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THE MAXIMAL IDEAL SPACE OF $H^\infty + C$ ON THE BALL IN $\mathbf{C}^n$

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Let S denote the unit sphere in $\mathbf{C}^n$, B the (open) unit ball in $\mathbf{C}^n$ and $H^\infty(B)$ the collection of all bounded holomorphic functions on B. For $f \in H^\infty(B)$ the limits

$$f^*(\zeta) = \lim_{r\to 1} f(r\zeta)$$

exist for almost every ζ in S, and the map $f \to f^*$ defines an isometric isomorphism from $H^\infty(B)$ onto a closed subalgebra of $L^\infty(S)$, denoted $H^\infty(S)$. (The only measure on S we will refer to in this paper is the Lebesgue measure, $d\sigma$, generated by Euclidean surface area.) Rudin has shown in [**4**] that the spaces $H^\infty(B) + C(\bar{B})$ and $H^\infty(S) + C(S)$ are Banach algebras in the sup norm. In this paper we will show that the maximal ideal space of $H^\infty(B) + C(\bar{B})$, $\sum (H^\infty(B) + C(\bar{B}))$, is naturally homeomorphic to $\sum (H^\infty(B))$, and that $\sum (H^\infty(S) + C(S))$ is naturally homeomorphic to $\sum (H^\infty(S))\backslash B$. This last result is an extension of Sarason's theorem for $H^\infty + C$ on the unit circle [**5**; p. 199].

1. Preliminaries. Let $A(B)$ denote the Banach algebra of all functions holomorphic on B and continuous on $\bar{B}$. The maximal ideal space $\sum (A(\bar{B}))$ consists precisely of the evaluation functionals e_ζ, where $\zeta \in \bar{B}$ and

$$e_\zeta(f) = f(\zeta), \quad f \in A(\bar{B}).$$

The map $\zeta \to e_\zeta$ is a homeomorphism from $\bar{B}$ onto $\sum (A(\bar{B}))$. With this in mind we will write $\bar{B}$ for $\sum (A(\bar{B}))$.

If $\zeta \in B$ we can consider e_ζ as an element of both $\sum (H^\infty(B))$ and

$$\sum (H^\infty(B) + C(\bar{B})).$$

We set

$$B_\infty = \left\{ m \in \sum (H^\infty(B)) : m = e_\zeta, \zeta \in B \right\},$$

and

$$B_{\infty+c} = \left\{ m \in \sum (H^\infty(B) + C(\bar{B})) : m = e_\zeta, \zeta \in B \right\}.$$

Since the map $f^* \to f$ is an isomorphism from $H^\infty(S)$ onto $H^\infty(B)$, there is an

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induced homeomorphism τ from $\sum(H^\infty(B))$ onto $\sum(H^\infty(S))$ defined by

$$\tau(m)(f^*) = m(f), \quad m \in \sum(H^\infty(B)), f \in H^\infty(B).$$

Let $B_{\infty,s} = \tau(B_\infty)$. It is evident that

$$B_{\infty,s} = \left\{m \in \sum(H^\infty(S)) : m(f^*) = f(\zeta), \text{ for some fixed } \zeta \text{ in } B \text{ and all } f^* \in H^\infty(S)\right\}.$$

2. Statement of theorems. The first two theorems below are more or less lemmas for the remaining two. They remain true if B is replaced by any strongly pseudoconvex domain in $\mathbf{C}^n$. We have been informed that Theorem 2 was proved previously by Garnett using methods apparently similar to our own. Since we were unable to find proofs of either theorem in the literature, we will prove them in the next section.

THEOREM 1. *The map* $p : \sum(H^\infty(B)) \to \bar{B}$ *defined by*

$$p(m) = m|A(\bar{B})$$

is continuous and onto. The map p^{-1} *is well-defined on* B *and is a homeomorphism onto an open subset of* $\sum(H^\infty(B))$. *This subset is precisely* B_∞.

For $\lambda \in S$, let $F_\lambda = \{p^{-1}(\lambda)\}$. It follows from Theorem 1 that $\sum(H^\infty(B))\backslash B_\infty$ is closed and that

$$\sum(H^\infty(B))\backslash B_\infty = \bigcup_{\lambda \in S} F_\lambda.$$

This says simply that the multiplicative linear functionals on $H^\infty(B)$ which are not evaluation at some point in B are precisely those which, when restricted to $A(\bar{B})$, are evaluation at a point on the boundary of B. It also follows that $\tau \circ p^{-1}$ is a homeomorphism of B onto $B_{\infty,s}$ and that the latter is an open subset of $\sum(H_\infty(S))$.

