SZEGÖ POLYNOMIALS ON A COMPACT GROUP WITH ORDERED DUAL

I. I. HIRSCHMAN, JR.

1. Introduction. The Szegö polynomials are defined on T, the real numbers modulo 1. In this paper and in its sequel we give a generalization of Szegö polynomials in which T is replaced by an arbitrary locally compact abelian group Θ on whose dual Ξ there has been distinguished a measurable order relation compatible with the group structure. The present paper is devoted to the case where Θ is compact and Ξ therefore discrete. The general case will be taken up in the sequel mentioned above. It is desirable to proceed in this way because the case Θ compact is much simpler and much more like the classical situation than is the general case, in which various measure-theoretic difficulties obtrude. Moreover, as it happens, it is possible to develop the theory in this way with relatively little repetition.

Let $\nu(n)$ be a real function defined for $n \in I$, the integers, and satisfying

$$\nu(0) = 1, \quad \nu(n) \geqslant 1, \quad n \in I; \qquad \nu(n+m) \leqslant \nu(n)\nu(m), \quad m, n \in I.$$

If $\mathscr{G}$ is the set of those functions $f(\theta)$, $\theta \in T$, of the form

$$f(\theta) = \sum_{n=-\infty}^{\infty} \mathbf{f}(n) \exp(2\pi i n\theta),$$

for which $||f||$ is finite where

$$||f|| = \sum_{-\infty}^{\infty} |\mathbf{f}(n)| \, \nu(n),$$

then $\mathscr{G}$ is a Banach algebra of functions on T. We denote the class of all such Beurling–Gelfand algebras by $\mathfrak{G}$. For $f \in \mathscr{G}$ let

$$E^+(n)f\cdot(\theta) = \sum_{k\geqslant n} \mathbf{f}(k)\exp(2\pi ik\theta), \qquad E^-(n)f(\theta) = \sum_{k\leqslant n} \mathbf{f}(k)\exp(2\pi ik\theta),$$

for $n \in I$. Obviously $E^+(n)$ and $E^-(n)$ are linear operators on $\mathscr{G}$ (considered as a Banach space) and $||E^+(n)|| = ||E^-(n)|| = 1$. Let $c \in \mathscr{G}$; we associate with c two Wiener–Hopf operators acting on $E^+(0)\mathscr{G}$ and $E^-(0)\mathscr{G}$ respectively:

$$W_c^+ f\cdot(\theta) = E^+(0)c(\theta)f(\theta) \qquad \text{for } f \in E^+(0)\mathscr{G},$$

$$W_c^- f\cdot(\theta) = E^-(0)c(\theta)f(\theta) \qquad \text{for } f \in E^-(0)\mathscr{G}.$$

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We say that $c(\theta) \in WH(\mathscr{G})$ if both W_c^+ and W_c^- have bounded inverses. For each $n \geqslant 0$ we also define the finite-section Wiener–Hopf operators:

$$W_c^+(n)f\cdot(\theta) = E^-(n)E^+(0)c(\theta)f(\theta) \qquad \text{for } f \in E^-(n)E^+(0)\mathscr{G},$$

and

$$W_c^-(n)f\cdot(\theta) = E^+(-n)E^-(0)c(\theta)f(\theta) \qquad \text{for } f \in E^+(-n)E^-(0)\mathscr{G}.$$

The following basic result, due to Baxter (**3**), asserts that if the infinite-section Wiener–Hopf operators W_c^+ and W_c^- both have bounded inverses, then so do $W_c^+(n)$ and $W_c^-(n)$ if n is large enough.

THEOREM 1(a). *If $c \in WH(\mathscr{G})$, then there exists an integer N and a constant A both depending only upon c with the following property. If $n \geqslant N$ and if $f \in E^-(n)\, E^+(0)\, \mathscr{G}$, then*

$$||f|| \leqslant A||W_c^+(n)f||.$$

There is also a parallel result for $W_c^-(n)$.

Using this powerful tool and exploiting various special properties of the Beurling–Gelfand algebras $\mathfrak{G}$, Baxter created an extensive and detailed theory of generalized Szegö polynomials on T(see **1**; **2**; and also **5**).

Let Θ be a compact abelian topological group with dual Ξ on which there is a linear order relation "$<$" compatible with the group structure. For $\xi \in \Xi$ and $\theta \in \Theta$ we denote by (ξ, θ) the value of the character ξ at θ. $d\theta$ denotes Haar measure on Θ so normalized that Θ has measure 1. $\mathscr{A}_0$ is the class of those functions $f(\theta)$ of the form

$$f(\theta) = \textstyle\sum_\xi \mathbf{f}(\xi)(\xi, \theta)$$

for which $||f||_0$ is finite where

$$||f||_0 = \textstyle\sum_\xi |\mathbf{f}(\xi)|.$$

Note that

$$\mathbf{f}(\xi) = \textstyle\int_\Theta f(\theta)(-\xi, \theta)\, d\theta.$$

DEFINITION 1(b). *A Banach algebra $\mathscr{A}$ of complex functions on Θ is said to be of type $\mathfrak{S}$ if:*

1. *$\mathscr{A} \subset \mathscr{A}_0$, and $||f||_0 \leqslant ||f||$ for all $f \in \mathscr{A}$;*

2. *$(\xi, \theta) \in \mathscr{A}$ for every $\xi \in \Xi$, and finite linear combinations of (ξ, θ)'s are dense in $\mathscr{A}$;*

3. *$f \in \mathscr{A}$, $g \in \mathscr{A}_0$, and $|\mathbf{g}(\xi)| \leqslant |\mathbf{f}(\xi)|$ for all ξ implies $g \in \mathscr{A}$ and $||g|| \leqslant ||f||$.*

Henceforth every algebra $\mathscr{A}$ considered will be of type $\mathfrak{S}$.

Let us introduce the following families of operators:

$$E^+(\eta)f\cdot(\theta) = \sum_{\xi \geqslant \eta} \mathbf{f}(\xi)(\xi, \theta),$$

$$E^-(\eta)f\cdot(\theta) = \sum_{\xi \leqslant \eta} \mathbf{f}(\xi)(\xi, \theta).$$

It is apparent, using Property 3 of Definition 1(b), that for all $\eta \in \Xi$, $E^+(\eta)$ and $E^-(\eta)$ are linear operators on $\mathscr{A}$ (considered as a Banach space) of norm 1.

Let $c \in \mathscr{A}$. We define a linear operator W_c^+ on $E^+(0)\mathscr{A}$ by

$$W_c^+ f = E^+(0)cf, \qquad f \in E^+(0)\mathscr{A}.$$

Similarly

$$W_c^- f = E^-(0)cf, \qquad f \in E^-(0)\mathscr{A}.$$

W_c^+ and W_c^- are the Wiener–Hopf operators associated with c. We shall say that $c \in WH(\mathscr{A})$ if both W_c^+ and W_c^- have bounded inverses. We next introduce finite-section Wiener–Hopf operators. For $\eta \geqslant 0$ let

$$W_c^+(\eta) f = E^-(\eta)E^+(0)cf, \qquad f \in E^-(\eta)E^+(0)\mathscr{A},$$

and

$$W_c^-(\eta) f = E^+(-\eta)E^-(0)cf, \qquad f \in E^+(-\eta)E^-(0)\mathscr{A}.$$

$W_c^+(\eta)$ and $W_c^-(\eta)$ are bounded linear operators on the Banach spaces $E^-(\eta)E^+(0)\mathscr{A}$ and $E^+(-\eta)E^-(0)\mathscr{A}$ respectively.

We require some additional notation. For $f(\theta) \in \mathscr{A}$,

$$f(\theta) = \sum_\xi \mathbf{f}(\xi)(\xi, \theta),$$

let

$$f^\#(\theta) = \sum_\xi |\mathbf{f}(\xi)|(\xi, \theta).$$

It follows from 3 of Definition 1(b) that $f^\# \in \mathscr{A}$, and $||f^\#|| = ||f||$. We shall write

$$f \multimap g^\#$$

if $f, g \in \mathscr{A}$ and if

$$|\mathbf{f}(\xi)| \leqslant |\mathbf{g}(\xi)| \qquad \text{for all } \xi \in \Xi.$$

A basic result of the present paper is the following generalization of Theorem 1(a).

