MULTIPLIERS OF BANACH VALUED FUNCTION SPACES

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Abstract

Let A be a commutative Banach algebra with identity of norm 1, X a Banach A-module and G a locally compact abelian group with Haar measure. Then the multipliers from an A-valued function algebra into an X-valued function space is studied. We characterize the multiplier spaces as the following isometrically isomorphic relations under some appropriate conditions:

$$\text{Hom}_{L^1(G,A)}\big(L^1(G,A), L^p(G,X)\big) \cong L^p(G,X), \quad 1 < p < \infty;$$

$$\text{Hom}_{L^1(G,A)}\big(L^1(G,A), C_0(G,X)\big) \cong C_0(G,X);$$

$$\text{Hom}_{L^1(G,A)}\big(L^1(G,A), L^1(G,X)\big) \cong M(G,X).$$

1. Introduction and preliminaries

Let G be a locally compact abelian group with dual group $\hat{G}$. Suppose that A is a commutative Banach algebra and X a Banach space which is a Banach A-module, that is an A-module in the algebraic sense and satisfying

$$|ax|_X \leqslant |a|_A |x|_X \quad \text{for all } a \in A \text{ and } x \in X,$$

where $| \cdot |_E$ denotes the norm of a normed linear space E.

Throughout, an A-module means a Banach A-module.

If X is an A-module, then X^*, the topological dual of X, is an A-module under the adjoint action of A: $(ax^*)(x) = x^*(ax)$ for all $a \in A$, $x^* \in X^*$, $x \in X$.

If V and W are A-modules, then the A-module tensor product $V \otimes_A W$ is defined to be $(V \hat{\otimes}_\gamma W)/K$, where K is the closed linear subspace of the projective tensor product $V \hat{\otimes}_\gamma W$ generated by the elements of the form $av \otimes w - v \otimes aw$ for any $a \in A$, $v \in V$, $w \in W$. Rieffel [12] had shown that

(1) $\text{Hom}_A(V, W^*) \cong (V \otimes_A W)^*$

in which a linear functional ϕ on $V \otimes_A W$ corresponding to an operator $T \in \text{Hom}_A(V, W^*)$ is given by $(Tv)(w) = \phi(v \otimes w)$ for all $v \in V$, $w \in W$. Here $\cong$ means the isometric isomorphism between the two Banach spaces on each side. This T, in fact, satisfies

(2) $T(av) = aT(v)$ for all $a \in A$, $v \in V$

and is called an A-*module homomorphism* which is a continuous linear transform of V into W^*. We call this A-module homomorphism a *multiplier* (operator) from V to W^*. In particular if $V = A = W^*$, then the A-module homomorphism is the usual multiplier of the Banach algebra A, that is a bounded linear operator T on A such that

(3) $T(ab) = a \cdot Tb$ for all $a, b \in A$.

Some results have been established for the convolution $L^1(G)$-module spaces, for example

  (i) $\text{Hom}_G(L^1(G), L^1(G)) \cong M(G)$

where $M(G)$ denotes the bounded regular measure space on G and Hom_G means $\text{Hom}_{L^1(G)}$, and

  (ii) $\text{Hom}_G(L^1(G), L^p(G)) \cong L^p(G) \cong (L^1(G) \otimes_G L^q(G))^*$

for $1 < p < \infty$, $1/p + 1/q = 1$ where $\otimes_G$ means $\otimes_{L^1(G)}$. Moreover

  (iii) $\text{Hom}_G(L^p(G), L^p(G)) \cong (L^p(G) \otimes_G L^q(G))^* \cong A_p(G)^*$

for $1 < p < \infty$, $1/p + 1/q = 1$ where $A_p(G)$ is a Banach algebra generated by

$$\left\{ u = \sum_{i=1}^\infty f_i * g_i; f_i \in L^p, g_i \in L^q, \sum_{i=1}^\infty \|f_i\|_p \|g_i\|_q < \infty \right\}$$

under pointwise product and the norm given by

$$\|u\| = \inf\left\{ \sum_{i=1}^\infty \|f_i\|_p \|g_i\|_q; u = \sum f_i * g_i \in A_p(G) \right\}$$

etc. (cf. Larsen [10], Lai and Chen [9]).

