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A CROSSED-PRODUCT APPROACH TO THE CUNTZ–LI ALGEBRAS

S. KALISZEWSKI[1], MAGNUS B. LANDSTAD[2] AND JOHN QUIGG[1]

[1] *School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287* (quigg@asu.edu; kaliszewski@asu.edu)

[2] *Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway* (magnusla@math.ntnu.no)

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Abstract Cuntz and Li have defined a C^*-algebra associated to any integral domain, using generators and relations, and proved that it is simple and purely infinite and that it is stably isomorphic to a crossed product of a commutative C^*-algebra. We give an approach to a class of C^*-algebras containing those studied by Cuntz and Li, using the general theory of C^*-dynamical systems associated to certain semidirect product groups. Even for the special case of the Cuntz–Li algebras, our development is new.

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1. Introduction

For an integral domain R, in [**8**] Cuntz and Li define a remarkable C^*-algebra $\mathfrak{A}[R]$ via generators and relations. Using only the relations, they show that $\mathfrak{A}[R]$ is simple and purely infinite. They then show that $\mathfrak{A}[R]$ is isomorphic to a corner of the crossed product of a commutative C^*-algebra by an action of the $ax + b$ group of the quotient field $Q(R)$.

Our purpose here is to give an alternative approach to $\mathfrak{A}[R]$, essentially the reverse of that in [**8**]: we *begin* with a crossed product, which we show is simple and purely infinite using the theory of C^*-dynamical systems. Then we show that a certain corner of this crossed product is generated by elements satisfying relations similar to those of Cuntz and Li, and that this corner C^*-algebra is universal for these relations. It follows that this corner is in fact isomorphic to the algebra $\mathfrak{A}[R]$ of [**8**]. Of course, this isomorphism is not surprising, since $\mathfrak{A}[R]$ is simple by the results of [**8**]. The point is, though, that we do not *assume* the simplicity result of [**8**]; rather, we *deduce* it as a consequence of our results. The embedding of the Bost–Connes algebra in $\mathfrak{A}[R]$ is now easy to explain, since it is a corner of a subalgebra of the crossed product obtained by cutting down with the same projection.

Other approaches to the Cuntz–Li algebra are given in [**2**, **6**, **9**, **12**, **21**–**23**, **26**].

We actually do most of our work in a somewhat more general context than [**8**]: we use a semidirect product that incorporates the essential features of the $ax + b$ group of the integral domain R, but there is no ring. More precisely, we have a semidirect product $G = N \rtimes H$ satisfying certain mild conditions regarding a certain normal subgroup M of N. We make heavy use of the completion $\bar{N}$ relative to the subgroup topology generated by the conjugates of M.

In § 3 we use standard C^*-crossed-product theory, specifically results of Archbold, Laca and Spielberg [**1**, **19**], to prove that if D denotes the C^*-algebra generated by the characteristic functions of the cosets in N of the conjugates of M, then $D = C_0(\bar{N})$ and the reduced crossed product $D \rtimes_r G$ is simple and purely infinite.

In § 4 we show that the full crossed product $D \rtimes G$ is the enveloping C^*-algebra of an algebraic crossed product $D_0 \rtimes G$, which in turn is universal for appropriate covariant representations of (D_0, G). Assuming that the action of G on D is 'regular' in the sense that $D \rtimes G = D \rtimes_r G$, we prove that $D_0 \rtimes G$ has a unique C^*-norm, and consequently the corner $p(D \rtimes G)p$ is the enveloping C^*-algebra of $p(D_0 \rtimes G)p$, where p is the characteristic function of M.

In § 5 we show that M is a Hecke subgroup of G and the associated Hecke algebra $\mathcal{M}$ has a universal C^*-algebra $C^*(\mathcal{M})$, which in turn embeds as a full corner of $C^*(\bar{G})$ (where $\bar{G}$ is the completion relative to the subgroup topology). When N is appropriately self-dual, we conclude that $C^*(\mathcal{M})$ embeds faithfully in the corner $p(D \rtimes G)p$.

In § 6 we specialize to the case where G is the $ax+b$ group of the field of fractions of an integral domain R. In this case G is amenable, so the full crossed product $D \rtimes G \cong D \rtimes_r G$ is simple and purely infinite.

In § 7 we specialize even further, as Cuntz and Li do, so that R is the ring of integers of an algebraic number field $K = \mathbb{Q}(\theta)$. If $R = \mathbb{Z}[\theta]$, then $C^*(\mathcal{M})$ is the generalized Bost–Connes C^*-algebra. Anticipating the isomorphism $p(D \rtimes G)p \cong \mathfrak{A}[R]$ of § 10, the results of § 5 allow us to recover the embedding of Cuntz and Li of the generalized Bost–Connes algebra into $\mathfrak{A}[R]$.

In § 8 we give an example in a different direction from the Cuntz–Li work, namely with G a lamplighter group. Cuntz and Li define their algebras using generators and relations. In § 9 we show that the corners $p(D \rtimes G)p$ and $p(D \rtimes H)p$ are universal C^*-algebras for similar generators and relations. We use standard dilation techniques as presented by Douglas [**11**]. In § 10 we use the universal property of § 9 to see that the corner $p(D \rtimes G)p$ is isomorphic to the ring C^*-algebra $\mathfrak{A}[R]$ of Cuntz and Li. We thus recover their result that $\mathfrak{A}[R]$ is simple and purely infinite.

2. Preliminaries

Throughout this paper we let $G = N \rtimes H$ be a (discrete) semidirect product group with normal subgroup N and quotient group H, and we let $M \neq \{e\}$ be a normal subgroup of N. We assume that the family

$$\mathcal{U} := \{aMa^{-1} : a \in H\}$$

of normal subgroups of N is a *filterbase* in the sense that

$$\text{for all } U, V \in \mathcal{U} \quad \text{there exists } W \in \mathcal{U} \text{ such that } W \subseteq U \cap V, \tag{2.1}$$

and that $\mathcal{U}$ is *separating* in the sense that

$$\bigcap_{U \in \mathcal{U}} U = \{e\}. \tag{2.2}$$

Because we require $M \neq \{e\}$, condition (2.2) implies that $M \neq N$ and $H \neq \{e\}$.

We shall at times also explicitly make one or both of the following assumptions: that $\mathcal{U}$ has *finite quotients* in the sense that

$$|U/V| < \infty \quad \text{whenever } V \subseteq U \in \mathcal{U}; \tag{2.3}$$

and that H acts *effectively* on M, so that

$$\text{for each } a \neq e \in H \quad \text{there exists } s \in M \text{ such that } asa^{-1} \neq s. \tag{2.4}$$

The following results are essentially [**4**, Chapter III, § 7.3, Corollary 2].

Lemma 2.1. *Let N be as above with the separating filter base $\mathcal{U}$ of normal subgroups. Then the inverse limit*

$$\varprojlim_{U \in \mathcal{U}} N/U$$

is a Hausdorff completion of N.

Lemma 2.2. *For each $U \in \mathcal{U}$, the map $xU \mapsto x\bar{U}$ gives a group isomorphism of N/U onto $\bar{N}/\bar{U}$.*

Lemma 2.3. *The completion $\bar{N}$ is locally compact if and only if there exists $U \in \mathcal{U}$ such that*

$$|U/V| < \infty \quad \textit{for all } V \in \mathcal{U} \textit{ with } V \subseteq U.$$

Proof. First assume that we have $U \in \mathcal{U}$ such that $|U/V| < \infty$ whenever $V \in \mathcal{U}$ and $V \subseteq U$, and set

$$\mathcal{U}' = \{V \in \mathcal{U} \colon V \subseteq U\}.$$

Since $\mathcal{U}'$ is cofinal in $\mathcal{U}$, we have a natural isomorphism

$$\bar{N} \cong \varprojlim_{V \in \mathcal{U}'} N/V.$$

The closure $\bar{U}$ is a compact (and open) subgroup of the completion $\bar{N}$, because it is an inverse limit of the finite groups $\{U/V \colon V \in \mathcal{U}'\}$. It follows that $\bar{N}$ is locally compact, because it contains a compact open subgroup.

Conversely, assume that $\bar{N}$ is locally compact, and choose a compact neighbourhood W of e. Since $\{\bar{U} \colon U \in \mathcal{U}\}$ is a local base of $\bar{N}$, there exists $U \in \mathcal{U}$ such that $\bar{U} \subseteq W$. Thus, $\bar{U}$ is compact. Let $V \in \mathcal{U}$ with $V \subseteq U$. Then $\bar{V}$ is an open normal subgroup of $\bar{U}$, so the quotient group $\bar{U}/\bar{V}$ is compact and discrete, and hence is finite. Since $U/V \cong \bar{U}/\bar{V}$ by Lemma 2.2, we have $|U/V| < \infty$. □

3. The $ax+b$ group action

We define the following '$ax+b$'-type action β of G on N: for $g \in G$, let $\beta_g\colon N \to N$ by

$$\beta_g(y) = xaya^{-1}$$

whenever $g = xa$ with $x \in N$ and $a \in H$; thus, $N \subseteq G$ acts on N by left translation and $H \subseteq G$ acts by conjugation. It is not hard to check that β_g is a well-defined bijection of N onto itself (note that β_g is a group homomorphism of N if and only if $g \in H$), and that $g \mapsto \beta_g$ is a group homomorphism. In fact, β corresponds to the left action of G on G/H under the natural identification of $G/H = (NH)/H$ with N.

Note that if $g = xa$ as above, for $y \in N$ and $U \in \mathcal{U}$ we have

$$\beta_g(yU) = (xaya^{-1})(aUa^{-1}). \tag{3.1}$$

Thus, the set

$$\mathcal{P} = \{\chi_{xU} : x \in N,\ U \in \mathcal{U}\} \subseteq \ell^\infty(N)$$

is invariant under the action α of G on $\ell^\infty(N)$ associated to β in the usual way by

$$\alpha_g(f) = f \circ \beta_{g^{-1}}.$$

If we further define

$$D = C^*(\mathcal{P}) \subseteq \ell^\infty(N),$$

it follows that α restricts to an action, still denoted by α, of G on the C^*-algebra D.

Theorem 3.1. *With the above notation and assumptions, suppose that $\mathcal{U}$ has finite quotients (condition (2.3)) and that H acts effectively on M (condition (2.4)). Then the reduced crossed product $D \rtimes_{\alpha,r} G$ is simple and purely infinite.*

The first step of the proof is to identify the commutative C^*-algebra D. Notice that, with $g = xa$ as at (3.1), β_g induces a bijection of $\mathcal{U}$ onto $\{xU : U \in \mathcal{U}\}$; thus, β_g is a uniform isomorphism of N (with the subgroup topology determined by $\mathcal{U}$) onto itself. It follows that the action β of G on N extends uniquely to an action $\bar\beta$ of G by homeomorphisms of the completion $\bar N$. We let $\bar\alpha$ denote the associated action of G on $C_0(\bar N)$, so that

$$\bar\alpha_g(f) = f \circ \bar\beta_{g^{-1}} \quad \text{for } f \in C_0(\bar N) \text{ and } g \in G.$$

The following result is elementary, and we claim no originality; however, we could not find it in the literature, so we include the proof for completeness.