THEOREM 1(c). *Let $c(\theta) \in WH(\mathscr{A})$; there exists $\zeta_1 \geqslant 0$ in Ξ, and $C_+ = C_+^\#$ in $\mathscr{A}$, such that if $\eta \geqslant \zeta_1$ and if $f \in E^-(\eta)\,E^+(0)\,\mathscr{A}$, then*

a. $$f \multimap [W_c^+(\eta)f]^\# C_+^\#$$

and if $\eta \geqslant \zeta_1$, the range of $W_c^+(\eta)$ is $E^-(\eta)\,E^+(0)\,\mathscr{A}$. This implies in particular that $W_c^+(\eta)^{-1}$ exists and

b. $$||W_c^+(\eta)^{-1}|| \leqslant ||C_+^\#||.$$

There is a similar result associated with $W_c^-(\eta)$.

Using Theorem 1(c) we can extend Baxter's theory to the groups and Banach algebra described above.

2. Introduction—Szegö polynomials. In this section we shall give briefly and in outline the definition and some of the principal properties of the Szegö polynomials. We assume throughout that $c \in WH(\mathscr{A})$. For $\eta \geqslant \zeta_1$ we set

$$u(\eta, \theta) = W_c^+(\eta)^{-1}1, \qquad v(\eta, \theta) = W_c^-(\eta)^{-1}1.$$

By Theorem 1(c) $u(\eta, \theta)$ and $v(\eta, \theta)$ are well defined. Let

$$u(\eta, \theta) = \sum_{0 \leqslant \xi \leqslant \eta} \mathbf{u}(\eta, \xi)(\xi, \theta),$$

$$v(\eta, \theta) = \sum_{-\eta \leqslant \xi \leqslant 0} \mathbf{v}(\eta, \xi)(\xi, \theta).$$

A simple argument shows that $\mathbf{u}(\eta, 0) = \mathbf{v}(\eta, 0)$. For each $\eta \geqslant \zeta_1$ define $d(\eta)$ by

$$d(\eta)^2 = \mathbf{u}(\eta, 0) = \mathbf{v}(\eta, 0).$$

It follows from a second application of Theorem 1(c) that if

$$U(\theta) = (W_c^+)^{-1}1, \qquad V(\theta) = (W_c^-)^{-1}1,$$

then

$$\lim_{\eta \to \infty} ||U(\theta) - u(\eta, \theta)|| = \lim_{\eta \to \infty} ||V(\theta) - v(\eta, \theta)|| = 0.$$

This implies in particular that if $\zeta_2 \geqslant \zeta_1$ is sufficiently large, then

$$d(\eta) \neq 0, \qquad \eta \geqslant \zeta_2.$$

We now define the Szegö polynomials $\phi(\eta, \theta)$ and $\psi(\eta, \theta)$ for all $\eta \geqslant \zeta_2$ by the formulas

$$\phi(\eta, \theta) = d(\eta)^{-1}(\eta, \theta)v(\eta, \theta),$$

$$\psi(\eta, \theta) = d(\eta)^{-1}(-\eta, \theta)u(\eta, \theta).$$

Note that except in the case $\Xi = I$ the ϕ's and ψ's are *not* finite sums of characters. It follows almost immediately from the definition that if η_1, $\eta_2 \geqslant \zeta_2$, then

$$\int_\Theta \psi(\eta_1, \theta)\phi(\eta_2, \theta)c(\theta)\, d\theta = \begin{cases} 1, & \eta_1 = \eta_2, \\ 0, & \eta_1 \neq \eta_2, \end{cases}$$

so that the ϕ's and ψ's are biorthonormal. (If $c(\theta)$ is real, then

$$\psi(\eta, \theta) = \overline{\phi(\eta, \theta)}.)$$

Any attempt to represent functions in $E^+(0)\mathscr{A}$ as series of ϕ's, or functions in $E^-(0)\mathscr{A}$ as series of ψ's, must take into account the fact that the ϕ's and ψ's are defined only for $\xi \geqslant \zeta_2$. This is reflected in the statement of the following expansion theorem.

THEOREM 2(a). *There exists $\zeta_4 \geqslant \zeta_2$ in Ξ such that if $f(\theta) \in E^+(0)\mathscr{A}$, and if*

$$\int_\Theta f(\theta)(-\xi, \theta)c(\theta)\, d\theta = 0$$

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for all ξ, $0 \leqslant \xi \leqslant \zeta_4$, *then*

(1) $$f(\theta) = \sum_{\xi > \zeta_4} \mathbf{f}^{\sim}(\xi)\,\phi(\xi, \theta),$$

where

$$\mathbf{f}^{\sim}(\xi) = \int_{\Theta} f(\theta)\psi(\xi, \theta)c(\theta)\, d\theta.$$

Here the series (1) is defined as the limit in $\mathscr{A}$ of the net of sums over the finite subsets of $\{\xi | \xi > \zeta_4\}$.

There is, of course, a parallel result associated with the ψ's.

Using Theorem 2(a) we can derive several interesting structural relations involving the u's and v's. Let $\boldsymbol{\alpha}(\xi)$ and $\boldsymbol{\beta}(\xi)$ be defined by

$$\begin{aligned} \boldsymbol{\alpha}(\xi)d(\xi)^2 &= \mathbf{u}(\xi, \xi), \\ \boldsymbol{\beta}(\xi)d(\xi)^2 &= \mathbf{v}(\xi, -\xi), \end{aligned} \qquad \xi \geqslant \zeta_2.$$

Then for $\eta_2 > \eta_1 \geqslant \zeta_4$ we have

(2) $$\begin{aligned} u(\eta_2, \theta) - u(\eta_1, \theta) &= \sum_{\eta_1 < \xi \leqslant \eta_2} \boldsymbol{\alpha}(\xi) v(\xi, \theta)(\xi, \theta), \\ v(\eta_2, \theta) - v(\eta_1, \theta) &= \sum_{\eta_1 < \xi \leqslant \eta_2} \boldsymbol{\beta}(\xi) u(\xi, \theta)(-\xi, \theta). \end{aligned}$$

If $\Xi = I$ and if $\eta_2 = n$, $\eta_1 = n - 1$, these reduce to the recursion relations

(2′) $$\begin{aligned} u(n, \theta) - u(n-1, \theta) &= \boldsymbol{\alpha}(n) v(n, \theta) \exp(2\pi i n\theta), \\ v(n, \theta) - v(n-1, \theta) &= \boldsymbol{\beta}(n) u(n, \theta) \exp(-2\pi i n\theta), \end{aligned}$$

obtained in **(1)**. It is rather curious that whereas in the case $\Xi = I$ (2′) follows immediately from the definition of $u(n, \theta)$ and $v(n, \theta)$, in the general case (2) can be derived only after a great deal of previous work. As a final result we mention that starting from (2) it is possible to prove that

(3) $$\begin{bmatrix} u(\eta_1, \theta) \\ v(\eta_1, \theta) \end{bmatrix} = \prod_{\eta_1 < \xi \leqslant \eta_2} \begin{bmatrix} 1 & -\boldsymbol{\alpha}(\xi)(\xi, \theta) \\ -\boldsymbol{\beta}(\xi)(-\xi, \theta) & 1 \end{bmatrix} \begin{bmatrix} u(\eta_2, \theta) \\ u(\eta_1, \theta) \end{bmatrix}.$$

The formulas (2) and (3) show that the u's and v's (and therefore the ϕ's and ψ's) are essentially determined by the $\boldsymbol{\alpha}(\xi)$'s and $\boldsymbol{\beta}(\xi)$'s.

In conclusion it is a pleasure to express my indebtedness to Professor Baxter's very interesting work.

3. Wiener–Hopf equations. We assume henceforth that $\mathscr{A} \in \mathfrak{S}$. Let $c \in \mathscr{A}$. We recall that the operator W_c^+ on $E^+(0)\mathscr{A}$ is defined by

$$W_c^+ f = E^+(0)cf \qquad \text{for } f \in E^+(0)\mathscr{A}$$

and that the operator W_c^- on $E^-(0)A$ is defined by

$$W_c^- f = E^-(0)cf \qquad \text{for } f \in E^-(0)\mathscr{A}.$$

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Clearly $||W_c^+||, ||W_c^-|| \leqslant ||c||$. As before we write $c \in WH(\mathscr{A})$ if both W_c^+ and W_c^- have bounded inverses.