In this paper we characterize the multipliers of vector valued function spaces. This is closely related to the Banach space valued vector measure (cf. [1, 2, 3, 14]). Sometimes a space of multipliers is characterized as the dual of tensor products of the form (1). We shall characterize the dual of tensor products spaces as the spaces of vector valued functions on a given locally compact abelian group. The

concept of multipliers of a vector valued function space is somewhat different from the scalar case, and so we have to be careful in the characterization of multipliers of vector-valued case. Recently Tewari, Dutta and Vaidya [13] studied the multipliers of group algebras of vector valued functions and proved that

$$\operatorname{Hom}_G\big(L^1(G), L^1(G, X)\big) \cong M(G, X)$$

and

$$\operatorname{Hom}_{L^1(G,A)}\big(L^1(G, A), L^1(G, A)\big) \cong M(G, A)$$

where $M(G, X)$ and $M(G, A)$ denote respectively the X-valued and A-valued bounded measure spaces. In this paper we treat more generally, and prove that

(4) $$\operatorname{Hom}_{L^1(G,A)}\big(L^1(G, A), L^1(G, X)\big) \cong M(G, X)$$

and if X^* as well as X^{**} have the Radon Nikodym property then

(5) $$\operatorname{Hom}_{L^1(G, A)}\big(L^1(G, A), L^p(G, X)\big) \cong L^p(G, X)$$

for $1 < p < \infty$, when A is a commutative Banach algebra with identity of norm 1 and X is an A-module. Some known results can be deduced from these results as corollaries, for example, one can consult Tewari, Dutta and Vaidya [13, Theorems 1 and 4].

2. $L^1(G, A)$-module spaces

In the general theory, a vector measure continuous with respect to a scalar measure will have a density of a vector-valued (Bochner) integrable function. It is discussed in Thomas [14], Dinculeanu [1, 2], Diestel and Uhl [3]. In order that a vector-valued function is a density of a vector measure with respect to a positive scalar measure, it is essential that the vector function must be approximable by a finite sum of simple functions in a linear subspace of a given topological linear space. In this paper we take a locally compact abelian Hausdorff group G with Haar measure as the special case of a locally compact Hausdorff space with a positive Radon measure, and consider the Bochner integrable functions defined from G to X or to A. We assume that X is a Banach A-module, in which the A-valued functions act on the X-valued functions. This is well defined as the X-valued functions. Some elementary results of A-valued Bochner integral on G was obtained by Johnson [6]. Based on Dinculeanu [1, 2], Thomas [14] and Johnson [6], the spaces $L^1(G, A)$, $M(G, A)$, $M(G, X)$, $L^p(G, X)$, $1 \leqslant p \leqslant \infty$, are defined in the usual sense. The space $L^1(G, A)$ of all A-valued Bochner integrable

functions on G is a commutative Banach algebra under the convolution formula given by

$$f * g(t) = \int_G f(ts^{-1}) g(s) \, ds = \int_G f(s) g(ts^{-1}) \, ds$$

for $f, g \in L^1(G, A)$ and the norm given by

$$\|f\|_1 = \int_G |f(g)|_A \, dt \quad \text{for } f \in L^1(G, A).$$

Here $f * g$ is approximable by a finite sum of simple functions (see Johnson [6], Theorem 1.1).

We assume throughout that A is a commutative Banach algebra with identity of norm 1. The approximate identity of $L^1(G, A)$ is given in the following sense.

DEFINITION. A directed system $\{v_\alpha\}$ of functions in $L^1(G, A)$ is called an *approximate identity* if

$$\lim_\alpha \|v_\alpha * f - f\|_1 = 0 \quad \text{for every } f \in L^1(G, A).$$

Note that this definition of approximate identity is different from the definition of Johnson [6, page 414] but is in the usual Banach algebra sense.