Proposition 3.2. *If $\mathcal{U}$ has finite quotients, then the restriction map ρ of $C_0(\bar N)$ into $\ell^\infty(N)$ given by*

$$\rho(f) = f|_N$$

is an $\bar\alpha - \alpha$ equivariant isomorphism of $C_0(\bar N)$ onto D.

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Proof. Since N is dense in $\bar{N}$, ρ gives an isometric homomorphism of $C_0(\bar{N})$ into $\ell^\infty(N)$. The cosets $\{x\bar{U} : x \in \bar{N},\ U \in \mathcal{U}\}$ form a base for the topology of $\bar{N}$ consisting of compact sets (see the proof of Lemma 2.3), so $C_0(\bar{N})$ is generated by the set $\mathcal{S} = \{\chi_{x\bar{U}} : x \in \bar{N},\ U \in \mathcal{U}\}$ of characteristic functions. Again, since N is dense, for each $x \in \bar{N}$ and $U \in \mathcal{U}$ there exists $y \in N$ such that $x\bar{U} = y\bar{U}$. Since $(\chi_{y\bar{U}})|_N = \chi_{yU}$ for such y and U, we have

$$\rho(\mathcal{S}) = \{\chi_{yU} : y \in N,\ U \in \mathcal{U}\} = \mathcal{P},$$

and it follows that ρ maps $C_0(\bar{N})$ onto D.

For the equivariance, since each $\bar{\beta}_g$ is the extension to $\bar{N}$ of β_g, for $f \in C_0(\bar{N})$ we have

$$\rho(\bar{\alpha}_g(f)) = (f \circ \bar{\beta}_{g^{-1}})|_N = (f|_N) \circ \beta_{g^{-1}} = \alpha_g(\rho(f)).$$

□

Recall from [**1**,**19**] that an action γ of a discrete group G on a locally compact Hausdorff space X is

- *minimal* if for every $x \in X$ the orbit $\{\gamma_g(x) : g \in G\}$ is dense,
- *locally contractive* if for every non-empty open set $O \subseteq X$ there exist $g \in G$ and a non-empty open set $O' \subseteq O$ such that
$$\gamma_g(\bar{O'}) \subsetneq O',$$
- *topologically free* if for every $g \in G \setminus \{e\}$ the set
$$\{x \in X : \gamma_g(x) = x\}$$
of fixed points has empty interior.

In [**1**] the term 'local boundary action' is used instead of 'locally contractive action'.

Lemma 3.3. *If $\mathcal{U}$ has finite quotients, then $\bar{N}$ is locally compact and the action $\bar{\beta}$ of G on $\bar{N}$ is minimal and locally contractive.*

Proof. Local compactness follows from Lemma 2.3. For minimality, let $x \in \bar{N}$. Then the orbit of x under the action of G contains the coset Nx, which is dense in $\bar{N}x = \bar{N}$.

For local contractivity, since the cosets $\{y\bar{U} : y \in N,\ U \in \mathcal{U}\}$ form a base for the topology of $\bar{N}$ consisting of closed sets, it suffices to show that for every $y \in N$ and $U \in \mathcal{U}$ there exists $g \in G$ such that

$$\bar{\beta}_g(y\bar{U}) \subsetneq y\bar{U}.$$

For this, first note that since $M \neq \{e\}$, by (2.2) there exists $d \in H$ such that $dMd^{-1} \neq M$, and then by (2.1) we can choose $c \in H$ such that $cMc^{-1} \subseteq M \cap dMd^{-1} \subsetneq M$.

Now fix $b \in H$, so that $U = bMb^{-1}$ is an arbitrary element of $\mathcal{U}$. Setting $a = bcb^{-1} \in H$ gives

$$aUa^{-1} = b(cMc^{-1})b^{-1} \subsetneq bMb^{-1} = U,$$

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and since the map $U \mapsto \bar{U}$ is injective on $\mathcal{U}$ (Lemma 2.2), it follows that $\overline{aUa^{-1}} \subsetneq \bar{U}$. Thus, for any $y \in N$, if we further set $x = y(ay^{-1}a^{-1}) \in N$ and then $g = xa$, we have

$$\bar{\beta}_g(y\bar{U}) = \overline{\beta_g(yU)} = \overline{(xaya^{-1})(aUa^{-1})} = y\overline{aUa^{-1}} \subsetneq y\bar{U},$$

as desired. □

Lemma 3.4. *If H acts effectively on M, then the action $\bar{\beta}$ of G on $\bar{N}$ is topologically free.*

Proof. Let $g \in G \setminus \{e\}$. We must show that the fixed-point set

$$\{y \in \bar{N} : \bar{\beta}_g(y) = y\}$$

has empty interior. Suppose the contrary. Since N is dense in $\bar{N}$ there is a non-empty open set $O \subseteq N$ that is fixed pointwise by β_g. Let $g = xa$ with $x \in N$ and $a \in H$. Now, for any $y, z \in O$ we have

$$xaya^{-1} = y \quad \text{and} \quad xaza^{-1} = z,$$

so

$$x = yay^{-1}a^{-1} = zaz^{-1}a^{-1},$$

and hence

$$y^{-1}z = ay^{-1}a^{-1}aza^{-1} = ay^{-1}za^{-1},$$

i.e. the open neighbourhood $O^{-1}O$ of e in N consists of fixed points for conjugation by the element a of H. By definition of the subgroup topology, we have $bMb^{-1} \subseteq O^{-1}O$ for some $b \in H$. Thus, for all $s \in M$, we have

$$absb^{-1}a^{-1} = bsb^{-1},$$

so $b^{-1}ab$ acts trivially by conjugation on M. If H acts effectively on M we must then have $b^{-1}ab = e$, so $a = e$, which in turn forces $xy = y$ for all $y \in O$, so $x = e$, and hence $g = e$, giving a contradiction. □

Proof of Theorem 3.1. Since $D \rtimes_{\alpha,r} G \cong C_0(\bar{N}) \rtimes_{\bar{\alpha},r} G$ by Proposition 3.2, it suffices to show that $C_0(\bar{N}) \rtimes_{\bar{\alpha},r} G$ is simple and purely infinite. Simplicity follows from [**1**, Corollary of Theorem 1] and [**1**, Theorem 2], since $\bar{\beta}$ is minimal by Lemma 3.3 and topologically free by Lemma 3.4. The crossed product is purely infinite by [**19**, Theorem 9], since $\bar{\beta}$ is locally contractive by Lemma 3.3. □

Remark 3.5. Note that in Theorem 3.1 the action of G on $\hat{D} = \bar{N}$ is not free: if $g \in H$, then $\beta_g(e) = e$.

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4. Universal C^*-algebras

In this section we show that $D \rtimes_\alpha G$ is the enveloping C^*-algebra of an 'algebraic crossed product' (Corollary 4.6). The main result of this section is that under an extra hypothesis (H1) on the action (namely that the full and reduced crossed products coincide), a certain corner of the algebraic crossed product has a unique C^*-norm, and moreover the corresponding corner of $D \rtimes_\alpha G$ is its enveloping C^*-algebra (Corollary 4.11).

Throughout this section we assume that $\mathcal{U}$ has finite quotients (condition (2.3)) and that H acts effectively on M (condition (2.4)). Recall that, by definition,

$$\mathcal{P} = \{\chi_{xU} \colon x \in N,\ U \in \mathcal{U}\} \quad \text{and} \quad D = C^*(\mathcal{P}) \subseteq \ell^\infty(N).$$

Now we further define

$$D_0 = \operatorname{span} \mathcal{P} \subseteq D.$$

Lemma 4.1. *D_0 is a $*$-subalgebra of $\ell^\infty(N)$, and consequently*

$$D = \bar{D}_0 = \overline{\operatorname{span}}\, \mathcal{P}.$$

Proof. Clearly, D_0 is a self-adjoint linear subspace of $\ell^\infty(N)$. Fix $U, V \in \mathcal{U}$, and choose $W \in \mathcal{U}$ with $W \subseteq U \cap V$. Then, for any $x, y \in N$, since $\mathcal{U}$ has finite quotients, we can write

$$\chi_{xU} = \sum_{zW \in xU/W} \chi_{zW} \quad \text{and} \quad \chi_{yV} = \sum_{wW \in yV/W} \chi_{wW}$$

where both sums are finite. Thus,

$$\chi_{xU}\chi_{yV} = \sum_{zW \in (xU \cap yV)/W} \chi_{zW} \in D_0,$$

and it follows that D_0 is closed under multiplication. □

Now we want to work with the elements of $\mathcal{P}$ as projections in the C^*-algebra D rather than as functions on N, so we introduce an alternative notation.

Notation 4.2. Define

$$p_c = \chi_c \quad \text{for } c \in N/U,\ U \in \mathcal{U}.$$

Lemma 4.3. *D_0 has a unique C^*-norm, so that D is the enveloping C^*-algebra of D_0.*

Proof. For any $*$-homomorphism π of D_0 into a C^*-algebra E, the norm of $\pi(q)$ is at most 1 for each $q \in \mathcal{P}$, so D_0 has a universal enveloping C^*-algebra $C^*(D_0)$. For the uniqueness of the C^*-norm, we must show that if π is a homomorphism of $C^*(D_0)$ into a C^*-algebra E, then π is faithful if the restriction $\pi|_{D_0}$ is faithful. Equivalently, we must show that if I is any non-zero ideal of $C^*(D_0)$, then $I \cap D_0 \neq \{0\}$.

The key fact is that $C^*(D_0)$ is the closure of the union of the family of C^*-subalgebras

$$D_U := \overline{\operatorname{span}}\{p_c \colon c \in N/U\}$$

for $U \in \mathcal{U}$ (where the closure is taken in $C^*(D_0)$), which is an upward-directed family because $\mathcal{U}$ is a filter base with finite quotients. By a standard argument, it follows that there exists $U \in \mathcal{U}$ such that $I \cap D_U \neq \{0\}$. Now, the map

$$f \mapsto \sum_{c \in N/U} f(c) p_c$$

gives an isomorphism of $c_0(N/U)$ onto D_U. It follows that

$$I \cap \operatorname{span}\{p_c \colon c \in N/U\} \neq \{0\}.$$

Since

$$\operatorname{span}\{p_c \colon c \in N/U\} \subseteq D_0,$$

this completes the proof. □

Since D_0 is evidently self-adjoint and G-invariant, the 'algebraic crossed product'

$$D_0 \rtimes_\alpha G := \operatorname{span} i_G(G) i_D(D_0)$$

is a $*$-subalgebra of the C^*-crossed product $D \rtimes_\alpha G$. We emphasize that we use the term 'algebraic crossed product' purely as shorthand for $D_0 \rtimes_\alpha G \subseteq D \rtimes_\alpha G$ as defined here. There are certainly other uses of the term in the literature, and they are generally different from ours; our approach is inherently C^*-algebraic.

