,s,k_i,q_i,s_i are functionals of μ, v^k and their derivatives.

In what follows it is convenient to use the invariant rest-mass density $\rho^* = \rho u^o \sqrt{-g}$, instead of μ. For ρ^* one has the exact equation of continuity (Fock, 1959) :

$$\rho^*_{,t} + (\rho^*\, v_k)_{,k} = 0 \qquad (2.6)$$

In addition, for any sufficiently smooth function $f(\vec{x},t)$, the following useful formula is valid :

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$$\frac{\partial}{\partial t} \int_{V_a} \rho^* \, f(\vec{x},t) d^3x = \int_{V_a} \rho^* \frac{d}{dt} f(\vec{x},t) d^3x$$

where $d/dt = \partial/\partial t + v^k \partial/\partial x^k$ and V_a means integration over the volume of a body "a". In all formulae we replace μ by ρ^* (see Kopejkin, 1985 for more details). The replacement of μ by ρ^* is accompanied by the expansion of U, W_i, χ, Φ, Π in powers of $1/c$ (Brumberg, 1972 or Kopejkin, 1985). For example :

$$U = \underset{0}{U} + c^{-2}\underset{2}{U} + c^{-4}\underset{4}{U} + O(c^{-6})$$

Further, one replaces μ by ρ^* in the equations (2.5). As a result one obtains (Kopejkin, 1985 ; Grishchuk and Kopejkin, 1983) :

$$(\rho^* v_i)_{,t} + (\rho^* v_i v_k)_{,k} + p_{,i} - \rho^* \underset{0}{U}_{,i} + \\ + c^{-2}k_i + c^{-4}q_i + c^{-5}s_i = O(c^{-6}) \tag{2.7}$$

where k_i, q_i, s_i are functionals of ρ^*, v^i and their derivatives.

The eq.m. for the center of mass of a body (defined below) are derived by integrating (2.7) over the volume of the body. In this way one describes the motion of the body as if it moves in ordinary flat space-time with respect to the Cartesian coordinates $\vec{x} = (x^1, x^2, x^3)$. From this point of view one may treat the PNA method as a power expansion of an exact field theory with the field variables $\gamma^{\alpha\beta}(\vec{x},t)$, $\rho(\vec{x},t)$, $v^i(\vec{x},t)$ given in the flat background space-time. A presentation of the general relativity as an exact field theory in an arbitrary background space-time can be found in Grishchuk et al., 1984.

Let us recall the characteristics of the various approximations :

$(v/c)^0$ (OPNA) : Newtonian approximation

$(v/c)^2$ (1PNA) : post-Newtonian approximation

$(v/c)^4$ (2PNA) : post-post-Newtonian approximation

$(v/c)^5$ (2 1/2 PNA) : radiative approximation.

3. RESTRICTIONS ON THE DIMENSIONLESS PARAMETERS

In what follows we consider a double system composed of bodies with the characteristic mass M, the linear size L and with the distance between them R. One can introduce three small dimensionless parameters :

$$\varepsilon_1 = v/c \sim (GM/c^2R)^{1/2}$$

$$\varepsilon_2 = (GM/c^2L)^{1/2}$$

$$\varepsilon_3 = L/R$$

(ε_1 describes the slowness of the orbital motion, ε_2 shows the weakness of the internal gravity and ε_3 indicates the smallness of the sizes of the bodies as compared with the distance between them).

We assume that the bodies maintain their spherically-symmetric shape and do not rotate at least with the accuracy which is needed for derivation of the eq.m. in (2-1/2PNA). This assumption establishes a certain relation between parameters ε_1 and ε_3 which we discuss below.