THEOREM 2. *Suppose* $f \in H^\infty(B)$, $\lambda \in S$, *and* $\alpha \in \mathbf{C}$. *There exists an* $m \in F_\lambda$ *such that* $m(f) = \alpha$ *if and only if there is a sequence* $\{\zeta_k\}$ *in* B *converging to* λ *such that* $f(\zeta_k)$ *converges to* α.

We define $p_B : \sum(H^\infty(B) + C(\bar{B})) \to \sum(H^\infty(B))$ by

$$p_B(m) = m|H^\infty(B),$$

and $p_S : \sum(H^\infty(S) + C(S)) \to \sum(H^\infty(S))$ by

$$p_S(m) = m|H^\infty(S).$$

THEOREM 3. *The map* p_B *is a homeomorphism from* $\sum(H^\infty(B) + C(\bar{B}))$ *onto* $\sum(H^\infty(B))$.

THEOREM 4. *The map* p_S *is a homeomorphism from* $\sum(H^\infty(S) + C(S))$ *onto* $\sum(H^\infty(S))\backslash B_{\infty,s}$.

These results can be written as $\sum (H^\infty(B) + C(\bar{B})) = \sum (H^\infty(B))$ and $\sum (H^\infty(S) + C(S)) = \sum (H^\infty(S))\backslash B$.

Theorem 3 will be proved in Section 4. The main difficulty in the proof lies in showing that each element of $\sum (H^\infty(B))\backslash B_\infty$ is extendable to a multiplicative linear functional on $H^\infty(B) + C(\bar{B})$. This is where we need Theorem 2. Theorem 4 will be proved in Section 5. We will show that the "natural" homeomorphism from $H^\infty(B) + C(\bar{B})$ to $H^\infty(S) + C(S)$ defined by

$$\varphi + f \to \varphi^* + (f|S)$$

induces a homeomorphism from $\sum (H^\infty(S) + C(S))$ onto $\sum (H^\infty(S))\backslash B_{\infty,s}$ and that this induced map is p_S.

3. Proof of Theorems 1 and 2. The proof of Theorem 1 when B is the unit disc in $\mathbf{C}^1$ can be found in [**6**] and [**1**; pp. 160–161]. The proofs of the continuity of p and p^{-1} are independent of the dimension of B, and we omit them. To see that p is onto, note that $p(B_\infty) = B$, and hence

$$B \subset p\left(\sum (H^\infty(B))\right) \subset \bar{B}.$$

Since p is continuous and $\sum (H^\infty(B))$ is compact, $p(\sum (H^\infty(B)))$ must be compact, and so closed. Thus $p(\sum (H^\infty(B))) = \bar{B}$. Again since p is continuous, $p^{-1}(B)$ must be an open subset of $\sum (H^\infty(B))$. It remains to show that p^{-1} is in fact well-defined on B. To do this we need the following lemma due to Kerzman and Nagel [**3**; p. 215(4)]. In the case $n = 1$ the lemma can be proved using Blaschke products. For $n > 1$ sheaf-theoretic methods are used.

LEMMA 5. *Let B be the unit ball in $\mathbf{C}^n$, $n \geqq 1$. If $\varphi \in H^\infty(B)$ has a zero at $\zeta = (\zeta_1, \ldots, \zeta_n)$ in B, then there exist $h_1, \ldots, h_n$ in $H^\infty(B)$ such that*

$$\varphi = (z_1 - \zeta_1)h_1 + \ldots + (z_n - \zeta_n)h_n.$$

It now follows that p^{-1} is well-defined over B. For suppose $p(m) = \zeta$ for m in $\sum (H^\infty(B))$. Choose an arbitrary φ in $H^\infty(B)$. By Lemma 5 there are functions $h_1, \ldots, h_n$ in $H^\infty(B)$ such that

$$\varphi - \varphi(\zeta) = (z_1 - \zeta_1)h_1 + \ldots + (z_n - \zeta_n)h_n.$$

By hypothesis $m(z_j - \zeta_j) = 0$ for $j = 1, \ldots, n$, and therefore

$$\begin{aligned} m(\varphi) - \varphi(\zeta) &= m(\varphi - \varphi(\zeta)) \\ &= m(z_1 - \zeta_1)m(h_1) + \ldots + m(z_n - \zeta_n)m(h_n) \\ &= 0. \end{aligned}$$

Since φ was arbitrary we must have $m = e_\zeta$ on $H^\infty(B)$.

As with Theorem 1, the proof of Theorem 2 is exactly the same for B of arbitrary dimension as it is for the unit disc ([**6**] and [**1**; pp. 161–162]), with the exception of Lemma 7 below. On the unit disc the lemma is proved by using Blaschke products. We need first a result due to Kerzman [**2**; pp. 342–345].