We shall suppress the maps i_G and i_D, thus identifying G and D_0 with their images in $M(D \rtimes_\alpha G)$. For $g \in G$, $U \in \mathcal{U}$ and $c \in N/U$, covariance becomes

$$g p_c g^{-1} = p_{\beta_g(c)},$$

where β is as in (3.1).

We first note that $D_0 \rtimes_\alpha G$ is universal for covariant representations. (A *representation* of a $*$-algebra B on a Hilbert space X is a $*$-homomorphism from B to $\mathcal{B}(X)$.)

Definition 4.4. If π and u are representations of D_0 and G on a Hilbert space X, we say that (π, u) is a *covariant representation of* (D_0, G) if

$$u_g \pi(f) u_g^* = \pi(\alpha_g(f)) \quad \text{for } g \in G,\ f \in D_0.$$

Corollary 4.5. *For every covariant representation (π, u) of (D_0, G) on a Hilbert space X, there is a unique representation Π of $D_0 \rtimes_\alpha G$ on X such that*

$$\Pi(gf) = u_g \pi(f) \quad \text{for } g \in G,\ f \in D_0. \tag{4.1}$$

Proof. Uniqueness is clear, since $D_0 \rtimes_\alpha G$ is spanned by the products gf for $g \in G$, $f \in D_0$. Given (π, u), by Lemma 4.3 π extends uniquely to a representation $\tilde{\pi}$ of D on X. By density and continuity, the pair $(\tilde{\pi}, u)$ is a covariant representation of (D, G), so there is a unique representation $\tilde{\Pi}$ of the C^*-crossed product $D \rtimes_\alpha G$ on X such that

$$\tilde{\Pi}(gf) = u_g \tilde{\pi}(f) \quad \text{for } g \in G,\ f \in D.$$

Then the restriction $\Pi := \tilde{\Pi}|_{D_0 \rtimes_\alpha G}$ is a representation of $D_0 \rtimes_\alpha G$ on X satisfying (4.1). □

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Corollary 4.6. *$D \rtimes_\alpha G$ is the enveloping C^*-algebra of $D_0 \rtimes_\alpha G$.*

Proof. This follows from Lemma 4.3 and [**13**, Lemma 2.3]. □

(H1) For the remainder of this section we assume that the action α of G on D is 'regular' in the sense that the regular representation of $D \rtimes_\alpha G$ onto $D \rtimes_{\alpha,r} G$ is an isomorphism.

Note that hypothesis (H1) is satisfied in particular whenever G is amenable, which is the case in all our examples.

As a consequence of Theorem 3.1 and (H1), the full crossed product $D \rtimes_\alpha G$ is simple and purely infinite.

Corollary 4.7. *The $*$-algebra $D_0 \rtimes_\alpha G$ has a unique C^*-norm.*

Proof. This follows from Corollary 4.6 and simplicity of $D \rtimes_\alpha G$. □

Notation 4.8. For notational simplicity, let

(i) $A_0 = D_0 \rtimes_\alpha G$ (the algebraic crossed product),

(ii) $A = D \rtimes_\alpha G$ (the C^*-crossed product).

Also, let
$$p = p_M.$$

The following lemma shows that p is 'algebraically full' in A_0.

Lemma 4.9. $A_0 = \operatorname{span} A_0 p A_0$.

Proof. Since $gA_0 = A_0$ for all $g \in G$, it suffices to show that for every $U \in \mathcal{U}$ and $c \in N/U$ we have
$$p_c \in \operatorname{span} A_0 p A_0.$$
Choose $V \in \mathcal{U}$ such that $V \subseteq U \cap M$. Then
$$\begin{aligned} p_c &= \sum \{p_{xV} : x \in N,\ xV \subseteq c\} \\ &= \sum \{x p_V : x \in N,\ xV \subseteq c\} \\ &= \sum \{x p_V p p_V : x \in N,\ xV \subseteq c\} \quad (\text{because } p_V p = p p_V = p_V) \\ &\in \operatorname{span} A_0 p A_0. \end{aligned}$$
□

Now we see that $A_0 p$ is an $A_0 - pA_0 p$ imprimitivity bimodule in the sense of Fell and Doran [**14**, Definition XI.6.2]. Since $p \in A_0$, the left inner product ${}_L\langle \cdot, \cdot \rangle$ on $A_0 p$ is *positive* in the sense that for all $b \in A_0$ we have
$${}_L\langle bp, bp \rangle = (bp)(bp)^*.$$

We need to know that the right inner product is also positive.

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Lemma 4.10. *For all $b \in A_0$ there exist $c_1, \dots, c_n \in pA_0p$ such that*

$$\langle bp, bp \rangle_R = pb^*bp = \sum_{i=1}^{n} c_i^* c_i.$$

Proof. The proof is almost identical to an argument in [**16**, Proof of Theorem 5.13], so we shall omit it. □

Corollary 4.11. *pA_0p has a unique C^*-norm, and pAp is its enveloping C^*-algebra.*

Proof. A is simple by Theorem 3.1 and (H1), and is the enveloping C^*-algebra of A_0 by Corollary 4.6. Since A is Morita–Rieffel equivalent (in Rieffel's sense, i.e. the inner products are positive) to pAp via the $A - pAp$ imprimitivity bimodule Ap, it follows that pAp is simple.

Since the $A_0 - pA_0p$ imprimitivity bimodule A_0p is dense in the $A - pAp$ imprimitivity bimodule Ap, and the right inner product $\langle \cdot, \cdot \rangle_R$ on A_0p is positive by Lemma 4.10, an application of [**16**, Proposition 5.5 (iii)] shows that pAp is the enveloping C^*-algebra of pA_0p, and hence pA_0p has a unique C^*-norm because pAp is simple. □

5. Embedding the Hecke algebra

We continue to assume that $\mathcal{U}$ has finite quotients (condition (2.3)) and that H acts effectively on M (condition (2.4)); this implies that M is in fact a Hecke subgroup of G. To see this, fix $g \in G$ and choose $a \in H$ and $x \in N$ such that $g = ax$. Then $gMg^{-1} = aMa^{-1} \in \mathcal{U}$ (since M is normal in N), so by (2.1) we can choose $U \in \mathcal{U}$ such that $U \subseteq M \cap aMa^{-1}$, and it follows from (2.3) that

$$|M/(M \cap gMg^{-1})| \leqslant |M/U| < \infty.$$

Furthermore, by (2.2) we have

$$\bigcap_{g \in G} gMg^{-1} = \bigcap_{U \in \mathcal{U}} U = \{e\};$$

so the pair (G, M) is *reduced* in the sense of [**16**], and by [**16**, Lemma 6.3], condition (2.1) means that the pair is *directed*.

The *Hecke algebra* $\mathcal{M}$ of the pair (G, M) is a convolution $*$-algebra generated by the double cosets of M in G. By [**16**, Theorem 6.4], $\mathcal{M}$ has a universal enveloping C^*-algebra $C^*(\mathcal{M})$; this is the *Hecke C^*-algebra* of the pair (G, M).

In this section, we shall give sufficient conditions for $C^*(\mathcal{M})$ to embed in the Cuntz–Li algebra. In § 7, this will be applied to the generalized Bost–Connes algebra.

Our embedding will require $\bar{N}$ to be self-dual.

Theorem 5.1. *Let N be abelian, and assume that there exists an isomorphism $\theta \colon \bar{N} \to \hat{\bar{N}}$ such that*

$$\theta \circ \beta_a(n) = \theta(n) \circ \beta_a^{-1} \quad \text{for } a \in H,\ n \in \bar{N} \tag{5.1}$$

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and

$$\theta(\bar{M}) = (\bar{M})^{\perp}. \tag{5.2}$$

Then the Hecke C^*-algebra $C^*(\mathcal{M})$ embeds faithfully in the corner $p(D \rtimes_\alpha G)p$, where $p = i_D(\chi_M)$.

Proof. The subgroup topology on G determined by $\mathcal{U}$ is precisely the *Hecke topology* of the pair (G, M) [**16**, Definition 3.3], so the Hausdorff completion $\bar{G}$ of G with respect to this topology is a locally compact (and totally disconnected) group, the closure $\bar{M}$ of M in $\bar{G}$ is a compact open subgroup of $\bar{G}$ and the Hausdorff completion $\bar{N}$ of N is (identified with) the closure of N in $\bar{G}$ [**16**, § 3]. We normalize the left Haar measure on $\bar{G}$ so that $\bar{M}$ has measure 1; thus, $\chi_{\bar{M}}$ is a projection in the convolution $*$-algebra $C_c(\bar{G})$, and $\mathcal{M}$ can be identified with the corner $\chi_{\bar{M}} C_c(\bar{G}) \chi_{\bar{M}}$. Moreover, since (G, M) is directed, $\chi_{\bar{M}}$ is a full projection in the group C^*-algebra $C^*(\bar{G})$, and we can identify $C^*(\mathcal{M})$ with the full corner $\chi_{\bar{M}} C^*(\bar{G}) \chi_{\bar{M}}$ [**16**, Theorem 6.4].

Now, the $ax + b$ group action β of the discrete semidirect product $G = N \rtimes H$ by homeomorphisms of the space $\bar{N}$ (extended from the $ax + b$ group action on N, where we now drop the bar on the notation for the extended action) restricts to the action of H by automorphisms of $\bar{N}$ that defines the semidirect product $\bar{G} = \bar{N} \rtimes H$, and we continue to denote this action by $\beta\colon H \to \operatorname{Aut} \bar{N}$. This in turn determines an action $\gamma\colon H \to \operatorname{Aut} C^*(\bar{N})$ such that

$$C^*(\bar{G}) \cong C^*(\bar{N}) \rtimes_\gamma H,$$

and the isomorphism carries $\chi_{\bar{M}}$ to $i_{C^*(\bar{N})}(\chi_{\bar{M}})$.

On the other hand, we shall replace D with the isomorphic C^*-algebra $C_0(\bar{N})$ (thus replacing $i_D(\chi_M)$ with $i_{C_0(\bar{N})}(\chi_{\bar{M}})$), and we shall denote the associated action of G on $C_0(\bar{N})$ by α (rather than $\bar{\alpha}$ as we did previously). Thus, to prove the theorem it suffices to find an isomorphism

$$C^*(\bar{N}) \rtimes_\gamma H \cong C_0(\bar{N}) \rtimes_\alpha H$$

that takes $i_{C^*(\bar{N})}(\chi_{\bar{M}})$ to $i_{C_0(\bar{N})}(\chi_{\bar{M}})$, and for this it suffices to find a $\gamma - \alpha$ equivariant isomorphism that takes $C^*(\bar{N})$ onto $C_0(\bar{N})$ and preserves $\chi_{\bar{M}}$.