It is known that if two Banach algebras A and B have bounded approximate identities $\{a_\alpha\}$ and $\{b_\beta\}$ then $A \hat{\otimes}_\gamma B$ has a bounded approximate identity $\{a_\alpha \otimes b_\beta\}_{(\alpha, \beta)}$ where $\hat{\otimes}_\gamma$ denotes the completion of the usual tensor product of Banach spaces with respect to the projective tensor norm γ. Since $L^1(G, A) = L^1(G) \hat{\otimes}_\gamma A$, the following fact is easy to see.

LEMMA 1. *The algebra $L^1(G, A)$ has an approximate identity $\{v_\alpha\}$ in $L^1(G, A)$ with $\|v_\alpha\|_1 = 1$.*

Evidently, the Banach algebra $L^1(G, A)$ and the Banach space $L^1(G, X)$ are $L^1(G)$ modules under convolution. Furthermore, it can be shown that $L^p(G, X)$, $1 \leqslant p \leqslant \infty$ is an $L^1(G, A)$-module under convolution. Here the space $L^p(G, X)$ is the set of all measurable functions $f: G \to X$ such that $|f(t)|_X^p$ is integrable for $1 \leqslant p < \infty$, that is, $|f(t)|_X^p \in L^1(G)$. The norm of a function f in $L^p(G, X)$ is given by

$$\|f\|_{pX} = \left(\int_G |f(t)|_X^p \, dt \right)^{1/p}, \qquad 1 \leqslant p < \infty,$$

and $f = g$ if $f(t) = g(t)$ in X almost everywhere $t \in G$. It follows that $L^p(G, X)$ is a Banach space for $1 \leqslant p < \infty$. If $X = A$, we write $\|f\|_p$ instead of $\|f\|_{pA}$. We

denote by $L^\infty(G, X)$ the set of all measurable X-valued functions f with norm given by

$$\|f\|_{\infty X} = \operatorname*{ess\,sup}_{t \in G} |f(t)|_X,$$

that is, the least upper bound of $|f(t)|_X$ outside the negligible set. We denote by $C_0(G, X)$ the space of all X-valued continuous functions vanishing at infinity of G, and supply the norm

$$\|f\|_{\infty X} = \sup_{t \in G} |f(t)|_X \quad \text{for every } f \in C_0(G, X).$$

Then $C_0(G, X)$ is a Banach space whose dual is denoted by $M(G, X^*)$ in usual form (cf. [1, 13]), where X^* is the Banach dual of X.

For $1 < p < \infty$, $1/p + 1/q = 1$, the dual space of $L^p(G, X)$ is isometrically isomorphic to the space $L^q(G, X^*)$ if and only if X^* has the Radon Nikodym property in the wide sense (see Lai [7]). Throughout we assume that the dual space X^* has the Radon Nikodym property. It follows that if $f \in L^p(G, X)$ and $g \in L^q(G, X^*)$ then $\langle f(\cdot), g(\cdot) \rangle \in L^1(G)$ and by the Hölder inequality, it would have

$$\int_G |\langle f(t), g(t) \rangle| \, dt \leq \|f\|_{p X} \|g\|_{q X^*}.$$

Analogous to the scalar function case, we can easily obtain the following

THEOREM 2. *Let* $1 < p < \infty$, $1/p + 1/q = 1$, $f \in L^p(G, A)$ *and* $g \in L^q(G, A)$. *Then* $f * g \in C_0(G, A)$ *which is defined by*

$$f * g(t) = \int_G f(ts^{-1}) g(s) \, ds,$$

and so

$$\|f * g\|_\infty \leq \|f\|_p \|g\|_q.$$

PROOF. The convolution is approximate by a finite sum of A multiples of translates of f which can be shown, *mutatis mutandis*, as in the proof of Johnson [6, Theorem 1.1]. The inequality and norm continuity for $f * g$ follow from the Hölder inequality, while the convolution $f * g$ vanishing at the point of infinity is not difficult to show. For $f \in L^p(G, A)$ and $g \in L^q(G, A)$, there exist compact subsets K_f and K_g such that $(\int_{K_f'} |f(t)|_A^p \, dt)^{1/p}$ and $(\int_{K_g'} |g(t)|_A^q \, dt)^{1/q}$ are small where K' denotes the complement set of K and so there exist continuous functions f_c and g_c with compact support in K_f and K_g respectively such that $\|f - f_c\|_p$ and $\|f - g_c\|_q$ are small. It follows that $f_c * g_c$ vanishes outside the