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LEMMA 6. *If ϕ is a smooth, bounded, closed $(0, 1)$-form on B then there exists a smooth function u on B, continuous on $\bar{B}$, such that $\bar{\partial} u = \phi$.*

By bounded we mean that if $\phi = \sum \varphi_i d\bar{z}_i$, then each φ_i is a bounded function on B. This result is only part of a much more general result. In particular B may be replaced by any strongly pseudoconvex domain in $\mathbf{C}^n$ with suitably smooth boundary. We will need, however, nothing stronger than Lemma 6 in this paper.

LEMMA 7. *Let $\varphi \in H^\infty(B)$ and $\lambda \in \partial B$. If φ is bounded away from zero on a neighborhood of λ in B, then there exists $\psi \in H^\infty(B)$ such that for any sequence $\{\zeta_k\}$ in B with $\zeta_k \to \lambda$,*

$$\varphi\psi(\zeta_k) \to 1.$$

Proof. Choose a neighborhood V of λ in B on which φ is bounded away from zero and let f be a smooth function on $\mathbf{C}^n$ which is identically 1 on a neighborhood of λ in $\mathbf{C}^n$ with $\operatorname{supp} f \cap B$ contained in V. Thus $(1/\varphi)f$ is defined and bounded on B, and

$$\bar{\partial}\left(\frac{1}{\varphi} f\right) = \frac{1}{\varphi}\bar{\partial} f.$$

It follows from Lemma 6 that there exists u in $C(\bar{B})$ such that

$$\bar{\partial} u = \bar{\partial}\left(\frac{1}{\varphi} f\right),$$

and thus $(1/\varphi)f - u \in H^\infty(B)$. Nothing changes if we subtract a constant from u, so we can assume $u(\lambda) = 0$. Since $f \equiv 1$ on a neighborhood of λ in B, on the same neighborhood we have

$$\varphi\left(\frac{1}{\varphi} f - u\right) = 1 - \varphi u.$$

Since u is continuous on $\bar{B}$ and equal to 0 at λ, and since φ is bounded, we have

$$(1 - \varphi u)(\zeta_k) \to 1$$

for any sequence $\{\zeta_k\}$ in B converging to λ, and hence

$$\varphi\left(\frac{1}{\varphi} f - u\right)(\zeta_k) \to 1.$$

Let $\psi = (1/\varphi)f - u$. This completes the proof of Lemma 7 and hence the proof of Theorem 2.

4. Proof of Theorem 3. Let $\{m_\alpha\}$ be a net in $\sum(H^\infty(B) + C(\bar{B}))$ converging to m. By definition of the topology on $\sum (H^\infty(B) + C(\bar{B}))$ this means

$$m_\alpha(f) \to m(f)$$

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for all f in $H^\infty(B) + C(\bar{B})$, and so for all f in $H^\infty(B)$. Therefore

$$p_B(m_\alpha) = m_\alpha|H^\infty(B) \to m|H^\infty(B) = p_B(m),$$

and so p_B is continuous. Since the maximal ideal spaces are compact and Hausdorff, the theorem will be established once we have shown that p_B is bijective. This means showing that every multiplicative linear functional on $H^\infty(B)$ has a unique extension to a multiplicative linear functional on $H^\infty(B) + C(\bar{B})$.

We first show that every m in $\sum (H^\infty)$ can be extended. If $m = e_\zeta$ belongs to B_∞ then m extends to e_ζ in $B_{\infty+c}$. If m belongs to $\sum (H^\infty(B))\backslash B_\infty$ then $m \in F_\lambda$ for some $\lambda \in \partial B$. Let

$$\mathscr{W} = \left\{ \sum_{|\beta| \leqq k} \varphi_\beta \bar{z}^\beta : \beta \text{ a multi-index, } k \text{ a positive integer, } \varphi_\beta \in H^\infty(B) \right\}.$$

$\mathscr{W}$ is obviously an algebra, and is dense in $H^\infty(B) + C(\bar{B})$ by the Stone–Weierstrass Theorem. Define $\tilde{m}$ on $\mathscr{W}$ by

$$\tilde{m}\left(\sum \varphi_\beta \bar{z}^\beta\right) = m\left(\sum \varphi_\beta \bar{\lambda}^\beta\right).$$

By the continuity of the $\bar{z}^\beta$ at λ and the boundedness of the φ_β, given $\sum \varphi_\beta \bar{z}^\beta$ in $\mathscr{W}$ and $\epsilon > 0$, we can find a neighborhood U of λ in B such that

$$\sup_U \left|\sum \varphi_\beta \bar{\lambda}^\beta\right| < \epsilon + \sup_U \left|\sum \varphi_\beta \bar{z}^\beta\right|.$$