We claim that the isomorphism ρ defined by the commutative diagram

$$\begin{tikzcd} C^*(\bar{N}) \arrow[r, "\rho"] \arrow[d, "f \mapsto \hat{f}"'] & C_0(\bar{N}) \\ C_0(\hat{\bar{N}}) \arrow[ur, "f \mapsto f \circ \theta"'] & \end{tikzcd}$$

does the job, where $\hat{f}$ denotes the Fourier transform of f, for which we use the convention (again normalizing so that $\int_{\bar{N}} \chi_{\bar{M}}(n)\,\mathrm{d}n = 1$) that

$$\hat{f}(\chi) = \int_{\bar{N}} f(n)\overline{\chi(n)}\,\mathrm{d}n.$$

In preparation for the verification of this claim, we record the formula for γ: for $a \in H$ and $f \in C_c(\bar{N})$ we have

$$\begin{aligned}\gamma_a(f) &= \int_{\bar{N}} f(n)\beta_a(n)\,\mathrm{d}n \\ &= \Delta_\beta(a)\int_{\bar{N}} f(\beta_a^{-1}(n))n\,\mathrm{d}n \quad (\text{for some scalar } \Delta_\beta(a)) \\ &= \Delta_\beta(a) f\circ\beta_a^{-1}.\end{aligned}$$

Then, for the same a, f, and for $n \in \bar{N}$, we have

$$\begin{aligned}\rho(\gamma_a(f))(n) &= (\gamma_a(f)\widehat{\ }\circ\theta)(n) \\ &= \Delta_\beta(a)(f\circ\beta_a^{-1})\widehat{\ }(\theta(n)) \\ &= \Delta_\beta(a)\Delta_\beta(a)^{-1}\hat{f}(\theta(n)\circ\beta_a) \\ &= \hat{f}((\theta\circ\beta_a^{-1})(n)) \quad (\text{by } (5.1)) \\ &= (\hat{f}\circ\theta)(\beta_a^{-1}(n)) \\ &= \rho(f)(\beta_a^{-1}(n)) \\ &= \alpha_a(\rho(f))(n),\end{aligned}$$

where the third equality follows from the following calculation: for $\chi \in \hat{\bar{N}}$ we have

$$\begin{aligned}(f\circ\beta_a^{-1})\widehat{\ }(\chi) &= \int_{\bar{N}} f\circ\beta_a^{-1}(n)\overline{\chi(n)}\,\mathrm{d}n \\ &= \Delta_\beta(a)^{-1}\int_{\bar{N}} f(n)\chi(\beta_a^{-}(n))\,\mathrm{d}n \\ &= \Delta_\beta(a)^{-1}\hat{f}(\chi\circ\beta_a).\end{aligned}$$

Thus, ρ is a $\gamma - \alpha$ equivariant isomorphism. Since the Fourier transform of $\chi_{\bar{M}}$ is $\chi_{\bar{M}^\perp}$, and since our hypothesis (5.2) implies that $\chi_{\bar{M}^\perp}\circ\theta = \chi_{\bar{M}}$, we have

$$\rho(\chi_{\bar{M}}) = \chi_{\bar{M}},$$

as required. □

5.1. The abelian case

We now assume that H is abelian, and derive an alternative sufficient condition for embedding.

Theorem 5.2. *Let both N and H be abelian, and assume that there exists an isomorphism $\theta\colon \bar{N} \to \hat{\bar{N}}$ such that*

$$\theta\circ\beta_a(n) = \theta(n)\circ\beta_a \quad \text{for } a \in H,\ n \in \bar{N} \tag{5.3}$$

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and

$$\theta(\bar{M}) = (\bar{M})^{\perp}. \tag{5.4}$$

Then the Hecke C^*-algebra $C^*(\mathcal{M})$ embeds faithfully in the corner $p(D \rtimes_\alpha G)p$, where $p = i_D(\chi_M)$.

Proof. Recall from the proof of Theorem 5.1 that we have an action $\beta\colon H \to \operatorname{Aut} \bar{N}$ giving the semidirect product $\bar{G} = \bar{N} \rtimes_\beta H$, and that it suffices to find an embedding of $C^*(\bar{G})$ in $C_0(\bar{N}) \rtimes_\alpha H$ taking $\chi_{\bar{M}}$ to $i_{C_0(\bar{N})}(\chi_{\bar{M}})$. Since H is abelian, we can define another action $\beta'\colon H \to \operatorname{Aut} \bar{N}$ by

$$\beta'_a = \beta_a^{-1}.$$

A routine calculation shows that the assignment

$$(n, a) \mapsto (n, a^{-1})$$

gives an isomorphism of semidirect products:

$$\bar{N} \rtimes_\beta H \cong \bar{N} \rtimes_{\beta'} H,$$

so it suffices to embed

$$C^*(\bar{N} \rtimes_{\beta'} H) \cong C^*(\bar{N}) \rtimes_{\gamma'} H$$

in $C_0(\bar{N}) \rtimes_\alpha H$, where $\gamma'\colon H \to \operatorname{Aut} C^*(N)$ is the action associated to β'. Again, it suffices to show that the same isomorphism $\rho\colon C^*(N) \cong C_0(N)$ as we used in the proof of Theorem 5.1 is now $\gamma' - \alpha$ equivariant and preserves $\chi_{\bar{M}}$. But the same calculations as in that proof accomplish this, using the modified hypothesis (5.3) rather than (5.1). □

Remark 5.3. Conditions (5.1)–(5.4) can be expressed using a bicharacter: assuming that N is abelian, and that there is an isomorphism $\theta\colon \bar{N} \to \hat{\bar{N}}$, we can define a bicharacter on $\bar{N}$ by

$$B(x, y) = \theta(y)(x) \quad \text{for } x, y \in \bar{N}.$$

Then (5.1) is equivalent to

$$B(\beta_a(x), \beta_a(y)) = B(x, y) \quad \text{for } a \in H,\ x, y \in \bar{N}. \tag{5.5}$$

Condition (5.3) is equivalent to symmetry of the automorphisms β_a:

$$B(\beta_a(x), y) = B(x, \beta_a(y)) \quad \text{for } a \in H,\ x, y \in \bar{N}, \tag{5.6}$$

and (5.2) and (5.4) are equivalent to

$$x \in \bar{M} \quad \iff \quad B(x, y) = 1 \quad \text{for all } y \in \bar{M}. \tag{5.7}$$

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6. Integral domains

In §§ 2 and 3 we introduced a general context where $G = N \rtimes H$ is a semidirect product with certain properties, and α is an action of G on a commutative C^*-algebra D. A source of examples comes from integral domains, as in the work of Cuntz and Li [**8**]. Let R be an integral domain that is not a field, so that, in particular, its group of units is strictly contained in the set $R \setminus \{0\}$ of non-zero elements, and assume that R has *finite quotients* in the sense that

$$|R/aR| < \infty \quad \text{for all } a \in R \setminus \{0\}. \tag{6.1}$$

Let $Q = Q(R)$ be the field of fractions of R, and in the notation of the previous sections, take

(i) $N = Q$ (the additive group of Q),

(ii) $H = Q^\times$ (the multiplicative group of the field Q),

(iii) $M = R$ (the additive group of the ring R).

Thus, $G = N \rtimes H = Q \rtimes Q^\times$ is the $ax + b$ group of Q, and our assumptions (2.2)–(2.4) hold in this situation. Moreover, because G is amenable in this context we have $D \rtimes_{\alpha,r} G = D \rtimes_\alpha G$.

Theorem 6.1. *With the above notation, the crossed product $D \rtimes_\alpha G$ is simple and purely infinite.*

Proof. This follows immediately from Theorem 3.1 and amenability of G. □

In § 10 we shall show that the Cuntz–Li algebra $\mathfrak{A}[R]$ of [**8**] is isomorphic to a corner of $D \rtimes_\alpha G$.

7. Embedding the generalized Bost–Connes algebra

Let $K = \mathbb{Q}(\theta)$ be an algebraic number field with θ an algebraic integer, let N and M be the additive groups of $\mathbb{Q}(\theta)$ and $\mathbb{Z}[\theta]$, respectively, and let H be the multiplicative group of K. In this situation the Hecke C^*-algebra $C^*(\mathcal{M})$, as defined in § 5, is the generalized Bost–Connes C^*-algebra. In combination with Theorem 10.1, Corollary 7.1 will recover Cuntz and Li's embedding of the Bost–Connes algebra in $\mathfrak{A}[R]$.

Sometimes, M equals the ring R of integers in K; if so, K is called a *monogenic* field. In general there is $s \in \mathbb{N}$ such that $sR \subset M \subset R$. Therefore, the topologies defined by M and R will be the same, in particular the completion $\bar{N}$ is the ring $\mathcal{A}_f$ of finite adeles of K (cf. [**3**, Chapter VII, § 2, No. 4, Proposition 3] and the preceding discussion therein).

Corollary 7.1. *If $M = R$ and with the above notation, the generalized Bost–Connes algebra $C^*(\mathcal{M})$ embeds faithfully in the corner $p(D \rtimes_\alpha G)p$, where $p = i_D(\chi_M)$.*

For this we need the following.

Lemma 7.2. *There is a $\mathbb{Q}$-linear map $\phi\colon K \to \mathbb{Q}$ such that*

$$x \in M \iff \phi(xy) \in \mathbb{Z} \quad \text{for all } y \in M.$$

Proof. Let $p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1}x^{n-1} - x^n$ with all $\alpha_i \in \mathbb{Z}$ be θ's minimal polynomial. Define $\phi\colon K \to \mathbb{Q}$ by

$$\phi(a_0 + a_1\theta + \cdots + a_{n-1}\theta^{n-1}) = a_{n-1}.$$

Clearly,

$$\phi(\theta^i) = \begin{cases} 0 & \text{for } 0 \leqslant i < n-1, \\ 1 & \text{for } i = n-1, \\ s_i \in \mathbb{Z} & \text{for } n \leqslant i, \end{cases}$$

so $\phi(xy) \in \mathbb{Z}$ for all $x, y \in M$.