THEOREM 3(a). *$c \in WH(\mathscr{A})$ if and only if*

$$c(\theta) = d^2 U(\theta)^{-1} V(\theta)^{-1},$$

where $d \neq 0$ and where

$$U(\theta),\ U(\theta)^{-1} \in E^+(0)\mathscr{A}, \qquad V(\theta),\ V(\theta)^{-1} \in E^-(0)\mathscr{A}.$$

Proof. Necessity. By assumption there are functions $U(\theta) \in E^+(0)\mathscr{A}$ and $V(\theta) \in E^-(0)\mathscr{A}$ such that

$$W_c^+ U = 1, \qquad W_c^- V = 1.$$

For future use we introduce the notation

$$U(\theta) = \sum_{\xi \geqslant 0} \mathbf{U}(\xi)(\xi, \theta),$$

$$V(\theta) = \sum_{\xi \leqslant 0} \mathbf{V}(\xi)(\xi, \theta).$$

Let

$$V_1(\theta) = c(\theta) U(\theta), \qquad U_1(\theta) = c(\theta) V(\theta).$$

Then $U_1, V_1 \in \mathscr{A}$ and

$$V_1(\theta) = \sum_{\xi \leqslant 0} \mathbf{V}_1(\xi)(\xi, \theta), \qquad \mathbf{V}_1(0) = 1,$$

$$U_1(\theta) = \sum_{\xi \geqslant 0} \mathbf{U}_1(\xi)(\xi, \theta), \qquad \mathbf{U}_1(0) = 1.$$

We have

$$V(\theta) V_1(\theta) = c(\theta) U(\theta) V(\theta) = U(\theta) U_1(\theta).$$

Now $V(\theta) V_1(\theta)$ involves only (ξ, θ)'s with $\xi \leqslant 0$ and $U(\theta) U_1(\theta)$ only (ξ, θ)'s with $\xi \geqslant 0$. By the uniqueness of Fourier expansions, there exists a constant d such that

$$V(\theta) V_1(\theta) = d^2 = U(\theta) U_1(\theta). \tag{1}$$

We have $c(\theta) U(\theta) \overline{U_1(\theta)} = V_1(\theta) \overline{U_1(\theta)}$. Since

$$\int_\Theta V_1(\theta) \overline{U_1(\theta)}\, d\theta = 1,$$

it follows that $V_1(\theta)\overline{U_1(\theta)} \not\equiv 0$ and hence that $U(\theta)\overline{U_1(\theta)} \not\equiv 0$. Consequently, $U(\theta) U_1(\theta) \not\equiv 0$ and $d \neq 0$. Since $U(\theta)^{-1} = d^{-2} U_1(\theta)$, we have

$$U(\theta)^{-1} \in E^+(0)\mathscr{A},$$

and similarly $V(\theta)^{-1} \in E^-(0)\mathscr{A}$.

Sufficiency. We define operators X_c^+ and X_c^-, on $E^+(0)\mathscr{A}$ and $E^-(0)\mathscr{A}$ respectively, by the formulas

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$$X_c^+f\cdot(\theta) = d^{-2}U(\theta)E^+(0)\,f(\theta)\,V(\theta), \qquad f \in E^+(0)\mathscr{A},$$
$$X_c^-f\cdot(\theta) = d^{-2}V(\theta)E^-(0)\,f(\theta)\,U(\theta), \qquad f \in E^-(0)\mathscr{A}.$$

We claim that X_c^+ is the inverse of W_c^+ and X_c^- is the inverse of W_c^-; that is

$$W_c^+X_c^+ = X_c^+W_c^+ = I, \qquad W_c^-X_c^- = X_c^-W_c^- = I.$$

In addition to $E^+(\eta)$ and $E^-(\eta)$ defined in §1 we shall require the (related) projection operators

$$E^+(\eta+)f\cdot(\theta) = \sum_{\xi>\eta} \mathbf{f}(\xi)(\xi,\theta),$$

and

$$E^-(\eta-)f\cdot(\theta) = \sum_{\xi<\eta} \mathbf{f}(\xi)(\xi,\theta).$$

Consider for $f \in E^+(0)\mathscr{A}$,

$$\begin{aligned} X_c^+W_c^+f &= d^{-2}U(\theta)E^+(0)[V(\theta)W_c^+f\cdot(\theta)] \\ &= d^{-2}U(\theta)E^+(0)[V(\theta)E^+(0)c(\theta)f(\theta)] \\ &= U(\theta)E^+(0)[V(\theta)E^+(0)\,U(\theta)^{-1}V(\theta)^{-1}f(\theta)] \\ &= U(\theta)E^+(0)[V(\theta)\,U(\theta)^{-1}V(\theta)^{-1}f(\theta)] \\ &\qquad - U(\theta)E^+(0)[V(\theta)E^-(0-)\,U(\theta)^{-1}V(\theta)^{-1}f(\theta)] \\ &= f(\theta). \end{aligned}$$

The remaining relations can be verified similarly. It follows from **(1)** that

$$\mathbf{U}(0) = \mathbf{V}(0) = d^2, \tag{2}$$

a fact we shall need later.

4. A fundamental identity. Throughout the present section it will be convenient to regard $W_c^+(\eta)$ and $W_c^-(\eta)$ as operators defined on all of $\mathscr{A}$ by the formulas

$$W_c^+(\eta)f = E^+(0)E^-(\eta)cE^+(0)E^-(\eta)f,$$
$$W_c^-(\eta)f = E^-(0)E^+(-\eta)cE^-(0)E^+(-\eta)f.$$

The following identity was obtained by M. Shinbrot **(8)**; see also **(4)**. For the reader's convenience the proof is repeated here.

THEOREM 4(a). *Let* $c(\theta) = u(\theta)^{-1}v(\theta)^{-1}$ *where*

$$u(\theta) \in E^+(0)E^-(\zeta)\mathscr{A}, \qquad u(\theta)^{-1} \in E^+(0)\mathscr{A},$$
$$v(\theta) \in E^-(0)E^+(-\zeta)\mathscr{A}, \qquad v(\theta)^{-1} \in E^-(0)\mathscr{A}.$$

We set

$$Y_c^+(\xi)f = vE^-(\xi)uv^{-1}E^+(0)vf,$$
$$Y_c^-(\xi)f = uE^+(-\xi)vu^{-1}E^-(0)uf.$$

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Then for $\xi \geqslant \zeta \geqslant 0$ *we have*

(1) $$\begin{aligned} Y_c^+(\xi)W_c^+(\xi) &= W_c^+(\xi)Y_c^+(\xi) = E^+(0)E^-(\xi), \\ Y_c^-(\xi)W_c^-(\xi) &= W_c^-(\xi)Y_c^-(\xi) = E^-(0)E^+(-\xi). \end{aligned}$$

Proof. We first assert that

(2) $$\mathscr{R}[Y_c^+(\xi)] \subset E^+(0)E^-(\xi)\mathscr{A},$$

where $\mathscr{R}[Y_c^+(\xi)]$ is the range of $Y_c^+(\xi)$. Since $\mathscr{R}[vE^-(\xi)] \subset E^-(\xi)\mathscr{A}$, we see that $\mathscr{R}[Y_c^+(\xi)] \subset E^-(\xi)\mathscr{A}$. We have

$$\begin{aligned} Y_c^+(\xi) &= v[I - E^+(\xi+)]uv^{-1}E^+(0)v \\ &= uE^+(0)v - vE^+(\xi+)uv^{-1}E^+(0)v. \end{aligned}$$

It is evident that $\mathscr{R}[uE^+(0)] \subset E^+(0)\mathscr{A}$, and since $\xi \geqslant \zeta$

$$\mathscr{R}[vE^+(\xi+)] \subset E^+(0+)\mathscr{A} \subset E^+(0)\mathscr{A}.$$

Thus $\mathscr{R}[Y_c^+(\xi)] \subset E^+(0)\mathscr{A}$, and (2) holds.