It is known that the shape of a body "a" is slightly deformed by the gravitational field of a neighboring body "b". In order of magnitude this tidal deformation is (Alexander, 1973) :

$$(\Delta L/L)_{tid.} \cong \kappa\ (L/R)^3$$

where κ depends on the elasticity of the body "a" (usually, $\kappa \cong 10^{-3}$ to 10^{-1}).The deformed body "a" acts on the body "b" with the additional force

$$F_{tid.} \cong \kappa (L/R)^5\ GM^2/R^2$$

On the other hand, the expected radiation-reaction force evaluated from (2.7) is :

$$F_{rad.} \cong (v/c)^5\, GM^2/R^2$$

So we have to assume that $F_{tid.} \ll F_{rad.}$ which yields the relation :

$$\varepsilon_3 \ll \kappa^{-1/5}.\ \varepsilon_1 \qquad (3.1)$$

According to Brumberg (1972) an initially non-rotating body may acquire a rotation in 1PNA with the angular velocity $\omega_{in} \cong (v/c)^2 (GM/R^3)^{1/2}$. The rotation of the body deforms its shape and creates an additional force $F_{rot.}$ However, it can be shown that $F_{rot.} \ll F_{rad.}$, if (3.1) is satisfied. The rotation gives also rise to additional forces resulting from spin-orbit and spin-spin interactions between the bodies. However, these forces are even smaller than $F_{rot.}$ Thus, if (3.1) is fulfilled, we can safely regard the bodies as spherically-symmetric and non-rotating.

We did not discuss any restrictions on the parameter ε_2. This is because ε_2, as it will be seen below, does not enter the eq.m. in the considered approximations.

4. THE EQUATIONS OF MOTION

Let us define the rest mass of the body "a" (a = 1,2) by the relation :

$$M_a = \int_{V_a} \rho^* \, d^3x$$

Due to (2.6), M_a is constant, i.e. $dM_a/dt = 0$. The coordinates of the center of mass x^i_a are defined by the formula :

$$M_a x^i_a = \int_{V_a} \rho^* \, x^i d^3x$$

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The velocity of the center of mass v_a^i , its acceleration a_a^i and the higher derivatives of x_a^i are defined by :

$$M_a v_a^i = M_a dx_a^i/dt = \int_{V_a} \rho^* \; v^i d^3x,$$

$$M_a a_a^i = M_a dv_a^i/dt = \int_{V_a} \rho^* \; dv^i/dt \; d^3x$$

$$M_a d^k x_a^i/dt = \int_{V_a} \rho^* \; d^k x^i/dt^k d^3x \qquad (k = 3,4,...)$$

Restrictions on ε_1 and ε_3 imposed in §3 permit us to confine attention to such motions where each fluid element of the body "a" moves with the same speed as the center of mass of that body. In other words, we assume that each body is spherically-symmetric and static in the frame of reference connected with the center of mass of that body. For this reason one can take v^i from under the sign of the integral over V_a and replace v^i with v_a^i. This significantly simplifies the derivation of the virial theorem in 1PNA which can be derived by multiplying equations (2.7) by $R_a^i = x^i - x_a^i$ and further integrating them over V_a. The virial theorem is used in the subsequent rearrangements (see §3 from Kopejkin, 1985).

Now we can proceed directly to the derivation of the eq.m. for the center of mass. Let us integrate (2.7) over V_a. When calculating the integrals which determine the eq.m. of the body "a" one decomposes the potentials U, W_i, χ , Φ into two parts : the internal part governed by the body "a" itself, and the external part produced by the body "b". One makes sure that all the terms describing the interaction of the elements of the body "a" between themselves cancel in the eq.m. of that body (Breuer and Rudolph, 1981b ; Kopejkin, 1985). The external part of the potentials is expanded in powers of L/R. When doing this one uses various expansions and formulae from §5 of Kopejkin (1985).

In 2PNA and 2 1/2 PNA the quantities $\gamma^{\alpha\beta}$ contain the "metric generates metric" terms, i.d. the integrals over the whole space from the non-linear combinations of metric coefficients of the preceding approximations. These terms complicate the derivation of the eq.m., especially in 2PNA.