compact set $K_g K_f$ and

$$|f * g(t) - f_c * g_c(t)|_A = |f * g(t) - g_c * g(t) + f_c * g(t) - f_c * g_c(t)|_A$$
$$\leqslant \|f - f_c\|_p \|g\|_q + \|g - g_c\|_q \|f_c\|_p$$

is small for all t. Hence $f * g(t)$ is approximately by $f_c * g_c(t)$ so that $|f * g(t)|_A$ is small enough if t is outside the compact set $K_f K_g$. This proves $f * g \in C_0(G, A)$.

The convolution of $f \in L^1(G, A)$ and $g \in L^p(G, X)$, $1 \leqslant p < \infty$, can be defined by

$$f * g(t) = \int_G f(ts^{-1}) g(s) \, ds.$$

This $f * g$ is a measurable X-valued function such that $|f * g|_X^p \in L^1(G)$. In fact, $f * g$ is approximable by a finite sum of translates of f acting on X in $L^p(G, X)$. This will imply that the convolution is a density of a vector measure with respect to Haar measure (cf. Thomas [14]). The following theorem accounts essentially for the convolution approximation. First we establish

PROPOSITION 3. *Let X^* have the Radon Nikodym property in the wide sense. Then the Banach space $L^p(G, X)$ is an essential $L^1(G, A)$-module under convolution such that for $f \in L^1(G, A)$ and $g \in L^p(G, X)$, $1 \leqslant p < \infty$, we have*

(6) $$\|f * g\|_{pX} \leqslant \|f\|_1 \|g\|_{pX}.$$

PROOF. Evidently the convolution $f * g(t)$ defines an element of X if X is an A-module. Since $L^1(G, A)$ has a bounded approximate identity, $L^p(G, X)$ is an essential $L^1(G, A)$-module (cf. Hewitt [5]). The proof of inequality (6) parallels the classical case since $L^p(G, X)^* \equiv L^q(G, X^*)$, $1 \leqslant p < \infty$, as X^* has the Radon-Nikodym property in the wide sense (cf. Lai [7]).

THEOREM 4. *If $f \in L^1(G, A)$, $g \in L^p(G, X)$, then for any $\varepsilon > 0$, there exists a finite sum k of translates of f acting on X such that*

(7) $$\|f * g - k\|_{pX} < \varepsilon.$$

PROOF. It is, *mutatis mutandis*, as in the proof of Johnson [6], that if $f \in C_c(G, A)$ and $g \in C_c(G, X)$, where $C_c(G, \cdot)$ denotes the space of continuous functions with compact support in G, then f and g have respectively the compact support K_f and K_g, thus $f * g$ is a continuous function of $G \rightarrow X$ with compact support $K = K_f K_g$. It can be shown that any $h \in C_c(G, X)$ with compact support K can be approximated by a finite sum of X-valued step functions, then, since C_c is dense in L^p, one concludes that this theorem holds.

The following corollary is obvious (cf. Hewitt [5]).

COROLLARY 5. $C_0(G, X)$ is an essential $L^1(G, A)$-module under convolution, and

$$C_0(G) \,\hat{\otimes}_\lambda\, X \cong C_0(G, X) \cong L^1(G, A) \otimes_{L^1(G,A)} C_0(G, X)$$
$$\cong L^1(G, A) * C_0(G, X)$$

where $\hat{\otimes}_\lambda$ denotes the completion of the space of tensor product in the smallest cross norm λ.

3. Multipliers of $L^1(G, A)$ to $L^p(G, X)$, $1 < p < \infty$

We have seen that $L^p(G, X)$ is an essential $L^1(G, A)$-module under convolution. A bounded linear operator T in $\mathscr{L}(L^1(G, A), L^p(G, X))$ is called a *multiplier* if $T(f * g) = f * Tg$ for all $f, g \in L^1(G, A)$. If T is a multiplier in this sense, it is invariant, that is, T commutes with translation. In the case of scalar function space, a bounded linear operator on a convolution algebra is a multiplier if and only if it is invariant. But in a vector valued convolution algebra, a bounded linear invariant operator need not commute with convolution, that is, an invariant operator is not a multiplier (see [13, Theorem 3]). In this section we shall characterize the multipliers with ranges not necessary in the same Banach algebra but in a Banach space of vector valued functions. We assume throughout that the Banach space X is an A-module where A is a commutative Banach algebra with identity of norm 1.