We have

$$\begin{aligned} Y_c^+(\xi)W_c^+(\xi) &= vE^-(\xi)v^{-1}uE^+(0)vE^+(0)E^-(\xi)u^{-1}v^{-1}E^+(0)E^-(\xi) \\ &= vE^-(\xi)v^{-1}uE^+(0)v[I - E^-(0-)]E^-(\xi)u^{-1}v^{-1}E^+(0)E^-(\xi) \\ &= vE^-(\xi)v^{-1}uE^+(0)vE^-(\xi)u^{-1}v^{-1}E^+(0)E^-(\xi), \end{aligned}$$

since $E^+(0) = 0$ on $\mathscr{R}[vE^-(0-)]$. Similarly

$$Y_c^+(\xi)W_c^+(\xi) = vE^-(\xi)v^{-1}uE^+(0)v[I - E^+(\xi+)]u^{-1}v^{-1}E^+(0)E^-(\xi).$$

Now

$$\begin{aligned} &vE^-(\xi)v^{-1}uE^+(0)vE^+(\xi+)u^{-1}v^{-1}E^+(0)E^-(\xi) \\ &\quad = vE^-(\xi)v^{-1}uvE^+(\xi+)u^{-1}v^{-1}E^+(0)E^-(\xi) \\ &\quad = vE^-(\xi)uE^+(\xi+)u^{-1}v^{-1}E^+(0)E^-(\xi) = 0, \end{aligned}$$

where we have used $E^+(0) = I$ on $\mathscr{R}[vE^+(\xi+)]$ and $E^-(\xi) = 0$ on $\mathscr{R}[uE^+(\xi+)]$. Thus

$$\begin{aligned} Y_c^+(\xi)W_c^+(\xi) &= vE^-(\xi)v^{-1}uE^+(0)vu^{-1}v^{-1}E^+(0)E^-(\xi) \\ &= vE^-(\xi)v^{-1}uE^+(0)u^{-1}E^+(0)E^-(\xi) \\ &= vE^-(\xi)v^{-1}uu^{-1}E^+(0)E^-(\xi) \\ &= vE^-(\xi)v^{-1}E^+(0)E^-(\xi), \end{aligned}$$

since $E^+(0) = I$ on $\mathscr{R}[u^{-1}E^+(0)]$. Continuing

$$\begin{aligned} Y_c^+(\xi)W_c^+(\xi) &= v[I - E^+(\xi+)]v^{-1}E^+(0)E^-(\xi) \\ &= vv^{-1}E^+(0)E^-(\xi) - vE^+(\xi+)v^{-1}E^-(\xi)E^+(0) \\ &= E^+(0)E^-(\xi), \end{aligned}$$

since $E^+(\xi+) = 0$ on $\mathscr{R}[v^{-1}E^-(\xi)]$.

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On the other hand we find using (2) that

$$\begin{aligned} W_c^+(\xi)Y_c^+(\xi) &= E^+(0)E^-(\xi)u^{-1}v^{-1}E^+(0)E^-(\xi)vE^-(\xi)uv^{-1}E^+(0)v \\ &= E^+(0)E^-(\xi)u^{-1}v^{-1}vE^-(\xi)uv^{-1}E^+(0)v \\ &= E^+(0)E^-(\xi)u^{-1}E^-(\xi)uv^{-1}E^+(0)v \\ &= E^+(0)E^-(\xi)u^{-1}[I - E^+(\xi+)]uv^{-1}E^+(0)v \\ &= E^+(0)E^-(\xi)u^{-1}uv^{-1}E^+(0)v \\ &= E^+(0)E^-(\xi)v^{-1}E^+(0)v, \end{aligned}$$

since $E^-(\xi) = 0$ on $\mathscr{R}[u^{-1}E^+(\xi+)]$. Thus

$$\begin{aligned} W_c^+(\xi)Y_c^+(\xi) &= E^-(\xi)E^+(0)v^{-1}[I - E^-(0-)]v \\ &= E^+(0)E^-(\xi)v^{-1}v = E^+(0)E^-(\xi), \end{aligned}$$

since $E^+(0) = 0$ on $\mathscr{R}[v^{-1}E^-(0-)]$.

The second formula in (1) can be proved similarly.

5. Finite-section Wiener–Hopf equations. We shall need the following simple result.

LEMMA 5(a). *Let $\mathscr{B}$ be a not necessarily commutative Banach algebra with unit. If $S, T \in \mathscr{B}$ and if*

(i) *S^{-1} exists,*

(ii) $||T - S|| \leqslant \frac{1}{2}||S^{-1}||^{-1}$,

then T^{-1} exists and

$$||T^{-1} - S^{-1}|| \leqslant 2||S^{-1}||^2||T - S||.$$

Proof. See Hille and Phillips, *Functional analysis and semi-groups* (New York, 1957), p. 118.

We recall from §1 that for $f(\theta) \in \mathscr{A}$,

$$f(\theta) = \sum_\xi \mathbf{f}(\xi)(\xi, \theta),$$

we define

$$f(\theta)^\# = \sum_\xi |\mathbf{f}(\xi)|(\xi, \theta).$$

Then $f^\# \in \mathscr{A}$ and $||f^\#|| = ||f||$. We write

$$f \prec g^\#$$

if

$$|\mathbf{f}(\xi)| \leqslant |\mathbf{g}(\xi)| \qquad \text{for all } \xi \in \Xi.$$

LEMMA 5(b). *Let $f, a, b \in \mathscr{A}$. If*

(i) $f \prec a^\# f^\# + b^\#$,

(ii) $||a^\#|| < 1$,

then

$$f^\# \prec \left\{\sum_{j=0}^{\infty} (a^\#)^j\right\} b^\#.$$

Proof. Iterating (i) we find successively that

$$f \prec (a^{\#})^2 f^{\#} + a^{\#} b^{\#} + b^{\#},$$
$$f \prec (a^{\#})^3 f^{\#} + (a^{\#})^2 b^{\#} + a^{\#} b^{\#} + b^{\#},$$
$$\vdots$$
$$f \prec (a^{\#})^n f^{\#} + \left\{ \sum_{j=0}^{n-1} (a^{\#})^j \right\} b^{\#}.$$

If we now let $n \to \infty$ we obtain our desired result.

THEOREM 5(c). *Let $c(\theta) \in WH(\mathscr{A})$. Then there exists $\zeta_1 \geqslant 0$, $C_+ = C_+^{\#}$ in $\mathscr{A}$ and $C_- = C_-^{\#}$ in $\mathscr{A}$ such that if $\eta \geqslant \zeta_1$ then*

(1) $$f^{\#} \prec [W_c^+(\eta)f]^{\#} C_+ \qquad \text{for } f \in E^-(\eta)E^+(0)\mathscr{A},$$

(2) $$f^{\#} \prec [W_c^-(\eta)f]^{\#} C_- \qquad \text{for } f \in E^+(-\eta)E^-(0)\mathscr{A},$$

and if $\eta \geqslant \zeta_1$ the range of $W_c^+(\eta)$ is $E^-(\eta)E^+(0)\mathscr{A}$, and the range of $W_c^-(\eta)$ is $E^+(-\eta)E^-(0)\mathscr{A}$.

Proof. Choose M so that

$$M > \max\{d^{-1}||U||, d^{-1}||V||, d||U^{-1}||, d||V^{-1}||\}.$$

It follows from Lemma 5(a) that we can choose ζ_1 so large that if

$$u = d^{-1}E^-(\xi_1)U, \qquad v = d^{-1}E^+(-\xi_1)V,$$

then $u^{-1} \in E^+(0)\mathscr{A}$, $v^{-1} \in E^+(0)\mathscr{A}$, and

$$M \geqslant \max\{||u||, ||v||, ||u^{-1}||, ||v^{-1}||\},$$

and so large that if

$$c_1(\theta) = u(\theta)^{-1}v(\theta)^{-1},$$

then

$$||c - c_1|| \leqslant \tfrac{1}{2}M^{-4}.$$

Let us now regard $W_c^+(\eta)$ once again as an operator with domain

$$E^+(0)E^-(\eta)\mathscr{A}.$$

For $\eta \geqslant \zeta_1$ we know from Theorem 4(a) that $W_{c_1}(\eta)^{-1}$ exists and that

$$||W_{c_1}(\eta)^{-1}|| \leqslant M^4.$$

Since

$$||W_c^+(\eta) - W_{c_1}^+(\eta)|| \leqslant ||c - c_1|| \leqslant \tfrac{1}{2}M^{-4},$$

it follows from a second application of Lemma 5(a) that $W_c^+(\eta)^{-1}$ exists (and that $||W_c^+(\eta)^{-1}|| \leqslant 2M^4$). For $f \in E^+(0)E^-(\eta)\mathscr{A}$ let $g = W_c^+(\eta)f$. We have

$$W_{c_1}^+(\eta)f = [W_{c_1}^+(\eta) - W_c^+(\eta)]f + g,$$
$$f = Y_{c_1}^+(\eta)[W_{c_1}^+(\eta) - W_c^+(\eta)]f + Y_{c_1}^+(\eta)g,$$

which implies that

$$f \prec a^{\#}f^{\#} + b^{\#},$$

where

$$a^{\#} = v^{\#}(v^{-1})^{\#}u^{\#}v^{\#}[c_1 - c]^{\#},$$
$$b^{\#} = v^{\#}(v^{-1})^{\#}u^{\#}v^{\#}g^{\#}.$$

Since $||a^{\#}|| \leqslant \frac{1}{2}$, it follows from Lemma 5(b) that

$$f \prec \left\{\sum_{j=0}^{\infty} (a^{\#})^j\right\} b^{\#},$$

which we can rewrite as

$$f \prec C_+[W_c^+(\eta)f]^{\#},$$

where

$$C_+ = \left\{\sum_{0}^{\infty} (a^{\#})^j\right\} v^{\#}(v^{-1})^{\#}u^{\#}v^{\#}.$$

We have thus proved all of our assertions concerning $W_c^+(\eta)$. The same arguments suffice for $W_c^-(\eta)$.