In the course of the tedious computations of these terms, we have extensively used the rule for differentiating generalized homogeneous functions (Gel'Fand and Shilov, 1959) and the formula for calculating integrals over the whole space from the products of the functions $R_a = |x-x_a|$ and $R_b = |x-x_b|$ (see Damour, 1983 and, for more details, Kopejkin, 1985). In the 2 1/2 PNA, the integrals are not so complicated and are computed similarly to 1PNA (Kopejkin, 1985 and Grishchuk and Kopejkin, 1983). The final form of the equations of motion is as follows (same references) :

$$ma^i = \underset{0}{F^i} + c^{-2}\underset{2}{F^i} + c^{-4}\underset{4}{F^i} + c^{-5}\underset{5}{F^i} + O(c^{-6}) \qquad (4.1)$$

where $F^i_{0} = -(Gmm'/R^2)N^i$,

$$
\begin{aligned}
F^i_{2} = & -\frac{1}{2}mv^2a^i - m(va)v^i + (Gmm'/R)\left(\frac{7}{2}a'^i - 3a^i\right) + \\
& + (Gmm'/R^2)\Big\{N^i\Big(-\frac{3}{2}v^2 - 2v'^2 + 4(vv') + \frac{3}{2}(Nv')^2 + \frac{1}{2}R(Na') + \\
& \qquad + Gm/R + Gm'/R\Big) + 3(Nv)v^i - 3(Nv')v^i + \\
& \qquad + 3(Nv')v'^i - 4(Nv)v'^i\Big\}
\end{aligned}
$$

$$
\begin{aligned}
F^i_{4} = & -\frac{3}{8}mv^4a^i - \frac{3}{2}mv^2(va)v^i + \\
& + Gmm'\Big\{\frac{15}{8}R\ddot{a}'^i - \frac{1}{8}R(N\ddot{a}')N^i + 2(Nv)\dot{a}'^i - \frac{11}{2}(Nv')\dot{a}'^i - \\
& \qquad - \frac{1}{2}(Nv')(N\dot{a}')N^i - 2(v\dot{a}')N^i + 2(v'\dot{a}')N^i + \frac{3}{2}(N\dot{a}')v^i - \\
& \qquad - \frac{3}{2}(N\dot{a}')v'^i + \frac{3}{2}(Na')a^i - \frac{21}{4}(Na')a'^i + \\
& \qquad + \frac{15}{8}a'^2N^i - \frac{3}{8}(Na')^2N^i\Big\} + \\
& + (Gmm'/R)\Big\{a^i\Big(-\frac{7}{2}v^2 + 2v'^2 + 4(vv') + \frac{3}{2}(Nv')^2\Big) + a'^i\Big(\frac{5}{4}v^2 + \\
& \qquad + \frac{13}{2}v'^2 - 6(vv') + 4(Nv)(Nv') - \frac{21}{4}(Nv')^2\Big) + \\
& \qquad + v^i\Big(\frac{11}{2}(va') - \frac{11}{2}(v'a') - \frac{3}{2}(Nv)(Na') + \frac{9}{2}(Nv')(Na') - \\
& \qquad - 7(va) + 4(v'a)\Big) + v'^i\Big(-2(va') + \frac{11}{2}(v'a') + 2(Nv)(Na') - \\
& \qquad - \frac{9}{2}(Nv')(Na') + 4(va) - 4(v'a)\Big) + N^i\Big(\frac{9}{2}(Nv')(v'a') - \\
& \qquad - 4(Nv')(va') - \frac{9}{4}(Nv')^2(Na') + \frac{3}{4}v^2(Na') - 2(vv')(Na') + \\
& \qquad + \frac{3}{2}v'^2(Na') - \frac{7}{8}v^4 - 2v'^4 - 2(vv')^2 - v^2v'^2 - \frac{15}{8}(Nv')^4 + \\
& \qquad + 4(vv')v'^2 + 2(vv')v^2 - 6(vv')(Nv')^2 + \frac{9}{2}v'^2(Nv')^2 + \\
& \qquad + \frac{9}{4}v^2(Nv')^2\Big)\Big\} + \\
& + (Gmm'/R^2)\Big\{v^i\Big(\frac{9}{2}(Nv')^3 + 7(Nv')(vv') - 5v'^2(Nv') - \frac{9}{2}(Nv)(Nv')^2 + \\
& \qquad + 2(Nv)v'^2 + \frac{7}{2}(Nv)v^2 - \frac{7}{2}(Nv')v^2 - 4(Nv)(vv')\Big) +
\end{aligned}
$$