In view of (1), if we embed the space X into its second dual space X^{**}, then $L^p(G, X)$ is embedded in $L^p(G, X^{**})$. Evidently, $L^p(G, X^{**}) \subset (L^q(G, X^*))^*$, and it follows that

$$\mathrm{Hom}_{L^1(G, A)}\big(L^1(G, A), L^p(G, X)\big)$$
$$\subset \mathrm{Hom}_{L^1(G, A)}\big(L^1(G, A), L^p(G, X^{**})\big)$$
$$\subset \mathrm{Hom}_{L^1(G, A)}\big(L^1(G, A), (L^q(G, X^*))^*\big)$$
$$\cong \big(L^1(G, A) \otimes_{L^1(G,A)} L^q(G, X^*)\big)^*$$

for $1 < p < \infty$, $1/p + 1/q = 1$. For our purpose, we require that $L^p(G, X)^{**} \cong L^p(G, X^{**})$ and thus we ask whether such an equality holds. Fortunately, if X^* and X^{**} have the Radon Nikodym property, then we have $L^p(G, X)^* \cong L^q(G, X^*)$ and $L^q(G, X^*)^* \cong L^p(G, X^{**})$ (see Lai [7]), so that $L^p(G, X)^{**} \cong L^p(G, X^{**})$.

With the above preparation, we state one of our main results as follows.

THEOREM 6. *Let A be a commutative Banach algebra with identity of norm 1 and X an A-module Banach space. Suppose that the dual space X^* and the second dual space X^{**} have the Radon Nikodym property in the wide sense with respect to a locally compact abelian group G. Then for $1 < p < \infty$, the following two statements are equivalent*:

(i) $T \in \mathrm{Hom}_{L^1(G, A)}(L^1(G, A), L^p(G, X))$;

(ii) *there exists a unique $h \in L^p(G, X)$ such that $Tf = f * h$ for any $f \in L^1(G, A)$.*

Moreover,

$$(8) \qquad \mathrm{Hom}_{L^1(G, A)}(L^1(G, A), L^p(G, X)) \cong L^p(G, X).$$

PROOF. (ii) $\Rightarrow$ (i). For each $h \in L^p(G, X)$, we define $f \to Tf = f * h$ for all $f \in L^1(G, A)$. Since $L^p(G, X)$ is an $L^1(G, A)$-module under convolution, T defines a bounded linear operator of $L^1(G, A)$ into $L^p(G, X)$ and $\|Tf\|_{pX} = \|f * h\|_{pX} \le \|f\|_1 \|h\|_{pX}$ so that $\|T\| \le \|h\|_{pX}$. Moreover, for f, g in $L^1(G, A)$,

$$T(f * g) = (f * g) * h = f * Tg.$$

Hence $T \in \mathrm{Hom}_{L^1(G, A)}(L^1(G, A), L^p(G, X))$.

(i) $\Rightarrow$ (ii). By Lemma 1, we see that $L^1(G, A)$ has an approximate identity $\{v_\alpha\}$ of norm 1 provided that the approximate identity of $L^1(G)$ is of norm 1. Then for any $f \in L^1(G, A)$,

$$\|Tf - Tv_\alpha * f\|_{pX} = \|Tf - T(V * f)\|_{pX} \le \|T\| \|f - V * f\|_1 \to 0$$