In an earlier version of this paper a somewhat different proof was given for Theorem 5(c). It is interesting to note that using Lemma 5(b) and (with slight variations) the argument given by Baxter in **(3)**, it is possible to establish (1) and (2). However, since the dimension of $E^+(0)E^-(\eta)\mathscr{A}$ is infinite (except when $\Xi = I$), it does not follow that the range of $W_c^+(\eta)$ is all of $E^+(0)E^-(\eta)\mathscr{A}$, and further arguments are needed to take care of this point.

From this point on ζ_1 will always be the constant of Theorem 5(c).

COROLLARY 5(d). *Let $c(\theta) \in WH(\mathscr{A})$ and let $g(\theta) \in E^+(0)\mathscr{A}$. For each $\eta \geqslant \zeta_1$ let*

$$f(\eta, \theta) = \{W_c^+(\eta)\}^{-1}E^-(\eta)g(\theta), \qquad f(\theta) = \{W_c^+\}^{-1}g(\theta);$$

then

$$f(\eta, \theta) \prec C_+(\theta)^{\#}g(\theta)^{\#}, \tag{3}$$

$$\lim_{\eta \to +\infty} ||f(\theta) - f(\eta, \theta)|| = 0. \tag{4}$$

There is also a parallel result for $W_c^-(\eta)$.

Proof. Theorem 5(b) implies that for all $\eta \geqslant \zeta_1$

$$f(\eta, \theta) \prec C_+(\theta)^{\#}[E^-(\eta)g(\theta)]^{\#} \prec C_+(\theta)^{\#}g(\theta)^{\#},$$

which gives (3). We have

$$\begin{aligned} W_c^+(\eta)E^-(\eta)f\cdot(\theta) &= E^-(\eta)E^+(0)[c(\theta)E^-(\eta)f(\theta)] \\ &= E^-(\eta)g(\theta) - E^-(\eta)E^+(0)[c(\theta)E^+(\eta+)f(\theta)]. \end{aligned}$$

Thus

$$W_c^+(\eta)[f(\eta, \theta) - E^-(\eta)f(\theta)] = E^-(\eta)E^+(0)[c(\theta)E^+(\eta+)f(\theta)].$$

It follows from Theorem 5(c) that

$$||f(\eta, \theta) - E^-(\eta)f(\theta)|| \leqslant ||C_+^{\#}|| \, ||c|| \, ||E^+(\eta, +)f||$$

and consequently

$$(5) \qquad \begin{aligned} ||f(\eta, \theta) - f(\theta)|| &\leqslant ||f(\eta, \theta) - E^-(\eta)f(\theta)|| + ||E^+(\eta+)f(\theta)|| \\ &\leqslant \{1 + ||C_+^{\#}|| \, ||c||\}||E^+(\eta+)f||, \end{aligned}$$

from which (4) follows.

6. Szegö polynomials. Henceforth we shall always assume that $c \in WH(\mathscr{A})$ where $\mathscr{A}$ is an algebra of type $\mathfrak{S}$. Let ζ_1, C_+, and C_- be as in Theorem 5(c). For $\eta \geqslant \zeta_1$ we define

$$(1) \qquad u(\eta, \theta) = W_c^+(\eta)^{-1}1,$$

$$(2) \qquad v(\eta, \theta) = W_c^-(\eta)^{-1}1.$$

We have

$$(3) \qquad \begin{aligned} u(\eta, \theta) &= \sum_{0 \leqslant \xi \leqslant \eta} \mathbf{u}(\eta, \xi)(\xi, \theta), \\ v(\eta, \theta) &= \sum_{-\eta \leqslant \xi \leqslant 0} \mathbf{v}(\eta, \xi)(\xi, \theta). \end{aligned}$$

LEMMA 6(a). *If* $\eta \geqslant \zeta_1$, *then*

$$\int_\Theta u(\eta, \theta)v(\eta, \theta)c(\theta)\, d\theta = \mathbf{u}(\eta, 0) = \mathbf{v}(\eta, 0).$$

Proof. Let us write $(a, b]$ for any function $g \in \mathscr{A}$ for which $\sigma(g) \subset (a, b]$, etc. This notation is convenient in that it makes it unnecessary to name irrelevant terms. By assumption

$$v(\eta, \theta)c(\theta) = (-\infty, -\eta) + 1 + (0, \infty)$$

and thus

$$u(\eta, \theta)v(\eta, \theta)c(\theta) = (-\infty, 0) + \mathbf{u}(\eta, 0) + (0, \infty).$$

This implies that

$$\int_\Theta u(\eta, \theta)v(\eta, \theta)c(\theta)\, d\theta = \mathbf{u}(\eta, 0),$$

etc.

It follows from Corollary 5(d) applied to $g(\theta) = 1$ that

$$|\mathbf{U}(0) - \mathbf{u}(\eta, 0)| \leqslant ||U(\theta) - u(\eta, \theta)|| \to 0 \qquad \text{as } \eta \to \infty,$$

$$|\mathbf{V}(0) - \mathbf{v}(\eta, 0)| \leqslant ||V(\theta) - v(\eta, \theta)|| \to 0 \qquad \text{as } \eta \to \infty.$$

For $\eta > \zeta_1$ we define $d(\eta)$ by $d(\eta)^2 = \mathbf{u}(\eta, 0) = \mathbf{v}(\eta, 0)$. We may suppose that

$$(4) \qquad \lim_{\eta \to \infty} d(\eta) = d$$

where $d^2 = \mathbf{U}(0) = \mathbf{V}(0)$. We choose $\zeta_2 > \zeta_1$ so that

$$(4') \qquad d(\eta) \neq 0 \qquad \text{for } \eta \geqslant \zeta_2.$$

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We can now define the (generalized) Szegö polynomials associated with c. For $\eta \geqslant \zeta_2$ let

$$\phi(\eta, \theta) = d(\eta)^{-1}(\eta, \theta)v(\eta, \theta), \tag{5}$$

$$\psi(\eta, \theta) = d(\eta)^{-1}(-\eta, \theta)u(\eta, \theta). \tag{6}$$

Note that despite their name, $\phi(\eta, \theta)$ and $\psi(\eta, \theta)$ are not (except in the case $\Theta = T$ and $\Xi = I$) polynomials, that is finite linear combinations of characters. In general they are infinite linear combinations of characters.

It is apparent that

$$\phi(\eta, \theta) \in E^-(\eta)E^+(0)\mathscr{A} \quad \text{and} \quad \psi(\eta, \theta) \in E^+(-\eta)E^-(0)\mathscr{A}.$$

The basic property of the polynomials (5) and (6) is that they are biorthonormal.

THEOREM 6(b). *If* $\eta_1, \eta_2 \geqslant \zeta_2$ *and if* $\delta(\eta_1, \eta_2)$ *is* 1 *or* 0 *as* $\eta_1 = \eta_2$ *or* $\eta_1 \neq \eta_2$, *then*

$$\int_\Theta \phi(\eta_1, \theta)\psi(\eta_2, \theta)c(\theta)\, d\theta = \delta(\eta_1, \eta_2). \tag{7}$$

Proof. The case $\eta_1 = \eta_2$ follows from Lemma 6(a). Suppose next that $\eta_1 < \eta_2$. We have

$$\begin{aligned}\psi(\eta_2, \theta)c(\theta) &= d(\eta_2)^{-1}(-\eta_2, \theta)c(\theta)u(\eta_2, \theta)\\ &= d(\eta_2)^{-1}(-\eta_2, \theta)[(-\infty, 0) + 1 + (\eta_2, \infty)]\\ &= (-\infty, -\eta_2] + (0, \infty),\end{aligned}$$

and thus

$$\phi(\eta_1, \theta)\psi(\eta_2, \theta)c(\theta) = (-\infty, \eta_1 - \eta_2] + (0, \infty)$$

so that (7) holds in this case. The case $\eta_1 > \eta_2$ is similar.