Now suppose $x = a_0 + a_1\theta + \cdots + a_{n-1}\theta^{n-1} \in K$ satisfies $\phi(xy) \in \mathbb{Z}$ for all $y \in M$, in particular, $\phi(x\theta^i) \in \mathbb{Z}$ for all i. Checking this for $i = 0, 1, 2, \dots$, we get

$$\begin{aligned} a_{n-1} &\in \mathbb{Z}, \\ a_{n-2} + a_{n-1}\phi(\theta^n) &\in \mathbb{Z}, \\ a_{n-3} + a_{n-2}\phi(\theta^n) + a_{n-1}\phi(\theta^{n+1}) &\in \mathbb{Z}, \end{aligned}$$

etc. Since $\phi(\theta^i) \in \mathbb{Z}$ for all i, we get $a_{n-1} \in \mathbb{Z}$. Then $a_{n-2} \in \mathbb{Z}$, etc. So $x \in M$. □

Proof of Corollary 7.1. By Theorem 5.2, we only need to verify hypotheses (5.6) and (5.7). With ϕ as in Lemma 7.2, define a character λ on K by

$$\lambda(x) = \exp(2\pi \mathrm{i}\phi(x)),$$

and a bicharacter B on K by

$$B(x, y) = \lambda(xy).$$

Copying the proof of [**20**, Chapter XIV, § 1, Theorem 1], λ extends to a character of $\bar{N}$, and B extends to a bicharacter inducing a self-duality of $\bar{N}$.

Clearly, $B(x, y) = 1$ for all $x, y \in \bar{M}$. Conversely, given $x \in \bar{N}$, we have $x = x_0 + m$ with $x_0 \in N$ and $m \in \bar{M}$. Therefore,

$$\begin{aligned} B(x, y) = 1 \quad &\text{for all } y \in \bar{M} \\ &\implies \lambda((x_0 + m)y) = 1 \quad \text{for all } y \in M \\ &\implies \lambda(x_0 y) = 1 \quad \text{for all } y \in M \\ &\implies \phi(x_0 y) \in \mathbb{Z} \quad \text{for all } y \in M \\ &\implies x_0 \in M \\ &\implies x \in \bar{M}. \end{aligned}$$

Thus, (5.7) holds. Since $H = K^\times$ acts on $\bar{N}$ by multiplication (using the canonical embedding of K in $\bar{N}$), (5.6) is also satisfied. □

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8. Lamplighter groups

Suppose H is an infinite abelian group with H^+ an Ore subsemigroup, i.e.

$$(H^+)^{-1} \cap (H^+) = \{e\} \quad \text{and} \quad (H^+)^{-1}H^+ = H.$$

We also assume the following finiteness condition:

$$c \in H^+ \implies H^+ \setminus cH^+ \quad \text{is finite.}$$

Let F be a finite group and take

$$N = \bigoplus_H F = \{f\colon H \to F \mid \operatorname{supp}(f) \text{ is finite}\}$$

with pointwise multiplication, where $\operatorname{supp}(f) = \{x \in H \mid f(x) \neq e\}$.

Then take

$$M = \bigoplus_{H^+} F = \{f \in N \mid \operatorname{supp}(f) \subseteq H^+\},$$

which is clearly a normal subgroup of N. H acts on N by shifting

$${}_af(x) = f(a^{-1}x).$$

Then $G = N \rtimes H$ is called the *wreath product* of H and F (cf. [**25**, pp. 172–176]), or the *lamplighter group* (cf. [**10**]) if $H = \mathbb{Z}$ with $H^+ = \mathbb{N}$.

One checks that

$$aMa^{-1} = \{f \in N \mid \operatorname{supp}(f) \in aH^+\},$$

and that $a \in H^+ \iff aMa^{-1} \subseteq M$, so the notation is consistent with § 9.

If $a = b^{-1}c$ with $b, c \in H^+$, then $aMa^{-1} \cap M \supseteq cMc^{-1}$, so $\{aMa^{-1}\}$ is downward directed.

Furthermore, if $c \in H^+$ then

$$M/cMc^{-1} \cong \{f \in N \mid \operatorname{supp}(f) \in H^+ \setminus cH^+\},$$

which is finite.

If $f \in \bigcap aMa^{-1}$, then $\operatorname{supp}(f) \subseteq \cap aH^+ = \emptyset$ (the last equality is not obvious), so $\bigcap aMa^{-1} = \{e\}$.

H acts effectively on M, so all assumptions in § 2 are satisfied. In addition, it should be clear that $\bigcup aMa^{-1} = N$.

As to the completions, we see that

$$\bar{M} = \{f\colon H^+ \to F\}$$

and

$$\bar{N} = \bigcup_{a \in H^+} \{f\colon a^{-1}H^+ \to F\}.$$

So with F non-abelian, this gives examples with $\bar{N}$ non-abelian.

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If F is abelian, then $F \cong \hat{F}$ by some isomorphism θ. Take

$$B(f,g) = \prod_i \langle f(i), \theta(g(i^{-1}))\rangle$$

and note the non-trivial fact that for $f, g \in \bar{N}$ the product is finite. This bilinear form is symmetric, i.e. satisfies (5.6). Moreover,

$$f \in \bar{M} \quad \Longleftrightarrow \quad B(f,g) = 1 \quad \text{for all } g \in \bar{M}.$$

Thus, Theorem 5.2 can be applied in this situation.

9. Generators and relations

In this section we shall look at the crossed products $A = D \rtimes G$ and $C = D \rtimes H$ together with the corner subalgebras pAp and pCp, where $p = \chi_M$. We shall see that, under (H2), below, the corner algebras have generators satisfying relations à la [**7**, **8**] and that they in fact are universal for these generators and relations. It turns out that the result for pAp follows from the case of pCp, so we shall deal with pCp first. There is some commonality in our approach to the two cases: we start with a representation of the generators and relations on a Hilbert space X. We use a dilation technique to embed X in a larger Hilbert space $\tilde{X}$, where we can represent the generators of the full algebras C and A. C and A are universal for covariant representations, so finally we only have to cut down with the projection p to get the result.

We continue to assume condition (2.3), that $\mathcal{U}$ has finite quotients; condition (2.4), that H acts effectively on N, is not needed for this section.

To begin, let

$$H^+ = \{a \in H : aMa^{-1} \subset M\}.$$

From our assumptions it follows that given $h \in H$ there exists $a \in H$ such that

$$aMa^{-1} \subset hMh^{-1} \cap M,$$

so $a \in H^+$ and $h^{-1}a \in H^+$, and thus

$$h = a(h^{-1}a)^{-1} \in H^+(H^+)^{-1}.$$

It follows that H is directed by the relation

$$a \leqslant b \quad \text{if and only if} \quad b \in aH^+.$$

Observe that this relation is not a partial ordering in general, since $H^+ \cap (H^+)^{-1}$ can be non-trivial.

Remark 9.1. For $a \in H$ we have $a \in H^+$ if and only if $ap = pap$, where we recall that we identify elements of the groups H and G with their images in the multipliers of the crossed products $D \rtimes H$ and $D \rtimes G$.

(H2) Throughout this section we assume that H is abelian and $N = \bigcup_{a\in H^+} a^{-1}Ma$.

This hypothesis will be satisfied in all of our examples.

We adopt the standard conventions that in a unital C^*-algebra B, an *isometry* is an element s such that $s^*s = 1$, a *unitary* is an element u such that $u^*u = uu^* = 1$, and a *projection* is an element p such that $p = p^* = p^2$.

The following relations will be useful for constructing representations of pCp.

Definition 9.2. An *SP-family* in a unital C^*-algebra B consists of a set $\{S_a : a \in H^+\}$ of isometries in B and a set $\{P(a,m) : a \in H^+,\ m \in M\}$ of projections satisfying the relations

$$S_aS_b = S_{ab}, \tag{9.1}$$

$$P(e,e) = 1, \tag{9.2}$$

$$S_aP(b,m)S_a^* = P(ab, ama^{-1}), \tag{9.3}$$

$$P(a,k) = \sum_{mbMb^{-1}\in M/bMb^{-1}} P(ab, kama^{-1}) \quad \text{for all } b \in H^+. \tag{9.4}$$

We shall also need the following for constructing representations of D.

Definition 9.3. A *PN-family* in a C^*-algebra B consists of a set $\{P(a,n) : a \in H^+,\ n \in N\}$ of projections satisfying the relations

$$P(a,n)P(a,k) = 0 \quad \text{if } naMa^{-1} \neq kaMa^{-1}, \tag{9.5}$$

$$P(a,n) = \sum_{mbMb^{-1}\in M/bMb^{-1}} P(ab, nama^{-1}) \quad \text{for all } b \in H^+. \tag{9.6}$$

It will sometimes be convenient to have analogues of the above relations with H^+ replaced by H.

Definition 9.4. A *PNH-family* in a C^*-algebra B consists of a set $\{P(h,n) : h \in H,\ n \in N\}$ of projections satisfying the relations

$$P(h,n)P(h,k) = 0 \quad \text{if } nhMh^{-1} \neq khMh^{-1}; \tag{9.7}$$

$$P(h,n) = \sum_{mbMb^{-1}\in M/bMb^{-1}} p(hb, nhmh^{-1}) \quad \text{for all } b \in H^+. \tag{9.8}$$

The following relations are needed to get representations of pAp.

Definition 9.5. An *SU-family* in a unital C^*-algebra B consists of a set $\{S_a : a \in H^+\}$ of isometries in B and a set $\{U(m) : m \in M\}$ of unitaries in B satisfying the relations (9.1) and

$$S_aU(m) = U(ama^{-1})S_a, \tag{9.9}$$

$$\sum_{maMa^{-1}\in M/aMa^{-1}} U(m)S_aS_a^*U(m^{-1}) = 1 \tag{9.10}$$

$$U(m)U(k) = U(mk). \tag{9.11}$$

Thus, S is an isometric representation of H^+ in B, and also U is a unitary representation of M in B.

Note that the C^*-algebra pCp is unital (with unit $p = \chi_M$); we construct an SP-family in pCp.

Lemma 9.6. *For $a \in H^+$ and $m \in M$ define elements of pCp by*

(i) $s_a = pap$,

(ii) $p(a, m) = \chi_{maMa^{-1}}$.

Then $\{s_a, p(a, m)\colon a \in H^+,\ m \in M\}$ is an SP-family that generates pCp as a C^*-algebra.

Proof. The verification that $\{s_a, p(a, m)\}$ is an SP-family is routine, with the help of the elementary properties

$$p(a, m) = pmapa^{-1}m^{-1}p = pms_a s_a^* m^{-1}p.$$

To see that this SP-family generates pCp, note that

$$C = \overline{\operatorname{span}}\{a^{-1}b\chi_{ncMc^{-1}}\colon a, b, c \in H^+,\ n \in N\},$$

so

$$pCp = \overline{\operatorname{span}}\{pa^{-1}b\chi_{ncMc^{-1}}p\colon a, b, c \in H^+,\ n \in N\}.$$

Now,

$$\chi_{ncMc^{-1}}p = \begin{cases} p(c, n) & \text{if } n \in M, \\ 0 & \text{if } n \notin M. \end{cases}$$

Thus,

$$pCp = \overline{\operatorname{span}}\{s_a^* s_b p(c, m)\colon a, b, c \in H^+,\ m \in M\}.$$

□

We similarly construct an SU-family in the unital C^*-algebra pAp.