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$$+v'^i\left(-\frac{9}{2}(Nv')^3 - 4(Nv')(vv') + 5v'^2(Nv') + 6(Nv)(Nv')^2 - \right.$$

$$\left. - 4(Nv)v'^2 - 2(Nv)v^2 + \frac{1}{2}(Nv')v^2 + 4(Nv)(vv')\right)\} +$$

$$+(G^2m^2m'/R^2)\{-4a^i + \frac{7}{2}a'^i + N^i(-4v^2 - \frac{7}{2}v'^2 + 7(vv') - 2(Nv')^2) +$$

$$+ 8(Nv)v^i - 8(Nv')v^i - 7(Nv)v'^i + 8(Nv')v'^i\} +$$

$$+(G^2mm'^2/R^2)\{-\frac{7}{2}a^i + \frac{7}{2}a'^i + N^i(-(Na) - \frac{9}{2}v^2 - 4v'^2 + 8(vv') -$$

$$- 4(Nv)(Nv') + 2(Nv)^2) + 7(Nv)v^i - 7(Nv')v^i -$$

$$- 6(Nv)v'^i + 7(Nv')v'^i\} +$$

$$+(G^3mm'/R^4)(-\frac{3}{2}m^2 - \frac{3}{2}m'^2 - 9\,mm')N^i$$

$$F^i_5 = (Gm^2m'^2)/(m+m')^2\{-\frac{1}{30}R^2\overset{.....}{R}_i + ((RR) - \frac{4}{3}(R\ddot{R}))\dddot{R}_i + \frac{11}{3}(\dot{R}\ddot{R})\ddot{R}_i +$$

$$+ (3(R\ddddot{R}) + \frac{17}{2}(\dot{R}\dddot{R}) + \frac{2}{3}(\ddot{R}\ddot{R}))\dot{R}_i + (\frac{3}{5}(R\overset{.....}{R}) +$$

$$+ \frac{13}{6}(\dot{R}\ddddot{R}) + \frac{61}{30}(\ddot{R}\dddot{R}))R_i\} +$$

$$+(Gm^3m'/(m+m')^2)\{\frac{19}{30}R^2\overset{.....}{R}_i + 5(R\dot{R})\ddddot{R}_i + (7(\dot{R}\dot{R}) + \frac{22}{3}(R\ddot{R}))\dddot{R}_i +$$

$$+ 3(\dot{R}\ddot{R})\ddot{R}_i + (-\frac{2}{3}(\ddot{R}\ddot{R}) + \frac{1}{2}(\dot{R}\dddot{R}))\dot{R}_i +$$

$$+ (-\frac{61}{30}(\ddot{R}\dddot{R}) - \frac{17}{30}(\dot{R}\ddddot{R}))R_i\}$$

We denoted the quantities belonging to the body "b" by a prime (so that m_a=m, m_b=m', etc.) and $N^i = R^i/R$, $R^i = x^i_a - x^i_b$, $R = |x_a - x_b|$.

One should pay attention that we have introduced a new (relativistic) mass m_a according to the definition :

$$m_a = \int_{V_a} d^3\mathbf{x}\,(\rho^* + c^{-2}(\rho^*\underset{0}{\Pi} - \frac{1}{2}\rho^*\underset{0}{U}_a) + c^{-4}(\frac{1}{2}\rho^*\underset{0}{U}{}^2_a\underset{0}{\Pi} - \rho^*\underset{00}{U}_a - 3p\underset{0}{U}_a)) + 0(c^{-6})$$

where $\underset{0}{U}_a$ is the part of $\underset{0}{U}$ only over V_a. Note that $dm_a/dt = 0$. This expression for m_a is an expansion of the Tolman mass of the isolated body (Landau and Lifshitz, 1975) in powers of ε_2. Thus, the compactness parameter ε_2 is completely incorporated in the definition of the mass and does not appear in a direct way in the eq.m.(4.1). This indicates that the equations (4.1) can be equally well applied to the extended bodies and the compact bodies such as neutron stars or black holes.