We now proceed to the discussion of the expansion of functions

$$f(\theta) \in E^+(0)\mathscr{A}$$

in terms of the ϕ's. This will occupy §§ 6–8. For $\xi \geqslant \zeta_2$ set

$$\mathbf{f}^\sim(\xi) = \int_\Theta f(\theta)\psi(\xi, \theta)c(\theta)\, d\theta.$$

For "large" η we would *like* to define an operator $R_+(\eta)$ by the formula

$$R_+(\eta)f\cdot(\theta) = \sum_{\xi > \eta} \mathbf{f}^\sim(\xi)\phi(\xi, \theta). \tag{8}$$

However, we must first discuss infinite series of the sort appearing on the right-hand side of this relation.

Let $\mathscr{E}$ be a complete separated linear topological space over the complex numbers. For M any abstract set consider the (formal) series

$$\sum_{\mu \in M} \omega(\mu), \tag{9}$$

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where $\omega(\mu) \in \mathscr{E}$ for each $\mu \in M$. Let $F(M)$ be the family of all finite subsets m of M ordered by inclusion. The definition of (9) is that it is the limit in $\mathscr{E}$ of the net

$$\sum_{\mu \in m} \omega(\mu), \qquad m \in F(M),$$

if this limit exists.

LEMMA 6(c). *Let $\mathscr{A}$ be of type $\mathfrak{S}$. If for each $\mu \in M$, $\omega(\mu, \theta) \in \mathscr{A}$, and if there exists $w = w^{\#}$* in *$\mathscr{A}$ such that for every $m \in F(M)$*

$$\sum_{\mu \in m} [\omega(\mu, \theta)]^{\#} \prec w^{\#}(\theta), \tag{10}$$

then

$$\sum_{\mu \in M} \omega(\mu, \theta)$$

is convergent.

Proof. Let $b(\mu)$ be a complex number for each $\mu \in M$. If the sums

$$\sum_{\mu \in m} |b(\mu)|$$

have a finite upper bound independent of m, then since the complex numbers are complete, $\sum_M b(\mu)$ is a well-defined complex number. By (10) for each $\xi \in \Xi$ we have

$$\sum_{\mu \in m} |\omega(\mu, \xi)| \leqslant \mathbf{w}^{\#}(\xi),$$

and therefore for each $\xi \in \Xi$ the series of complex numbers

$$\sum_{\mu \in M} \boldsymbol{\omega}(\mu, \xi) \tag{11}$$

is convergent. Given $\epsilon > 0$, let Ξ' be a finite subset of Ξ such that if

$$\Xi'' = \Xi - \Xi',$$

then

$$\left\| \sum_{\Xi''} \mathbf{w}^{\#}(\xi)(\xi, \theta) \right\| \leqslant \tfrac{1}{2}\epsilon. \tag{12}$$

By definition

$$\begin{aligned} &\left\| \sum_{\mu \in m_1} \omega(\mu, \theta) - \sum_{\mu \in m_2} \omega(\mu, \theta) \right\| \\ &\qquad = \left\| \sum_{\Xi} (\xi, \theta) \left\{ \sum_{\mu \in m_1} \boldsymbol{\omega}(\mu, \xi) - \sum_{\mu \in m_2} \boldsymbol{\omega}(\mu, \xi) \right\} \right\| \\ &\qquad \leqslant \left\| \sum_{\Xi'} (\xi, \theta)\{\ldots\} \right\| + \left\| \sum_{\Xi''} (\xi, \theta)\{\ldots\} \right\|. \end{aligned}$$

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Using (11) we see that

$$\lim_{m_1, m_2} \left\| \sum_{\Xi'} (\xi, \theta)\{\ldots\} \right\| = 0,$$

while by (10) and (12)

$$\left\| \sum_{\Xi''} (\xi, \theta)\{\ldots\} \right\| \leqslant \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon$$

independently of m_1 and m_2, etc.

Note that

$$[\sum_M \omega(\mu, \theta)]^\# \prec w^\#(\theta) \tag{13}$$

is also valid.

Let us choose $\zeta_3 \geqslant \zeta_2$ so large that for some constant D, $0 < D < \infty$,

$$\inf_{\eta \geqslant \zeta_3} |d(\eta)| \geqslant D^{-1} > 0. \tag{14}$$

THEOREM 6(d). *Let $f \in E^+(0)\mathscr{A}$. If $\eta \geqslant \zeta_3$, then the series*

$$\sum_{\xi > \eta} \mathbf{f}^\sim(\xi)\phi(\xi, \theta) = R_+(\eta)f\cdot(\theta)$$

is convergent, and $||R_+(\eta)|| \leqslant D^2||C_+^\#||\ ||C_-^\#||\ ||c||$.

Proof. We have

$$\mathbf{f}^\sim(\xi) = \int_\Theta f(\theta)\psi(\xi, \theta)c(\theta)\, d\theta,$$

$$\mathbf{f}^\sim(\xi) = d(\xi)^{-1}\int_\Theta f(\theta)u(\xi, \theta)(-\xi, \theta)c(\theta)\, d\theta.$$

By Corollary 5(d) applied to the function 1

$$u(\eta, \theta)^\# \prec C_+(\theta)^\#, \tag{15$'$}$$

and

$$v(\eta, \theta)^\# \prec C_-(\theta)^\#. \tag{15$''$}$$

Using (15′) we see that

$$|\mathbf{f}^\sim(\xi)| \leqslant \mathbf{F}(\xi)$$

where $F = F^\# = Df^\# c^\# C_+^\#$. Again

$$\phi(\xi, \theta) = d(\xi)^{-1}v(\xi, \theta)(\xi, \theta),$$

and using (15″) we find that

$$[\phi(\xi, \theta)]^\# \prec DC_-(\theta)^\#(\xi, \theta).$$

Combining these results we find that if m is any finite subset of $\{\xi | \xi > \eta\}$, then

$$\sum_{\xi \in m} [\mathbf{f}^\sim(\xi)\phi(\xi, \theta)]^\# \prec D^2 f(\theta)^\# C_+(\theta)^\# C_-(\theta)^\# c(\theta)^\#.$$

We can now apply Lemma 6(c).

In what follows we are going to show that in some sense functions

$$f(\theta) \in E^{+}(0)\mathscr{A}$$

can be expanded in terms of the $\phi(\xi, \theta)$'s. However, our theory must take a somewhat oblique form since the $\phi(\xi, \theta)$'s cannot be assumed to exist for small ξ. Suppose we did have

$$f(\theta) = \sum_{\xi \geqslant 0} \mathbf{f}^{\sim}(\xi)\phi(\xi, \theta);$$

this would imply that

$$E^{+}(\eta+)f\cdot(\theta) = E^{+}(\eta+) \sum_{\xi > \eta} \mathbf{f}^{\sim}(\xi)\phi(\xi, \theta),$$

$$E^{+}(\eta+)f = E^{+}(\eta+)R_{+}(\eta)f, \tag{16}$$

where this formula has the advantage that the right-hand side is defined for $\xi > \zeta_3$. In the next section we shall show that if $\eta > \zeta_4$, where $\zeta_4 \geqslant \zeta_3$ depends only on c, then (16) does in fact hold.

7. The expansion theorem. We proceed to carry out the programme described at the end of §6.

Lemma 7(a). *If* $\xi > \xi_1 \geqslant 0$, $\xi \geqslant \zeta_2$, *then*

$$\int_{\Theta} (\xi_1, \theta)\psi(\xi, \theta)c(\theta)d\theta = 0,$$

$$\int_{\Theta} (-\xi_1, \theta)\phi(\xi, \theta)c(\theta)d\theta = 0.$$

Proof. The integrand of the first integral above is equal to

$$d(\xi)^{-1}(\xi_1, \theta)(-\xi, \theta)u(\xi, \theta)c(\theta) = (\xi_1 - \xi, \theta)[(-\infty, 0) + d(\xi)^{-1} + (\xi, \infty)],$$

etc.