and $\|Tv_\alpha\|_{pX} \le \|T\| \|v_\alpha\|_1 = \|T\|$. It follows from Alaoglu's Theorem that the bounded subset $\{Tv_\alpha\}$ of $L^p(G, X)$ contains a subnet $\{Tv_\beta\}$ which is weak*-convergent to h. Since X^* and X^{**} have assumed to have the Radon Nikodym property, thus $L^p(G, X)^* \cong L^q(G, X^*)$ and $L^q(G, X^*)^* \cong L^p(G, X^{**})$ for $1 < p < \infty$ and $1/p + 1/q = 1$ (see Lai [7]). Thus for any $k \in L^q(G, X^*)$, $\lim\langle Tv_\beta, k\rangle = \langle h, k\rangle$. It follows that for any $f \in L^1(G, A)$ and $g \in L^q(G, X^*)$,

$$\langle Tf, g\rangle = \lim_\beta \langle Tv_\beta * f, g\rangle = \lim_\beta \langle Tv_\beta, \tilde{f} * g\rangle$$

since $L^q(G, X^*)$ is also an $L^1(G, A)$-module when $L^p(G, X)$ is an $L^1(G, A)$-module, thus $\tilde{f} * g \in L^q(G, X^*)$ and so the last formula is equal to $\langle h, \tilde{f} * g\rangle = \langle f * h, g\rangle$ for all $g \in L^q(G, X^*)$. Hence $Tf = f * h \in L^p(G, X)$ for all $f \in L^1(G, A)$. Here the weak *-limit h of Tv_β is in $L^q(G, X^*)^* \cong L^p(G, X^{**}) \cong L^p(G, X)^{**}$. It remains to show that h belongs to $L^p(G, X)$.

To this end we need the following

LEMMA. *Let X be an A-module. Then X^{**} is also an A-module so that $Y = X^{**}/X$ is an A-module.*

PROOF. Since X is an A-module, it follows that X^* is an A-module under the adjoint action of A: $(ax^*)(x) = x^*(ax)$ for all $a \in A$, $X^* \in X^*$ and $x \in X$. Thus

X^{**} is also an A-module defined by $(ax^{**})(x^*) = x^{**}(ax^*)$ for $x^{**} \in X^{**}$, $x^* \in X^*$ and $a \in A$. Therefore for any $y \in X^{**}/X$, there is $x^{**} \in X^{**}$ such that $y = [x^{**}] = x^{**} + X$ and $ay = ax^{**} + aX = [ax^{**}] \in Y$ for all $x \in X$. Hence Y is an A-module.

To continue the proof of this theorem, we consider a canonical mapping ψ: $X^{**} \to X^{**}/X = Y$. Then for the weak $*$-limit $h \in L^p(G, X^{**})$, we have $\psi \circ h = g \in L^p(G, Y)$. Now for all $t \in G$ and $f \in L^1(G, A)$,

$$(9) \qquad f * g(t) = \int_G f(ts^{-1}) g(s)\, ds = \psi \circ \int_G f(ts^{-1}) h(s)\, ds$$

$$= \psi \circ (f * h)(t) = \psi \circ Tf(t) = 0$$

since, by definition of T, $Tf \in L^p(G, X)$. Thus for any $k \in L^q(G, Y^*)$, there exist $k' \in L^q(G, Y^*)$ and $f \in L^1(G, A)$ such that $k = \tilde{f} * k'$ since from the Lemma, $L^q(G, Y^*)$ is an essential $L^1(G, A)$-module, and $\langle g, k \rangle = \langle g, \tilde{f} * k \rangle = \langle f * g, k' \rangle = 0$ by (9), where $\tilde{f}(t) = f(t^{-1})$. This implies $g = 0$, that is, $g(t) = \psi \circ h(t) = 0$ for all $t \in G$. This means that $h(t)$ takes values in X only, and hence $h \in L^p(G, X)$.

Finally, we show that h is uniquely determined in $L^p(G, X)$. Indeed, if $f * h = 0$ for all $f \in L^1(G, A)$ then $\langle f * h, g \rangle = 0$ for all $g \in L^q(G, X^*)$. This implies that $\langle h, f * g \rangle = 0$. But $L^q(G, X^*)$ is an essential Banach $L^1(G, A)$-module since $L^p(G, X)$ is an essential $L^1(G, A)$-module (see Hewitt [5]); it then follows that $h = 0$.