Lemma 9.7. *For $a \in H^+$ and $m \in M$ define elements of pAp by*

(i) $s_a = pap$,

(ii) $u(m) = pmp$.

Then $\{s_a, u(m)\colon a \in H^+,\ m \in M\}$ is an SU-family that generates pAp as a C^*-algebra.

Proof. The verification that $\{s_a, u(m)\}$ is an SU-family is routine, using the elementary properties

(i) $pap = ap$,

(ii) $s_a s_a^* = apa^{-1} = \chi_{aMa^{-1}}$,

(iii) $u(m) = mp = pm$.

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To see that this SU-family generates pAp, note that

$$A = \overline{\operatorname{span}}\{a^{-1}br^{-1}mr\chi_F \colon a, b, r, c \in H^+,\ m \in M,\ F = ncMc^{-1},\ n \in N\},$$

so

$$pA_0p = \overline{\operatorname{span}}\{pa^{-1}br^{-1}mrp\chi_F \colon a, b, r, c \in H^+,\ F = m_1cMc^{-1},\ m, m_1 \in M\}.$$

The result now follows from

$$\begin{aligned} pa^{-1}br^{-1}mrp &= s_{ra}^* s_b u(m) s_r, \\ \chi_{m_1cMc^{-1}} &= u(m_1) s_c s_c^* u(m_1^{-1}). \end{aligned}$$

□

Remark 9.8. Our notation for the above SP-family and SU-family is similar to that in [**8**].

We also construct a PN-family in D.

Lemma 9.9. *For $a \in H^+$ and $n \in N$, define an element of D by*

$$p(a, n) = \chi_{naMa^{-1}}.$$

Then $\{p(a, n) \colon a \in H^+,\ n \in N\}$ is a PN-family that generates D as a C^*-algebra.

Proof. This is routine. □

The following two theorems are the main results of this section. The above relations give generators and relations for pCp and pAp, in a sense made precise below.

Theorem 9.10. *Assuming (H2), for every SP-family $\{S_a, P(a, m) \colon a \in H^+,\ m \in M\}$ in a unital C^*-algebra B there is a unique homomorphism $\Pi \colon pCp \to B$ such that*

$$\begin{aligned} \Pi(s_a) &= S_a && \text{for all } a \in H^+, \\ \Pi(p(a, m)) &= P(a, m) && \text{for all } a \in H^+,\ m \in M. \end{aligned}$$

In other words, Theorem 9.10 says that pCp is a universal C^*-algebra for relations (9.1)–(9.4).

The proof will go as follows: we can put the C^*-algebra B on Hilbert space, so that we have isometries S_a and projections $P(a, m)$ on a Hilbert space X satisfying (9.1)–(9.4). We shall construct a Hilbert space $\tilde{X}$ together with unitary operators $\tilde{S}_h$ for $h \in H$ and projections $\tilde{P}(h, n)$ for $h \in H$ and $n \in N$ such that $\lambda \colon \chi_{nhMh^{-1}} \mapsto \tilde{P}(h, n)$ is a representation of D_0 (defined in § 4) with $(\lambda, \tilde{S})$ covariant.

In fact, we use Douglas's construction [**11**] of the Hilbert space $\tilde{X}$, which depends only upon the isometries S_a, and Douglas's dilation theorem [**11**, Theorem 1] shows that we get a homomorphism $\tilde{S}$ from H^+ into the unitary group of $\tilde{X}$ and an isometric embedding $T \colon X \to \tilde{X}$ intertwining S and $\tilde{S}$:

$$\tilde{S}_a T = T S_a \quad \text{for } a \in H^+;$$

we shall extend $\tilde{S}$ in the obvious way to a unitary representation of H. The embedding T will also intertwine P and $\tilde{P}$:

$$\tilde{P}(a,m)T = TP(a,m) \quad \text{for } a \in H^+,\ m \in M.$$

$(\lambda, \tilde{S})$ extends to a representation $\tilde{\Pi}$ of C and we get the desired representation Π of pCp by $\Pi(z) = T^* \tilde{\Pi}(z) T$.

Theorem 9.11. *Assuming (H2), for every SU-family $\{S_a, U(m) \colon a \in H^+,\ m \in M\}$ in a unital C^*-algebra B there is a unique homomorphism $\Pi \colon pAp \to B$ such that*

$$\begin{aligned} \Pi(s_a) &= S_a && \text{for all } a \in H^+, \\ \Pi(u(m)) &= U(m) && \text{for all } m \in M. \end{aligned}$$

In other words, Theorem 9.11 says that pAp is a universal C^*-algebra for relations (9.1) and (9.9)–(9.11).

For this theorem we first appeal to Theorem 9.10, after applying Lemma 9.12 to get an SP-family, to get $\tilde{S}$ and $\tilde{P}$ as before, and then we construct unitary operators $\tilde{U}(n)$ for $n \in N$ such that $(h,n) \mapsto \tilde{S}_h \tilde{U}(n)$ is a unitary representation of G.

Here the same isometry $T \colon X \mapsto \tilde{X}$ also intertwines U and $\tilde{U}$:

$$\tilde{U}(m)T = TU(m) \quad \text{for } m \in M.$$

$\tilde{S}\tilde{U}$ will determine a representation $\tilde{\Pi}$ of A, and again we get the desired representation of pAp by $\Pi(z) = T^* \tilde{\Pi}(z) T$.

9.1. Consequences of the relations

Our proofs of Theorems 9.10 and 9.11 will use techniques that fall into two types: consequences of the relations and dilation techniques. Here we separate out the techniques that do not involve dilation.

Lemma 9.12. *Every SU-family $\{S_a, U(m) \colon a \in H^+,\ m \in M\}$ gives rise to an SP-family $\{S_a, P(a,m) \colon a \in H^+,\ m \in M\}$ via*

$$P(a,m) = U(m) S_a S_a^* U(m)^*.$$

Proof. This follows quickly from relations (9.1)–(9.4) and (9.9)–(9.11). □

Lemma 9.13. *Let B be a unital C^*-algebra. Suppose we have unitaries $\{S_a \colon a \in H^+\}$ in B satisfying relation (9.1). Then S extends uniquely to a unitary representation of H in B.*

Proof. If $h = ab^{-1}$ with $a, b \in H^+$, define

$$S_h = S_a S_b^*.$$

If $h = ab^{-1} = cd^{-1}$ with $a,b,c,d \in H^+$, then $ad = da = cb$, so

$$S_a S_b^* = S_a S_d S_d^* S_b^* = S_{ad} S_{bd}^* = S_{cb} S_{bd}^* = S_c S_b S_b^* S_d^* = S_c S_d^*;$$

thus, S_h is well defined. It is of course unitary, and so, because S_a and S_b are commuting unitaries, S_a and S_b^* commute for $a, b \in H^+$, whence S is a homomorphism. □

Lemma 9.14. *Let B be a unital C^*-algebra. Suppose we have unitaries $\{S_a : a \in H^+\}$ and projections $\{P(a,m) : a \in H^+,\ m \in M\}$ in B satisfying relations (9.1), (9.3) and (9.4). Then P extends uniquely to a PN-family $\{P(a,n) : a \in H^+,\ n \in N\}$ such that*

$$P(a, c^{-1}mc) = S_c^* P(ca, m) S_c \quad \text{for } c \in H^+. \tag{9.12}$$

Moreover, this extended family satisfies the analogue of (9.3):

$$S_a P(b,n) S_a^* = P(ab, ana^{-1}) \quad \text{for } a, b \in H^+,\ n \in N. \tag{9.13}$$

Proof. Since N is the union of the upward-directed family $\{c^{-1}Mc : c \in H^+\}$ (with $c^{-1}Mc \subset d^{-1}c^{-1}Mdc$), to construct a map from $H^+ \times N$ to the projections in B it suffices to find, for each $c \in H^+$, a map P^c from $H^+ \times c^{-1}Mc$ to the projections, such that

$$P^{dc}|_{H^+ \times c^{-1}Mc} = P^c.$$

For $a \in H^+$ and $m \in M$ we define $P^c(a, c^{-1}mc)$ by the right-hand side of (9.12). This is well defined, because $m \mapsto c^{-1}mc$ is a bijection of M onto $c^{-1}Mc$, and we have

$$\begin{aligned} P^c(a, c^{-1}mc) &= S_c^* P(ca, m) S_c \\ &= S_{dc}^* S_d P(ca, m) S_c^* S_{dc} \\ &= S_{dc}^* P(dca, dmd^{-1}) S_{dc} \\ &= P^{dc}(a, c^{-1}mc). \end{aligned}$$

In particular, $P^c|_{H^+ \times M} = P$. Thus, we have defined a family $\{P(a,n) : a \in H^+,\ n \in N\}$ of projections extending the given family $\{P(a,m) : a \in H^+,\ m \in M\}$ and satisfying (9.12), and moreover this relation gives the uniqueness. We expand (9.4) into the relation (9.6) for these projections: if $n = c^{-1}kc$ for $c \in H^+$ and $k \in M$, then

$$\begin{aligned} P(a, c^{-1}kc) &= S_c^* P(ca, k) S_c \\ &= S_c^* \sum_{mbMb^{-1} \in M/bMb^{-1}} P(cab, kcama^{-1}c^{-1}) S_c \\ &= \sum_{mbMb^{-1} \in M/bMb^{-1}} P(ab, c^{-1}kcama^{-1}). \end{aligned}$$

For (9.5), fix $a \in H^+$ and let $n, k \in N$ with $naMa^{-1} \neq kaMa^{-1}$. We must show that the projections $P(a,n)$ and $P(a,k)$ are orthogonal. Choose c in H^+ such that $cnc^{-1}, ckc^{-1} \in M$. It suffices to show that

$$S_c P(a,n) S_c^* = P(ca, cnc^{-1}) \perp S_c P(a,k) S_c^* = P(ca, ckc^{-1}).$$

Set $b = ca$. Since

$$cnc^{-1}bMb^{-1} = cnaMa^{-1}c^{-1} \neq ckaMa^{-1}c^{-1} = ckc^{-1}bMb^{-1},$$

the elements cnc^{-1} and ckc^{-1} of M are in distinct cosets of bMb^{-1}, so $P(b, cnc^{-1})$ and $P(b, ckc^{-1})$ are distinct terms in the expansion

$$P(e,e) = \sum_{mbMb^{-1} \in M/bMb^{-1}} P(e,m),$$

and are therefore orthogonal, as desired.