If one eliminates the higher time derivatives from (4.1) one obtains the reduced eq.m. which coincide precisely with those derived by Damour (1983) by EIH method. The expression for $\underset{0}{F}^i$ was derived by Breuer and Rudolph (1981b). $\underset{2}{F}^i$ and $\underset{4}{F}^i$ with higher time derivatives excluded were computed by Damour (1983), some terms of $\underset{5}{F}^i$ were found by Linet (1981), and the full expression for $\underset{5}{F}^i$ by Damour (1983). Schäfer (1982) has obtained $\underset{5}{F}^i$ in a non-harmonic frame of reference. The l.h.s. of (4.1) is given in Breuer and Rudolph (1981) and Damour (1983).

5. THE LAGRANGIAN, THE CONSERVATION LAWS AND THE BALANCE EQUATIONS.

The eq.m. up to the order $(v/c)^4$ have the form of the Euler-Lagrange equations :

$$\partial\mathcal{L}/\partial x^i_a - \frac{d}{dt}(\partial\mathcal{L}/\partial v^i_a) + \frac{d^2}{dt^2}(\partial\mathcal{L}/\partial a^i_a) = 0$$

The explicit expression for the Lagrangian $\mathcal{L}$ is as follows :

$$\mathcal{L}(z,v,a) = \underset{0}{\mathcal{L}}(z,v) + c^{-2}\underset{2}{\mathcal{L}}(z,v) + c^{-4}\underset{4}{\mathcal{L}}(z,v,a) \tag{5.1}$$

where $z^i = x^i_a$, $z'^i = x^i_b$ and :

$$\underset{0}{\mathcal{L}}(z,v) = \Sigma(\frac{1}{2}mv^2 + \frac{1}{2}Gmm'/R)$$

$$\underset{2}{\mathcal{L}}(z,v) = \Sigma\,\frac{1}{8}mv^4 + \Sigma\,\frac{Gmm'}{R}\{\frac{3}{2}v^2 - \frac{7}{4}(vv') - \frac{1}{4}(Nv)(Nv') - \frac{1Gm}{2R}\}$$

$$\begin{aligned}\underset{4}{\mathcal{L}}(z,v,a) = {} & \Sigma\,\frac{1}{16}mv^6 + \Sigma\,Gmm'\{\frac{15}{16}R(aa') - \frac{1}{16}R(Na)(Na') + \\ & + (Na)(\frac{7}{8}v'^2 - \frac{1}{8}(Nv')^2) - \frac{7}{4}(av')(Nv')\} + \\ & + \Sigma\,\frac{Gmm'}{R}(\frac{7}{8}v^4 + \frac{15}{16}v^2v'^2 - 2v^2(vv') + \frac{1}{8}(vv')^2 - \\ & - \frac{7}{8}(Nv)^2v'^2 + \frac{3}{4}(Nv)(Nv')(vv') + \frac{3}{16}(Nv)^2(Nv')^2) + \\ & + \Sigma\,\frac{G^2m^2m'}{R^2}(2v^2 + \frac{7}{4}v'^2 - \frac{7}{2}(vv') + \frac{1}{2}(Nv')^2) + \\ & + \Sigma\,\frac{G^3mm'}{R^3}(\frac{1}{2}m^2 + \frac{1}{2}m'^2 + 3mm')\end{aligned}$$

and Σ means the sum over bodies , the curly brackets mean the scalar product .The part $\underset{0}{\mathcal{L}} + c^{-2}\underset{2}{\mathcal{L}}$ is well known . (Fichtenholz, 1950 ; see also Droste ,1916). The term $\underset{4}{\mathcal{L}}$ obtained in this work differs from the one derived by Damour(1983) since this author neglected the "double-zero function" (Barker and O'Connel, 1980). Both expressions coincide if one adds the "double-zero" function and the total time derivative to Damour's expression. Damour was not able to derive the complete expression for $\underset{4}{\mathcal{L}}$ because he did not possess the non-reduced eq.m. (4.1). As a result of neglecting the "double-zero" function Damour's expression for $\mathcal{L}$ is only valid in a non-harmonic frame of reference (Schäfer, 1984).