Lemma 7(b). *If* $\eta \geqslant \zeta_3$, *then*

$$E^{+}(\eta+)R_{+}(\eta)E^{+}(\eta+) = E^{+}(\eta+)R^{+}(\eta).$$

Proof. It is enough to show that if $\xi > \eta$, then

$$\int_{\Theta} f(\theta)\psi(\xi, \theta)c(\theta)d\theta = \int_{\Theta} [E^{+}(\eta+)f\cdot(\theta)]\psi(\xi, \theta)c(\theta)d\theta,$$

or equivalently

$$\int_{\Theta}[E^{-}(\eta)f\cdot(\theta)]\psi(\xi, \theta)c(\theta)d\theta = 0.$$

This, however, follows from Lemma 7a.

Lemma 7(c). *If* $\eta > \zeta_3$, *then*

$$E^{+}(\eta+)R_{+}(\eta)E^{+}(\eta+)R_{+}(\eta) = E^{+}(\eta+)R_{+}(\eta).$$

Proof. By Lemma 7(b) we have

$$E^{+}(\eta+)R_{+}(\eta)E^{+}(\eta+)R_{+}(\eta) = E^{+}(\eta+)R_{+}(\eta)R_{+}(\eta).$$

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Let $f \in E^+(0)\mathscr{A}$, and let

$$g = R_+(\eta)f = \sum_{\xi > \eta} \phi(\xi, \theta)\mathbf{f}^\sim(\xi)$$

where the series on the right converges in $\mathscr{A}$, and therefore a fortiori converges uniformly. It follows from Theorem 6(b) that if $\xi > \eta \geqslant \zeta_3$, then

$$\mathbf{g}^\sim(\xi) = \mathbf{f}^\sim(\xi).$$

Our assertion is an immediate consequence of this.

Let Γ be a directed set with elements γ, γ', etc.

LEMMA 7(d). *If*
(i) $r(\theta) \in \mathscr{A}$, $r(\gamma, \theta) \in \mathscr{A}$ *for* $\gamma \in \Gamma$,
(ii) $r(\gamma, \theta) \prec s^\# $ *for all* $\gamma \in \Gamma$ *where* $s = s^\# \in \mathscr{A}$,
(iii) $\lim_\gamma r(\gamma, \theta) = r(\theta)$ *in* $\mathscr{A}$,
then given $\epsilon > 0$ *there exists* $\gamma' \in \Gamma$ *and* $t = t^\#$ *in* $\mathscr{A}$ *such that* $\|t\| \leqslant \epsilon$ *and*

$$[r(\theta) - r(\gamma, \theta)]^\# \prec t^\#(\theta) \qquad \textit{for } \gamma \geqslant \gamma'.$$

Proof. We can choose a finite subset Ξ' of Ξ such that if $\Xi'' = \Xi - \Xi'$, then

$$\|\textstyle\sum_{\Xi''}\mathbf{s}(\xi)(\xi, \theta)\| \leqslant \epsilon/4.$$

Let

$$N = \|\textstyle\sum_{\Xi'}(\xi, \theta)\|,$$

and set

$$\mathbf{t}(\xi) = \begin{cases} \epsilon/2N, & \xi \in \Xi', \\ 2\mathbf{s}(\xi), & \xi \in \Xi''. \end{cases}$$

Then $t = t^\# \in \mathscr{A}$ and $\|t\| \leqslant \epsilon$ since

$$\|t\| \leqslant \frac{\epsilon}{2N}\left\|\sum_{\Xi'} (\xi, \theta)\right\| + 2\left\| \sum_{\Xi''} \mathbf{s}(\xi)(\xi, \theta)\right\|,$$
$$\leqslant \frac{\epsilon}{2} + 2\frac{\epsilon}{4} = \epsilon.$$

Choose γ' so large that

$$\|r(\theta) - r(\gamma, \theta)\| \leqslant \epsilon/2N \qquad \text{for } \gamma \geqslant \gamma'.$$

If $\gamma \geqslant \gamma'$ and if $\xi \in \Xi'$, we have

$$|\mathbf{r}(\xi) - \mathbf{r}(\gamma, \xi)| \leqslant \|r(\theta) - r(\gamma, \theta)\| \leqslant \epsilon/2N = \mathbf{t}(\xi),$$

while if $\xi \in \Xi''$, we have

$$|\mathbf{r}(\xi) - \mathbf{r}(\gamma, \xi)| \leqslant 2\mathbf{s}(\xi) = \mathbf{t}(\xi).$$

THEOREM 7(e). *There exists* $\zeta_4 \geqslant \zeta_3$ *such that if* $f(\theta) \in E^+(0)\mathscr{A}$, *then*

$$E^+(\eta+)f(\theta) = E^+(\eta+)R_+(\eta)f\cdot(\theta) \qquad \textit{for } \eta \geqslant \zeta_4, \tag{1}$$

and if $g(\theta) \in E^-(0)\mathscr{A}$,

$$E^-(-\eta-)g\cdot(\theta) = E^-(-\eta-)R_-(\eta)g\cdot(\theta) \qquad \textit{for } \eta \geqslant \zeta_4. \tag{2}$$

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Here $R_-(\eta)$ is defined in the obvious way with the roles of the ϕ's and ψ's interchanged.

Proof. We shall prove only (1) since the proof of (2) is exactly the same. Let us set

$$I(\eta) = E^+(\eta+)|_{E^+(\eta+)\mathscr{A}}, \qquad Q(\eta) = E^+(\eta+)R(\eta+)|_{E^+(\eta+)\mathscr{A}};$$

that is, $I(\eta)$ is the restriction of $E^+(\eta+)$ to $E^+(\eta+)\mathscr{A}$, etc. $I(\eta)$ is the identity operator on $E^+(\eta+)\mathscr{A}$ and $Q(\eta)$ is an indempotent operator. We shall show that if ζ_4 is large enough, then

$$||I(\eta) - Q(\eta)|| < 1 \qquad \text{for } \eta > \zeta_4.$$

Since

$$[I(\eta) - Q(\eta)]^n = I(\eta) - Q(\eta),$$

it follows that

$$||I(\eta) - Q(\eta)|| \leqslant ||I(\eta) - Q(\eta)||^n,$$

and since n can be taken arbitrarily large this implies that $I(\eta) = Q(\eta)$, which is what we want to show.

Using $c = d^2/UV$ we see that $I(\eta)$ can be written in the form

$$I(\eta)f = E^+(\eta+) \sum_{\xi>\eta} d^{-2}V(\theta)(\xi, \theta) \int_\Theta f(\phi)(-\xi, \phi)U(\phi)c(\phi)d\phi,$$

where $f \in E^+(\eta+)\mathscr{A}$. Let

$$P(\eta) = E^+(\eta+) \sum_{\xi>\eta} d^{-1}d(\xi)^{-1}V(\theta)(\xi, \theta) \int_\Theta f(\phi)(-\xi, \phi)u(\xi, \phi)c(\phi)d\phi$$

for $f \in E^+(\eta+)\mathscr{A}$. Consider

$$[Q(\eta) - P(\eta)]f = E^+(\eta+)d^{-1} \sum_{\xi>\eta} [d^{-1}V(\theta) - d(\xi)^{-1}v(\xi, \theta)]\mathbf{f}^\sim(\xi)(\xi, \theta).$$

By Corollary 5(d) and by (4) of §6 we see that

$$\lim_{\xi\to+\infty} ||d^{-1}V(\theta) - d(\xi)^{-1}v(\xi, \theta)|| = 0,$$

and also that if $\xi \geqslant \zeta_3$

$$d^{-1}V(\theta) - d(\xi)^{-1}v(\xi, \theta) \prec 2DC_-^{\#}(\theta).$$

By Lemma 7(d), given $\epsilon \in 0$ there exists $\omega \geqslant \zeta_3$ and $t = t^{\#}$ in $\mathscr{A}$ such that $||t|| < \epsilon$ and

$$d^{-1}V(\theta) - d(\xi)^{-1}v(\xi, \theta) \prec t \qquad \text{if } \xi \geqslant \omega.$$