By Theorem 2 we have

$$\left|m\left(\sum \varphi_\beta \bar{\lambda}^\beta\right)\right| \leqq \sup_U \left|\sum \varphi_\beta \bar{\lambda}^\beta\right|,$$

since $\sum \varphi_\beta \bar{\lambda}^\beta$ is in $H^\infty(B)$ and m belongs to F_λ. Thus

$$\left|m\left(\sum \varphi_\beta \bar{\lambda}^\beta\right)\right| \leqq \left\|\sum \varphi_\beta \bar{z}^\beta\right\|_\infty.$$

The above inequalities show that $\tilde{m}$ is bounded on $\mathscr{W}$. It is clear that $\tilde{m}$ is a multiplicative linear functional on $\mathscr{W}$, and so, being bounded, it can be extended to a multiplicative linear functional on $H^\infty(B) + C(\bar{B})$, the closure of $\mathscr{W}$. This extension is an extension of m since $\tilde{m}$ is an extension of m.

We must now establish uniqueness. Suppose $m_1, m_2 \in \sum (H^\infty(B) + C(\bar{B}))$ and that

$$m_1|H^\infty(B) = m_2|H^\infty(B).$$

We want to show $m_1 = m_2$ on $H^\infty(B) + C(\bar{B})$. Since $\mathscr{W}$ is dense in $H^\infty(B) + C(\bar{B})$ it suffices to show that for all m in $\sum (H^\infty(B) + C(\bar{B}))$,

$$m(\bar{z}_j) = \overline{m(z_j)}, \quad j = 1, \ldots, n.$$

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The spectra of all the real coordinate functions are $[-1, 1]$, relative to $H^\infty(B) + C(\bar{B})$. Thus each $m(x_j)$, $m(y_j)$ is real for every m in $\sum (H^\infty(B) + C(\bar{B}))$, and consequently

$$\begin{aligned} m(\bar{z}_j) &= m(x_j - iy_j) \\ &= m(x_j) - im(y_j) \\ &= \overline{m(x_j) + im(y_j)} \\ &= \overline{m(z_j)}. \end{aligned}$$

5. Proof of Theorem 4. Let $\psi = \varphi + f$ belong to $H^\infty(B) + C(\bar{B})$, with $\varphi \in H^\infty(B)$ and $f \in C(\bar{B})$. For almost every ζ in S, $\lim_{r\to 1} \psi(r\zeta)$ exists, and

$$\lim_{r\to 1} \psi(r\zeta) = \lim_{r\to 1} (\varphi(r\zeta) + f(r\zeta)).$$

If we let

$$(1) \quad \psi^*(\zeta) = \lim_{r\to 1} \psi(r\zeta),$$

then ψ^* belongs to $H^\infty(S) + C(S)$, and

$$\psi^* = \varphi^* + f^*,$$

where $f^* = f|S$. Let $I_0 = \{f \in C(\bar{B}): f^* \equiv 0\}$.

LEMMA 8. *The map $\gamma(\psi) = \psi^*$ is an algebra homeomorphism from $H^\infty(B) + C(\bar{B})$ onto $H^\infty(S) + C(S)$ with kernel I_0.*

Proof. It is obvious from (1) that γ is a homeomorphism into $H^\infty(S) + C(S)$. From the definition of $H^\infty(S)$ it follows that γ is onto, and from the definition of I_0 it follows that ker γ contains I_0. Let P be the Poisson–Szegö kernel on B,

$$P(z, \zeta) = \frac{(n-1)!}{2\pi^n} \frac{(1 - |z|^2)^n}{|1 - z \cdot \bar{\zeta}|^{2n}}, \quad z \in B \subset \mathbf{C}^n, \zeta \in S.$$

If $\varphi \in H^\infty(B)$ and $f \in C(\bar{B})$, then for $z \in B$

$$(2) \quad \varphi(z) = \int_S P(z, \zeta)\varphi^*(\zeta)d\sigma(\zeta),$$

and

$$(3) \quad g(z) = \int_S P(z, \zeta) f^*(\zeta)d\sigma(\zeta),$$

where g is a continuous function on $\bar{B}$ with $g^* = f^*$. Details can be found in [**7**; p. 19]. Any element ψ in $H^\infty(B) + C(\bar{B})$ can be written in the form $\psi = \varphi - f$, φ and f as above. If $\gamma(\psi) = 0$ then $\varphi^* = f^*$, a.e. on S. Thus the integrals in (2) and (3) are equal. It follows that φ is continuous on $\bar{B}$ and hence that $\psi \in C(\bar{B})$.