It can be shown that the Lagrangian (5.1) is invariant with respect to the 10-parameter group of motions of the Minkowski space-time. This leads to the existence of 10 conserved quantities : energy E, momentum P, angular momentum L^i and the integrals of the center of mass K^i. They are defined by :

$$E = \underset{0}{E} + c^{-2}\underset{2}{E} + c^{-4}\underset{4}{E} = -\mathcal{L} + \Sigma(p_i v^i + q_i a^i)$$

$$P^i = \underset{0}{P^i} + c^{-2}\underset{2}{P^i} + c^{-4}\underset{4}{P^i} = \Sigma\, p^i$$

$$L^i = \underset{0}{L^i} + c^{-2}\underset{2}{L^i} + c^{-4}\underset{4}{L^i} = \varepsilon_{ikl}\,\Sigma\,(z^k p^l - z^l p^k + v^k q^l - v^l q^k)$$

$$K^i = \underset{0}{K^i} + c^{-2}\underset{2}{K^i} + c^{-4}\underset{4}{K^i} = C^i - tP^i$$

where $p^i = \mathcal{L}_{,v^i} - \frac{d}{dt}\,\mathcal{L}_{,a^i}$; $q^i = \mathcal{L}_{,a^i}$

The conserved quantities $\underset{n}{E}$, $\underset{n}{P^i}$, $\underset{n}{L^i}$, $\underset{n}{K^i}$ for n = 0,2 are known (Fichtenholtz, 1950).

For n = 4 these quantities can be easily obtained from (5.1). Being directly derived from (5.1) they contain the time derivatives of x^i_a up to the third order inclusively. One can exclude the higher time derivatives with the help of the eq.m. of the preceding orders. As a result, one obtains the set of conserved quantities E_{2PN}, $P^i{}_{2PN}$, $L^i{}_{2PN}$, $K^i{}_{2PN}$ which depend only on the coordinates and velocities of the bodies.

In order to derive the balance equations in 2 1/2 PNA one takes the time derivative from each of these quantities and then uses the equations of motion (4.1). This procedure yields :

$$\frac{d}{dt}\left(\underset{0}{E} + c^{-2}\underset{2}{E} + c^{-4}\underset{4}{E} + c^{-5}\underset{5}{E}\right) = -\frac{1}{45}\frac{G}{c^5}\,\dddot{D}_{ik}\,\dddot{D}_{ik} \tag{5.2}$$

$$\frac{d}{dt}\left(\underset{0}{L^i} + c^{-2}\underset{2}{L^i} + c^{-4}\underset{4}{L^i} + c^{-5}\underset{5}{L^i}\right) = -\frac{2}{45}\frac{G}{c^5}\,\varepsilon_{ikl}\,\ddot{D}_{kj}\,\dddot{D}_{jl} \tag{5.3}$$

$$\frac{d}{dt}\left(\underset{0}{P^i} + c^{-2}\underset{2}{P^i} + c^{-4}\underset{4}{P^i} + c^{-5}\underset{5}{P^i}\right) = 0 \tag{5.4}$$

$$\frac{d}{dt}\left(\underset{0}{K^i} + c^{-2}\underset{2}{K^i} + c^{-4}\underset{4}{K^i} + c^{-5}\underset{5}{K^i}\right) = 0 \tag{5.5}$$

where D_{ik} is the quadrupole moment of the system. The explicit expressions for $\underset{5}{E}$, $\underset{5}{P^i}$, $\underset{5}{L^i}$, $\underset{5}{K^i}$ are given is Grishchuk and Kopejkin(1983). See also Damour (1983) and McCrea (1981). It is seen from (5.2)(5.3) that the system loses its energy and angular momentum due to the quadrupole gravitational radiation. The momentum is radiated in the next octupole order (Bekenstein, 1979). It will show up in the eq.m. of order $(v/c)^7$. Since the momentum is conserved in this approximation the system moves with a constant speed. This fact is made clear by (5.4)(5.5).