Finally, for (9.13) we have

$$\begin{aligned} S_b P(a, c^{-1}mc) S_b^* &= S_b S_c^* P(ca, m) S_c S_b^* \\ &= S_c^* P(bca, bmb^{-1}) S_c \\ &= P(ba, bc^{-1}mcb^{-1}). \end{aligned}$$

□

Lemma 9.15. *Every PN-family can be extended uniquely to a PNH-family.*

Proof. Let $h \in H$ and $n \in N$. We want to define $P(h,n)$ as follows: choose $b \in H^+$ such that $hb \in H^+$, and set

$$P(h,n) = \sum_{mbMb^{-1} \in M/bMb^{-1}} P(hb, nhmh^{-1}). \tag{9.14}$$

We must show that this is well defined, and since H is directed by $h \leqslant k$ if and only if $k \in hH^+$ it suffices to show that if $c \in H^+$, then

$$\sum_{mbMb^{-1} \in M/bMb^{-1}} P(hb, nhmh^{-1}) = \sum_{kbcMc^{-1}b^{-1} \in M/bcMc^{-1}b^{-1}} P(hbc, nhkh^{-1}).$$

For each $m \in M$, by relation (9.6) we have

$$P(hb, nhmh^{-1}) = \sum_{lcMc^{-1} \in M/cMc^{-1}} P(hbc, nhmblb^{-1}h^{-1}),$$

and so we do have

$$\begin{aligned} &\sum_{mbMb^{-1} \in M/bMb^{-1}} P(hb, nhmh^{-1}) \\ &\qquad = \sum_{mbMb^{-1} \in M/bMb^{-1}} \sum_{lcMc^{-1} \in M/cMc^{-1}} P(hbc, nhmblb^{-1}h^{-1}), \end{aligned}$$

which suffices because, as m and l run through complete sets of representatives of cosets of bMb^{-1} and cMc^{-1}, respectively, the products $mblb^{-1}$ run through a complete set of representatives of cosets of $bcMc^{-1}b^{-1}$.

Thus, we have defined projections $\{P(h,n) \colon h \in H,\ n \in N\}$, and we turn to the relations. For (9.7), let $h \in H$ and $n, k \in N$, and assume that $nhMh^{-1} \neq khMh^{-1}$. Then, since $bMb^{-1} \subset M$, for all $m, l \in M$ we have

$$nhmbMb^{-1}h^{-1} \neq khlbMb^{-1}h^{-1},$$

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and therefore

$$P(hb, nhmh^{-1}) \perp P(hb, khlh^{-1}).$$

Summing over m and l (using (9.14)), we get

$$P(h, n) \perp P(h, k).$$

For the relation (9.8), we must now show that for $h \in H$, $n \in N$ and arbitrary $b \in H^+$ we have

$$P(h, n) = \sum_{mbMb^{-1}} P(hb, nhmh^{-1}). \tag{9.15}$$

This of course looks very much like the definition of the left-hand side, except that we are not assuming that $hb \in H^+$. Nevertheless, (9.15) can be proved using the same strategy as our proof that $P(h, n)$ is well defined: choose $c \in H^+$ such that $hbc \in H^+$, and compute that

$$\begin{aligned}\sum_{mbMb^{-1} \in M/bMb^{-1}} \sum_{lcMc^{-1} \in M/cMc^{-1}} & P(hbc, nhmblb^{-1}h^{-1}) \\ &= \sum_{kbcMc^{-1}b^{-1} \in M/bcMc^{-1}b^{-1}} P(hbc, nhkh^{-1}),\end{aligned}$$

as before.

The uniqueness of the family $\{P(h, n) : h \in H,\ n \in N\}$ follows from relations (9.7) and (9.8). □

Recall from Lemma 9.9 the PN-family $\{p(a, n) : a \in H^+,\ n \in N\}$ in D. We now show that D is a universal C^*-algebra for relations (9.5) and (9.6).

Proposition 9.16. *Assuming (H2), for every PN-family $\{P(a, n) : a \in H^+,\ n \in N\}$ in a C^*-algebra B there is a unique homomorphism $\pi : D \to B$ such that*

$$\pi(p(a, n)) = P(a, n) \quad \text{for all } a \in H^+,\ n \in N.$$

Proof. π will be unique if it exists, as the $p(a, n)$ generate D. Since D_0 is the union of the upward-directed family of $*$-subalgebras

$$D_a := \operatorname{span}\{p(a, n) : n \in N\}$$

indexed by $a \in H^+$, to construct a $*$-homomorphism $\pi : D_0 \to B$ it is enough by Lemma 4.3 to find $*$-homomorphisms $\pi_a : D_a \to B$ such that

$$\pi_{ab}|_{D_a} = \pi_a. \tag{9.16}$$

For each $a \in H^+$ we want to define π_a such that

$$\pi_a(p(a, n)) = P(a, n) \quad \text{for } n \in N.$$

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Once we verify that this is well defined, it will of course uniquely extend to a linear map $\pi_a\colon D_a \to B$. To see that it is well defined, suppose that we have $a, b \in H^+$ and $n, k \in N$ with $p(a,n) = p(b,k)$, i.e.

$$naMa^{-1} = kbMb^{-1}.$$

Then $aMa^{-1} = bMb^{-1}$ and $k^{-1}n \in aMa^{-1}$. Set $c = a^{-1}b$ and $m = a^{-1}n^{-1}ka$. Then c normalizes M and $m \in M$, so by the relation (9.6) we have

$$P(a,n) = P(ac, nama^{-1}) = P(b,k).$$

The relation (9.5) immediately implies that the linear map π_a is a $*$-homomorphism of D_a, and then the relation (9.6) implies the consistency condition (9.16). □

Lemma 9.17. *The unique extension, guaranteed by Lemma 9.15, of the PN-family $p(a,n) = \chi_{naMa^{-1}}$ in D to a PNH-family is given by $p(h,n) = \chi_{nhMh^{-1}}$. Also, the unique homomorphism $\pi\colon D \to B$, guaranteed by Proposition 9.16, determined by any PN-family in B takes $p(h,n)$ to the unique extension $P(h,n)$.*

Proof. This is routine. □

Lemma 9.18. *Let B be a unital C^*-algebra, and suppose we have unitaries $\{S_a\colon a \in H^+\}$ and projections $\{P(a,m)\colon a \in H^+,\ m \in M\}$ in B satisfying (9.1), (9.3) and (9.4). Extend P uniquely to a PN-family $\{P(a,n)\colon a \in H^+,\ n \in N\}$ using Lemma 9.14, and further extend uniquely to a PNH-family $\{P(h,n)\colon h \in H,\ n \in N\}$ using Lemma 9.15. Also let $\pi\colon D \to B$ be the unique $*$-homomorphism guaranteed by Proposition 9.16. Then (π, S) is a covariant representation of (D,H) in B.*

Proof. By density, covariance is equivalent to the relation

$$S_h P(a,n) S_h^* = P(ha, hnh^{-1}) \quad \text{for } h \in H,\ a \in H^+,\ n \in N,$$

which, because $H = H^+(H^+)^{-1}$, follows from (9.13) and the following computation: if $a, b \in H^+$ and $n \in N$, then

$$\begin{aligned} S_b^* P(a,n) S_b &= \sum_{mbMb^{-1} \in M/bMb^{-1}} S_b^* P(ab, nama^{-1}) S_b \\ &= \sum_{mbMb^{-1} \in M/bMb^{-1}} P(b^{-1}ab, b^{-1}nbb^{-1}ama^{-1}b) \\ &= P(b^{-1}a, b^{-1}nb). \end{aligned}$$

□

Lemma 9.19. *Let S and U be unitary representations of H and M, respectively, in a unital C^*-algebra B, and suppose relation (9.9) holds. Then U extends uniquely to a unitary representation of N in B such that*

$$S_h U(n) S_h^* = U(hnh^{-1}) \quad \text{for } h \in H,\ n \in N. \tag{9.17}$$

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Proof. Since N is the union of the upward-directed family of subgroups $a^{-1}Ma$ for $a \in H^+$, to define a representation of N it suffices to find representations $U_a\colon a^{-1}Ma \to \mathcal{B}(X)$ such that

$$U_{ab}|_{a^{-1}Ma} = U_a.$$

It is routine to verify that

$$U_a(a^{-1}ma) := S_a^* U(m) S_a$$

does the job, and then (9.17) follows from the following computation: for $a, b, c \in H^+$ and $m \in M$ we have

$$\begin{aligned} S_{ab^{-1}} U(c^{-1}mc) S_{ab^{-1}}^* &= S_{ab^{-1}} S_c^* U(m) S_c S_{ab^{-1}}^* \\ &= S_{cb}^* S_a U(m) S_a^* S_{cb} \\ &= S_{cb}^* U(ama^{-1}) S_{cb} \\ &= U(ab^{-1}c^{-1}mcba^{-1}). \end{aligned}$$

Finally, the uniqueness follows quickly from the relation (9.17). □

9.2. Proofs of the theorems

In both Theorems 9.10 and 9.11 we are given an isometric representation S of H^+ in a unital C^*-algebra B. We first show how to use the dilation technique originally developed for the abelian case (independently) by Brehmer [**5**] and Itô [**15**]. However, we follow closely the approach of Douglas in [**11**]. The result has been generalized to non-abelian groups in [**17, 18, 24**], but since we have no application in mind for the non-abelian case we have kept everything abelian in order to simplify the arguments.

First represent B faithfully and non-degenerately on a Hilbert space X. Theorem 1 of [**11**] constructs a Hilbert space $\tilde{X}$, a homomorphism $\tilde{S}$ from H^+ to the unitary group of $\tilde{X}$ and an isometric embedding $T\colon X \to \tilde{X}$ such that

$$\begin{aligned} \tilde{S}_a T &= T S_a \quad \text{for } a \in H^+, \\ \tilde{X} &= \overline{\bigcup_{a \in H^+} \tilde{S}_a^* T(X)}. \end{aligned}$$

Actually, we shall not need Douglas's construction of $\tilde{X}$; in fact, we only need the above properties.

Proof of Theorem 9.10. As above, dilate S to a homomorphism $\tilde{S}$ from H^+ to the unitary group of $\tilde{X}$, together with an isometry $T\colon X \to \tilde{X}$ intertwining S and $\tilde{S}$. Apply Lemma 9.13 to extend $\tilde{S}$ to a unitary representation of H on $\tilde{X}$.