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The induced map

$$\gamma^*(m)(f) = m(\gamma(f)), \quad m \in \sum (H^\infty(S) + C(S)), f \in H^\infty(B) + C(B),$$

is a homeomorphism from $\sum (H^\infty(S) + C(S))$ onto the subset of $\sum (H^\infty(B) + C(\bar{B}))$ consisting of those functionals which vanish on I_0.

We saw in the proof of Theorem 3 that each element of $\sum (H^\infty(B) + C(\bar{B}))\backslash B_{\infty+c}$, when restricted to $C(\bar{B})$, is evaluation at some point λ on S. These elements therefore vanish on I_0. Since no element of $B_{\infty+c}$ vanishes identically on I_0, γ^* is a homeomorphism onto $\sum(H^\infty(B) + C(\bar{B}))\backslash B_{\infty+c}$. Hence the map $\tau \circ p_B \circ \gamma^*$, τ as defined in the preliminaries, is a homeomorphism from $\sum (H^\infty(S) + C(S))$ onto $\sum H^\infty(S)\backslash B_{\infty,s}$. Unravelling the definitions will show that $p_s = \tau \circ p_B \circ \gamma^*$.

6. Conclusion. Lemma 6 played a central part in the preceding work. We would like to end this paper with one further example of the close relationship between this result and $H^\infty + C$.

Let D be a relatively compact open set in $\mathbf{C}^n$ and let $\mathscr{C}^\infty(D)$ denote the collection of smooth functions on D.

PROPOSITION 9. *If Lemma* 6 *is valid on D, then*

$$\text{clos}\,(H^\infty(D) + C(\bar{D})) = \text{clos}\,\{\varphi \in \mathscr{C}^\infty(D)\colon \varphi,\ \bar{\partial}\varphi \textit{ bounded}\}.$$

the closure being taken with respect to the sup norm. *The space*

$$\text{clos}\,(H^\infty(D) + C(\bar{D})) \textit{ is an algebra.}$$

Proof. Suppose Lemma 6 is valid and that φ and $\bar{\partial}\varphi$ are bounded. We can find a $u \in \mathscr{C}^\infty(D) \cap C(\bar{D})$ such that $\bar{\partial}u = \bar{\partial}\varphi$. Therefore $\varphi - u \in H^\infty(D)$ and $\varphi = (\varphi - u) + u \in H^\infty(D) + C(\bar{D})$, and so

$$\{\varphi \in \mathscr{C}^\infty(D)\colon \varphi,\ \bar{\partial}\varphi \text{ bounded}\} \subset H^\infty(D) + C(\bar{D}).$$

This proves inclusion in Proposition 9 in one direction. The set $\{\varphi + f\colon \varphi \in H^\infty(D),\ f$ a polynomial in the real coordinate functions$\}$ is dense in $H^\infty(D) + C(\bar{D})$, by the Stone–Weierstrass Theorem, and is obviously contained in $\{\varphi \in \mathscr{C}^\infty(D)\colon \varphi,\ \bar{\partial}\varphi$ bounded$\}$. Therefore inclusion follows in the other direction. The set $\{\varphi \in \mathscr{C}^\infty(D)\colon \varphi,\ \bar{\partial}\varphi$ bounded$\}$ is obviously an algebra, and so its closure is as well. Thus clos $(H^\infty(D) + C(\bar{D}))$ is an algebra.

Remarks. It follows immediately from Proposition 10 that $H^\infty(B) + C(\bar{B})$ is an algebra. This in turn implies $H^\infty(S) + C(S)$ is an algebra as well. The proof of these facts in [**4**; pp. 105–111] does not use Lemma 6 explicitly and is much longer. Finally, Rudin shows in [**4**; pp. 104–105] that if φ in $H^\infty(U^n)$, U^n the unit polydisc, does not have a continuous extension to $\bar{U}^n$, then

$$z_n\varphi \notin H^\infty(H^n) + C(\bar{U}^n),$$

for $n > 1$. We conclude from Proposition 10 that Lemma 6 is not valid on U^n,

$n > 1$. In fact, since

$$\bar{\partial}(z_n\varphi) = \varphi d\bar{z}_n,$$

it is clear that there does not exist a $u \in \mathscr{C}^\infty(U^n) \cap C(\bar{U}^n)$ such that

$$\bar{\partial}u = \varphi d\bar{z}_n.$$

Of course, by Dolbeault's Lemma there is a solution in $\mathscr{C}^\infty(U^n)$.

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