6. THE OSCULATING ELEMENTS

The eq.m. (4.1) can be treated as equations of ordinary celestial mechanics in the Euclidean space. In addition to the Newtonian force $\underset{0}{F}{}^i$ they contain the small disturbing force

$$F^i = c^{-2}\underset{2}{F}{}^i + c^{-4}\underset{4}{F}{}^i + c^{-5}\underset{5}{F}{}^i$$

The variation of Keplerian parameters can be found by the well known method of osculating elements (Duboshin, 1975).

Let us consider an elliptic orbit. The osculating elements are : Ω - the longitude of ascending node, i - the inclination of the orbit, ω - the angular distance of pericenter from ascending node, p - the semilatus rectum, e - eccentricity, m - the mean anomaly at t_0. Equations which determine the time evolution of the osculating elements depend on the disturbing force. The general form of these equations is well known (see Duboshin, 1975).

The analysis shows that Ω and i remain constant and, hence, the orbital plane stays fixed in the space. The equations for p, e, ω are solved by successive approximations. The results are presented in Grishchuk and Kopejkin (1983). From the expressions for p and e it is easy to derive the rate of change of the semi-major axis $a = p(1 - e^2)^{-1}$ and of the orbital period P. After averaging over the Newtonian period $P_0 = (a_0^3/Gm_0)^{1/2}$ (where $m_0 = m+m'$), one obtains (Grishchuk and Kopejkin, 1983):

$$\frac{1}{P}\left\langle\frac{dP}{dt}\right\rangle = \frac{3}{2}\frac{1}{a}\left\langle\frac{da}{dt}\right\rangle = -\frac{96}{5}\frac{G^3 mm' m_0}{a_0^4(1-e_0^2)^{7/2}}\left(1 + \frac{33}{24}e_0^2 + \frac{37}{96}e_0^4\right) \tag{6.1}$$

where a_0, e_0 are the "undisturbed" parameters.

This formula was derived long ago by a different method (Peters and Mathews, 1963). It is important to emphasize that we have derived (6.1) by a strictly celestial mechanical method without applying any additional notions such as the energy-momentum pseudotensor. As is known, the formula (6.1) is now confirmed observationally with the accuracy of a few percents (Weisberg and Taylor, 1984).

The results presented here conclude the exhaustive treatment of the motion of two bodies within the accepted assumptions.

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DISCUSSION

Will : in most post newtonian treatments of this problem, there occur divergent integrals of the source gravitational energy densities. How are these handled in your approch ?

Grishchuk : this problem is avoided in the approximation considered here by leaving the time derivatives of the source densities under the integral sign, according to a suggestion by Ehlers and others.

Bertotti : what is the relationship of your work and Damour's work ?

Grishchuk : Damour and his colleagues applied basically the E.I.H. method which necessitated the use of a rather complicated regularization procedure at the world lines of singularities. We preferred to give a "microscopic" description of the bodies by considering them as fluid drops. The equations of motion and the Lagrangian we gave are complete. They are not reduced by the substitution of the equations of motion of the preceeding orders and the Lagrangian does not ignore the so-called "double zero" function discussed by Barker and O'Connell. We have also found in an explicit form, the full set of osculating elements for the Keplerian orbit. The main conclusion of both works coincide precisely.

Bertotti : how did you deal with the matching problem between the nearby and the wave zones ?

Grishchuk : technically, we did not deal with the matching problem, since we have considered the near zone solutions. However, the boundary conditions are implied in the analysis and they are determined by using the retarded solutions only throughout the calculations.

Kristensen : the initial value problem depends on the coordinates and their first time derivatives. Your fourth order Lagrangian depends also on the accelerations. How do you avoid in higher order time derivatives in the equations of motion ?

Grishchuk : one cannot avoid the higher order derivatives and they really do appear in the equations of motion derived from our Lagrangian. However, they enter the equations in small terms of the order of c^{-4} so that they can be eliminated, if desired, by using the equations of motion of the preceeding approximations.

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