We now use the dilation method to construct projections $\{\tilde{P}(a, m)\colon a \in H^+,\ m \in M\}$ on $\tilde{X}$ such that $\tilde{S}$, $\tilde{P}$ satisfy (9.3) and (9.4) and

$$TP(a, m) = \tilde{P}(a, m) T \quad \text{for } a \in H^+,\ m \in M. \tag{9.18}$$

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Since $\tilde{X}$ is the closure of the union of the upward-directed family of closed subspaces $\tilde{X}_b = \tilde{S}_b^* T(X)$ indexed by $b \in H^+$, to construct a projection $\tilde{P}(a,m)$ on $\tilde{X}$ it suffices to find projections $\tilde{P}_b(a,m)$ on the subspaces $\tilde{X}_b$ such that

$$\tilde{P}_{bc}(a,m)|_{\tilde{X}_b} = \tilde{P}_b(a,m). \tag{9.19}$$

The appropriate definition is dictated by our desire for covariance: for $\xi \in X$ define

$$\tilde{P}_b(a,m)\tilde{S}_b^* T\xi = \tilde{S}_b^* T P(ba, bmb^{-1})\xi. \tag{9.20}$$

It follows quickly that $\tilde{P}_b(a,m)$ is a projection on $\tilde{X}_b$.

To see the consistency relation (9.19), let $\xi \in X$ and compute

$$\begin{aligned}
\tilde{P}_b(a,m)\tilde{S}_b^* T\xi &= \tilde{S}_b^* T P(ba, bmb^{-1})\xi \\
&= \tilde{S}_{cb}^* T S_c P(ba, bmb^{-1})\xi \\
&= \tilde{S}_{cb}^* T P(cba, cbmb^{-1}c^{-1}) S_c \xi \\
&= \tilde{P}_{cb}(a,m)\tilde{S}_{cb}^* T S_c \xi \\
&= \tilde{P}_{cb}(a,m)\tilde{S}_b^* T\xi.
\end{aligned}$$

Thus, there is a unique projection $\tilde{P}(a,m)$ on $\tilde{X}$ extending the operators $\tilde{P}_b(a,m)$ on the subspaces $\tilde{X}_b$. It suffices to verify relations (9.3) and (9.4) after post-multiplying with $\tilde{S}_c^* T$ for any $c \in H^+$. For this, we have

$$\begin{aligned}
\tilde{S}_b \tilde{P}(a,m)\tilde{S}_b^* \tilde{S}_c^* T &= \tilde{S}_b \tilde{S}_{cb}^* T P(cba, cbmb^{-1}c^{-1}) \\
&= \tilde{P}(ba, bmb^{-1})\tilde{S}_c^* T,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{P}(a,k)\tilde{S}_c^* T &= \tilde{S}_c^* T P(ca, ckc^{-1}) \\
&= \tilde{S}_c^* T \sum_{mbMb^{-1} \in M/bMb^{-1}} P(cab, ckc^{-1}cama^{-1}c^{-1}) \\
&= \sum_{mbMb^{-1} \in M/bMb^{-1}} \tilde{P}(ab, kama^{-1})\tilde{S}_c^* T.
\end{aligned}$$

Finally, (9.18) follows from (9.20) with $b = e$.

Next, apply Lemmas 9.14, (9.15), (9.18) and Proposition 9.16 to extend $\tilde{P}$ to a PNH-family $\{\tilde{P}(h,n) \colon h \in H,\ n \in N\}$ and get a representation π of D on $\tilde{X}$. The covariant representation $(\pi, \tilde{S})$ of (D,H) gives a representation $\tilde{\Pi} := \pi \times \tilde{S}$ of $C = D \rtimes H$ on $\tilde{X}$. Define $\Pi \colon pCp \to \mathcal{B}(X)$ by

$$\Pi(z) = T^* \tilde{\Pi}(z) T.$$

Since $\tilde{\Pi}$ takes $p = \chi_M$, the unit of the corner pCp, to $\tilde{P}(e,e) = TT^*$, it takes pCp to the corner $TT^*\mathcal{B}(\tilde{X})TT^*$, and it follows that Π is a representation. For $a \in H^+$ and $m \in M$ we have

$$\Pi(s_a) = T^* \tilde{S}_a T = S_a, \tag{9.21}$$

$$\Pi(p(a,m)) = T^* \tilde{P}(a,m) T = P(a,m). \tag{9.22}$$

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Since $\{s_a, p(a,m)\colon a \in H^+,\ m \in M\}$ generates pCp, it follows that Π maps pCp into the C^*-algebra B, and is moreover the unique homomorphism satisfying (9.21)–(9.22). $\square$

Proof of Theorem 9.11. Apply Lemma 9.12 to get an SP-family $\{S_a, P(a,m)\colon a \in H^+,\ m \in M\}$, and then apply Theorem 9.10 and its proof to get a covariant representation $(\pi, \tilde{S})$ of (D, H) on a Hilbert space $\tilde{X}$, together with an isometry $T\colon X \to \tilde{X}$ and a PHN-family $\{\tilde{P}(h,n)\colon h \in H,\ n \in N\}$.

We now construct a unique unitary representation $\tilde{U}$ of M on $\tilde{X}$ such that

$$\tilde{U}(m)T = TU(m), \tag{9.23}$$

$$\tilde{S}_a\tilde{U}(m)\tilde{S}_a^* = \tilde{U}(ama^{-1}) \tag{9.24}$$

for $a \in H^+$, $m \in M$. To construct a unitary operator $\tilde{U}(m)$ on $\tilde{X}$ it is enough to find a unitary $\tilde{U}_a(m)$ on each subspace $\tilde{X}_a = \tilde{S}_a^*T(X)$ such that

$$\tilde{U}_{ab}(m)|_{\tilde{X}_a} = \tilde{U}_a(m). \tag{9.25}$$

For $\xi \in X$ define

$$\tilde{U}_a(m)\tilde{S}_a^*T\xi = \tilde{S}_a^*TU(ama^{-1})\xi.$$

Since $\tilde{S}_a^*$ restricts to an isometric isomorphism of the subspace $T(X)$ onto the subspace $\tilde{X}_a$, with inverse the corresponding restriction of $\tilde{S}_a$, it follows that $\tilde{U}_a(m)\colon \tilde{X}_a \to \tilde{X}_a$ is unitary.

To see the consistency relation (9.25), let $\xi \in X$ and compute

$$\begin{aligned}\tilde{U}_a(m)\tilde{S}_a^*T\xi &= \tilde{S}_a^*TU(ama^{-1})\xi\\ &= \tilde{S}_{ab}^*TU(abmb^{-1}a^{-1})S_b\xi\\ &= \tilde{U}_{ab}(m)\tilde{S}_a^*T\xi.\end{aligned}$$

Thus, there is a unique unitary operator $\tilde{U}(m)$ on $\tilde{X}$ extending the operators $\tilde{U}_a(m)$ on the subspaces $\tilde{S}_a^*T(X)$. It follows quickly that this gives a unitary representation $\tilde{U}$ of M. Equation (9.23) follows from

$$\tilde{U}(m)T = \tilde{U}(m)\tilde{S}_e^*T = \tilde{S}_e^*TU(m) = TU(m).$$

For (9.24) it suffices to observe that, for all $b \in H^+$, we have

$$\begin{aligned}\tilde{S}_a\tilde{U}(m)\tilde{S}_a^*\tilde{S}_b^*T &= \tilde{S}_a\tilde{S}_{ba}^*TU(bama^{-1}b^{-1})\\ &= \tilde{S}_b^*TU(bama^{-1}b^{-1})\\ &= \tilde{U}(ama^{-1})\tilde{S}_b^*T.\end{aligned}$$

Now apply Lemma 9.19 to extend $\tilde{U}$ to a unitary representation of N on $\tilde{X}$ such that

$$\tilde{S}_h\tilde{U}(n)\tilde{S}_h^* = \tilde{U}(hnh^{-1}) \quad \text{for } h \in H,\ n \in N.$$

Then it is obvious that

$$hn \mapsto \tilde{S}_h \tilde{U}(n)$$

gives a unitary representation of G.

We shall show that $(\pi, \tilde{S}\tilde{U})$ is a covariant representation of (D, G): since $(\pi, \tilde{S})$ is a covariant representation of (D, H), by density it is enough to show

$$\tilde{U}(k)\tilde{P}(a,n)\tilde{U}(k)^* = \tilde{P}(a,kn) \quad \text{for } k \in N,$$

which follows immediately from the constructions.

The remainder of the proof follows the same lines as Theorem 9.10. □

As in [**8**, Remark 3.7], we deduce the following.

Corollary 9.20. *pAp is, up to isomorphism, the unique C^*-algebra generated by an SU-family.*

Proof. This follows immediately from Theorem 9.11 and Corollary 4.11. □

10. The Cuntz–Li ring C^*-algebras

As in § 6, let R be an integral domain that is not a field, and keep the other assumptions and notation from § 6. We shall apply our general theory to give an independent proof of the following result of Cuntz and Li.

Theorem 10.1. *The C^*-algebra $\mathfrak{A}[R]$ is a (full) corner of the crossed product $D \rtimes_\alpha G$, and hence is simple and purely infinite.*

Note that in [**8**] Cuntz and Li denote our group $G = N \rtimes H$ by $P_{Q(R)}$. Recall that Cuntz and Li studied $\mathfrak{A}[R]$ using auxiliary objects $\hat{R}$, $\mathcal{R}$ and $\mathfrak{D}(R)$.

Our $\bar{M}$ coincides with $\hat{R}$, since [**8**] constructs the topological group $\hat{R}$ as the completion of R. Further, our $\bar{N}$ can be identified with $\mathcal{R}$, since [**8**] constructs $\mathcal{R}$ as an inductive limit of the maps on $\hat{R}$ given by multiplication by elements of the directed multiplicative semigroup R. Thus, by [**8**, Observation 4.3], our D is isomorphic to $\mathfrak{D}(R)$.

Strictly speaking, to make the connection with [**8**] valid, we must note that this time we can get away with a slightly smaller family

$$\mathcal{U} = \{aR \colon a \in R \setminus \{0\}\},$$

rather than letting a run through all of $Q^\times$ as in the previous sections. Then from the properties of (non-field) integral domains, conditions (2.3) and (2.4) are satisfied. The fact that using this smaller version of $\mathcal{U}$ will not change our results follows from the next lemma.

Lemma 10.2. *$\mathcal{U}$ is cofinal in $\mathcal{U}' = \{aR\colon a \in Q \setminus \{0\}\}$.*

Proof. Clearly, $\mathcal{U} \subset \mathcal{U}'$. If

$$a = \frac{s}{t} \quad \text{with } s, t \in R,\ t \neq 0,$$

then

$$aR = \frac{s}{t}R \supset sR$$

and it follows that $\mathcal{U}$ is cofinal in $\mathcal{U}'$ and that they define the same group topology on Q. □

Proof of Theorem 10.1. Corollary 9.20 shows that the corner $p(D \rtimes_\alpha G)p$ is the unique C^*-algebra for the same generators and relations as [**8**, Definition 2.1] used to define $\mathfrak{A}[R]$, so we have $\mathfrak{A}[R] \cong p(D \rtimes_\alpha G)p$. By Theorem 6.1, the crossed product $D \rtimes_\alpha G$ is simple and purely infinite, and therefore so is the (full) corner $p(D \rtimes_\alpha G)p$. □

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