Moreover, $\|Tf\|_{pX} = \|f * h\|_{pX} \leqslant \|f\|_1 \|h\|_{pX}$ implies $\|T\| \leqslant \|h\|_{pX}$. On the other hand, $\|T\| = \sup_{\|f\|_1 \leqslant 1} \|Tf\|_{pX} = \sup_{\|f\|_1 \leqslant 1} \|f * h\|_{pX}$, and if we take $f = v_\alpha$, the approximate identity of $L^1(G, A)$, then $\|v_\alpha * h\|_{pX} \to \|h\|_{pX}$. Since $\{v_\alpha\}$ is an approximate identity of norm 1, for any $\varepsilon > 0$, there exists an α_0 such that $\|v_\alpha * h\|_{pX} > \|h\|_{pX} - \varepsilon$ whenever $\alpha > \alpha_0$. This implies that $\|T\| \geqslant \|h\|_{pX}$ since ε is arbitrary. Hence $\|T\| = \|h\|_{pX}$, and (8) is proved.

REMARK. In this paper, we always assume that A has identity of norm 1, because if the group G contains only one identity e and $X = A$, then (8) becomes $M(A) = A$ where $M(A)$ consists of the multipliers of the algebra A. But this equality holds if and only if A has identity.

The following corollary is obvious.

COROLLARY 7. *Under the same assumptions as in Theorem 6, for $1 < p < \infty$, $1/p + 1/q = 1$, we have*

(i) $\mathrm{Hom}_G(L^1(G), L^p(G, X)) \cong L^p(G, X)$,

(ii) $\mathrm{Hom}_G(L^1(G, A), L^p(G, X^*)) \cong (L^1(G, A) \otimes_G L^q(G, X))^*$,

(iii) $\mathrm{Hom}_{L^1(G, A)}(L^1(G, A), L^p(G, X^*)) \cong (L^1(G, A) \otimes_{L^1(G, A)} L^q(G, X))^*$

$$\cong L^p(G, X^*).$$

4. Multipliers of $L^1(G, A)$ to $C_0(G, X)$ and $L^1(G, X)$

In the theory of vector measures and integration, we can identity $C_0(G, X)^* \cong M(G, X^*)$ (see Section 19 of [1] and [2], cf. also [13, Section 2]). However, $M(G, X^{**})$ is isometrically isomorphic to $C_0(G, X^*)^*$. Hence in the following theorems, we need not assume that X^* and X^{**} have the Radon Nikodym property. As Corollary 5 has shown that $C_0(G, X)$ is an essential $L^1(G, A)$-module, we can establish the following

THEOREM 8. *Let A be a commutative Banach algebra with identity. If X is an A-module, then*

$$\mathrm{Hom}_{L^1(G, A)}\big(L^1(G, A), C_0(G, X)\big) \cong C_0(G, X)$$

under the correspondence of $T \in \mathrm{Hom}_{L^1(G,A)}(L^1(G, A), C_0(G, X))$ and $g \in C_0(G, X)$ defined by $Tf = f * g$ for all $f \in L^1(G, A)$.

PROOF. The proof is quite similar to that of Theorem 6, so we shall only give a brief sketch.

By Corollary 5, we see that $C_0(G, X)$ is an essential $L^1(G, A)$-module under convolution. For each $g \in C_0(G, A)$, define $Tf = f * g$, for all $f \in L^1(G, A)$; then T is a bounded linear operator from $L^1(G, A)$ to $C_0(G, X)$ which commutes with convolution, so that T is an $L^1(G, A)$-module homomorphism.

Conversely, suppose that $T \in \mathrm{Hom}_{L^1(G,A)} L^1(G, A), C_0(G, X)$. Using the same argument, *mutatis mutandis*, as in the proof of Theorem 6 with $C_0(G, X)$ instead of $L^p(G, X)$ and $L^p(G, X)^*$, $L^p(G, X)^{**}$ replaced by $M(G, X^*)$, $M(G, X^{**})$ respectively, we obtain that there exists an $h \in C_0(G, X)$ such that $Tf = f * h$ for all $f \in L^1(G, A)$. The isometry between T and h is easily established.

The following theorem is also essential in this paper.

THEOREM 9. *Let A be a commutative Banach algebra with an identity. Then the following two statements are equivalent*:
  (i) $T \in \mathrm{Hom}_{L^1(G, A)}(L^1(G, A), L^1(G, X))$;
  (ii) *there exists a unique X-valued vector measure $\mu \in M(G, X)$ such that*

$$Tf = f * \mu \quad \text{for all } f \in L^1(G, A).$$

Moreover,

$$\mathrm{Hom}_{L^1(G, A)}\big(L^1(G, A), L^1(G, X)\big) \cong M(G, X).$$

PROOF. For $f \in L^1(G, A)$ and $\mu \in M(G, X)$ the convolution

$$f * \mu(t) = \int_G f(ts^{-1}) \, d\mu(s)$$

defines an element of X. And it is easy to see that $f * \mu$ is an X-valued Bochner integrable function on G. In fact, by Lebesgue decomposition, $M(G, X)$ can be written as a direct sum of an absolutely continuous part and a singular part with respect to Haar measure, and if $L^1(G, A)$ acts on $M(G, X)$ under convolution then it vanishes on the singular part of $M(G, X)$. It follows that $f * \mu$ is an element of $L^1(G, X)$, that is, there is a function $g \in L^1(G, X)$ such that $f * \mu = f * g$ almost everywhere. Hence if X is an A-module, each $\mu \in M(G, X)$ defines an $L^1(G, A)$-homomorphism T from $L^1(G, A)$ to $L^1(G, X)$ which is given by $Tf = f * \mu$ for all $f \in L^1(G, A)$. The equivalence of (i) and (ii) is a straightforward and obvious generalization of the result of Tewari, Dutta and Vaidya [13]; we omit the details.

If $X = A$, Theorem 9 reduces to

COROLLARY 10 (Tewari, Dutta and Vaidya [13, Theorem 4]).

$$\text{Hom}_{L^1(G, A)}\big(L^1(G, A), L^1(G, A)\big) \cong M(G, A).$$

If we take $A = \mathbb{C}$ in Theorem 9, we obtain Theorem 1 of [13] as follows.

COROLLARY 11. $\text{Hom}_G(L^1(G), L^1(G, X)) \cong M(G, X)$.

It is remarkable that a Banach module in pointwise product and convolution product are different. For example, in the scalar function case, $C_0(G)$ is a Banach algebra with bounded approximate identity under pointwise product and supremum norm. Thus (cf. Hewitt [5]) each $L^p(G)$, $1 \leqslant p \leqslant \infty$, is a $C_0(G)$-module under pointwise multiplication and $C_0(G) \cdot L^p(G) = L^p(G)$. But $L^p(G)$ and $C_0(G)$ are $L^1(G)$-module under convolution, and so $L^1(G) * C_0(G) = C_0(G)$ and $L^1(G) * L^p(G) = L^p(G)$, $1 \leqslant p < \infty$. Therefore one can characterize the multipliers of $C_0(G)$ to be

$$\text{Hom}_{L^1(G)}(C_0(G), C_0(G)) \cong M(G)$$

(see Larsen [10]) if a multiplier of $C_0(G)$ is defined to be a bounded linear operator commuting with translation. But if we regard $C_0(G)$ as a Banach algebra in pointwise product, and the multiplier of $C_0(G)$ is defined to be a bounded linear operator commuting with the algebra product, then the multipliers of $C_0(G)$ are the space $C^b(G)$ of bounded continuous functions on G. In the $C_0(G)$-module homomorphisms, it can be characterized as

$$\text{Hom}_{L^1(G, A)}(C_0(G), C_0(G)) \cong C^b(G).$$

(cf. Lai [8, page 451]).

There are two open problems, namely, under what conditions can one characterize the multipliers in the form

$$\operatorname{Hom}_{L^1(G,\,A)}\big(C_0(G,\,X),\,C_0(G,\,X)\big) = ?$$

and

$$\operatorname{Hom}_{C_0(G,\,A)}\big(C_0(G,\,X),\,C_0(G,\,X)\big) = ?$$

where X is always a Banach A-module.

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