Using the estimate for $\mathbf{f}^\sim(\xi)$ given in the proof of Theorem 6(d), we see that if $\eta \geqslant \omega$

$$[Q(\eta) - P(\eta)]f \prec Dc^{\#}C_+^{\#}f^{\#}t^{\#},$$

$$||[Q(\eta) - P(\eta)]f|| \leqslant \epsilon||Dc^{\#}C_+^{\#}||\ ||f||.$$

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Since ϵ is arbitrary, this implies that

$$\lim_{\eta\to+\infty} ||Q(\eta) - P(\eta)|| = 0.$$

Similarly consider

$$[P(\eta) - I(\eta)]f = E^{+}(\eta+) \sum_{\xi>\eta} d^{-1}V(\theta)(\xi, \theta) \int_{\Theta} [d(\xi)^{-1}u(\xi, \phi) - d^{-1}U(\phi)](-\xi, \phi)f(\phi)c(\phi)d\phi.$$

By estimates altogether similar to those used above we can show that

$$\lim_{\eta\to+\infty} ||P(\eta) - I(\eta)|| = 0.$$

Choose ζ_4 so large that if $\eta \geqslant \zeta_4$

$$||Q(\eta) - P(\eta)|| < \tfrac{1}{2}, \qquad ||P(\eta) - I(\eta)|| < \tfrac{1}{2};$$

then

$$||I(\eta) - Q(\eta)|| < 1 \qquad \text{for } \eta \geqslant \zeta_4.$$

But this implies that $Q(\eta) = I(\eta)$ and our proof is complete.

8. The Christoffel–Darboux formula. In this section we shall show that if we define the operator $S_{+}(\eta)$ on $E^{+}(0)\mathscr{A}$ by the formula

(1) $$S_{+}(\eta) = I - R_{+}(\eta),$$

then $S_{+}(\eta)$ can be given a simple explicit closed form. For $\eta \geqslant \zeta_2$ let us set, for $f \in E^{+}(0)\mathscr{A}$,

(2) $$S'_{+}(\eta)f(\theta) = d(\eta)^{-2}[u(\eta, \theta)E^{+}(0)\{f(\theta)c(\theta)v(\eta, \theta)\} - v(\eta, \theta)E^{+}(\eta+)\{c(\theta)u(\eta, \theta)f(\theta)\}].$$

We shall show that $S_{+}(\eta) = S'_{+}(\eta)$ if $\eta \geqslant \zeta_2$. We remind the reader that we are assuming throughout that $\mathscr{A}$ is of type $\mathfrak{S}$ and that $c \in WH(\mathscr{A})$.

THEOREM 8(a). *Let $\eta \geqslant \zeta_2$. If $0 \leqslant \xi \leqslant \eta$, then*

(3) $$S'_{+}(\eta)(\xi, \theta) = (\xi, \theta),$$

and if $\xi > \eta$, then

(4) $$S'_{+}(\eta)\phi(\xi, \theta) = 0.$$

Proof. Note that (3) implies that $S'_{+}(\eta)\phi(\xi, \theta) = \phi(\xi, \theta)$ for any $\xi \leqslant \eta$ for which $\phi(\xi, \theta)$ is defined. Since

$$c(\theta)v(\eta, \theta) = (-\infty, -\eta) + 1 + (0, \infty),$$

if $0 \leqslant \xi \leqslant \eta$, we have

$$E^{+}(0)\{(\xi, \theta)c(\theta)v(\eta, \theta)\} = (\xi, \theta)E^{+}(0)\{c(\theta)v(\eta, \theta)\},$$

and

$$u(\eta, \theta)E^{+}(0)\{c(\theta)v(\eta, \theta)\} = E^{+}(0)\{c(\theta)u(\eta, \theta)v(\eta, \theta)\}.$$

Thus

$$d(\eta)^{-2}u(\eta,\theta)E^{+}(0)\{(\xi,\theta)c(\theta)v(\eta,\theta)\} = (\xi,\theta)E^{+}(0)\{d(\eta)^{-2}u(\eta,\theta)v(\eta,\theta)c(\theta)\}.$$

Similarly, since $c(\theta)u(\eta,\theta) = (-\infty, 0) + 1 + (\eta, \infty)$, we have

$$E^{+}(\eta+)\{(\xi,\theta)c(\theta)u(\eta,\theta)\} = (\xi,\theta)E^{+}(0)\{c(\theta)u(\eta,\theta) - 1\}$$

and

$$v(\eta,\theta)E^{+}(0)\{c(\theta)u(\eta,\theta) - 1\} = E^{+}(0)\{c(\theta)u(\eta,\theta)v(\eta,\theta) - v(\eta,\theta)\}.$$

It follows that

$$d(\eta)^{-2}v(\eta,\theta)E^{+}(\eta+)\{(\xi,\theta)c(\theta)u(\eta,\theta)\}$$
$$= (\xi,\theta)E^{+}(0)d(\eta)^{-2}\{c(\theta)u(\eta,\theta)v(\eta,\theta) - v(\eta,\theta\}.$$

Combining these results, we find that

$$S'_{+}(\eta)(\xi,\theta) = (\xi,\theta)E^{+}(0)\{d(\eta)^{-2}v(\eta,\theta)\} = (\xi,\theta),$$

and we have proved (3).

Suppose that $\xi > \eta$. Since $\phi(\xi,\theta)c(\theta) = (-\infty, 0) + [\xi, \infty)$, we find that

$$E^{+}(0)\{\phi(\xi,\theta)c(\theta)v(\eta,\theta)\} = v(\eta,\theta)E^{+}(0)\{\phi(\xi,\theta)c(\theta)\},$$

so that

$$d(\eta)^{-2}u(\eta,\theta)E^{+}(0)\{\phi(\xi,\theta)c(\theta)v(\eta,\theta)\} = d(\eta)^{-2}u(\eta,\theta)v(\eta,\theta)E^{+}(0)\{\phi(\xi,\theta)c(\theta)\}.$$

Again

$$E^{+}(\eta+)\{\phi(\xi,\theta)c(\theta)u(\eta,\theta)\} = u(\eta,\theta)E^{+}(0)\{\phi(\xi,\theta)c(\theta)\},$$

and hence

$$d(\eta)^{-2}v(\eta,\theta)E^{+}(\eta+)\{\phi(\xi,\theta)c(\theta)u(\eta,\theta)\} = d(\eta)^{-2}u(\eta,\theta)v(\eta,\theta)E^{+}(0)\{\phi(\xi,\theta)c(\theta)\}.$$

From these computations (4) follows immediately.

THEOREM 8(b). *If* $\eta \geqslant \zeta_4$, *then* $S_{+}(\eta) = S'_{+}(\eta)$.

Proof. By Theorem 7(e) *if* $\eta \geqslant \zeta_4$ *and if* $f \in E^{+}(0)\mathscr{A}$,

$$f(\theta) = \sum_{\xi>\eta} \mathbf{f}^{\sim}(\xi)\phi(\xi,\theta) + g(\theta),$$

where $g(\theta) \in E^{-}(\eta)E^{+}(0)\mathscr{A}$. Applying $S'_{+}(\eta)$ and using Theorem 8(a) we find that

$$S'_{+}(\eta)f\cdot(\theta) = g(\theta) = [I - R_{+}(\eta)]f\cdot(\theta) = S_{+}(\eta)f\cdot(\theta).$$

Note. It is easy to show directly from the definition that

$$\lim_{\eta} S'_{+}(\eta)f\cdot(\theta) = f(\theta) \qquad \text{for every } f \in E^{+}(0)\mathscr{A};$$

see (5). However, we do not need this fact in the present situation.

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9. Recursion formulas. For $\eta \geqslant \zeta_3$ we define $\boldsymbol{\alpha}(\eta)$ and $\boldsymbol{\beta}(\eta)$ by the formulas

$$(1)\qquad \begin{aligned} \boldsymbol{\alpha}(\eta)d(\eta) &= \int_\Theta U(\theta)\psi(\eta, \theta)c(\theta)d\theta, \\ \boldsymbol{\beta}(\eta)d(\eta) &= \int_\Theta V(\theta)\phi(\eta, \theta)c(\theta)d\theta. \end{aligned}$$

LEMMA 9(a). *If $\eta \geqslant \zeta_4$, then*

$$(2)\qquad \begin{aligned} S_+(\eta)U(\theta) &= u(\eta, \theta), \\ S_-(\eta)V(\theta) &= v(\eta, \theta). \end{aligned}$$

Proof. By Theorem 8(b)

$$S_+(\eta)U(\theta) = d(\eta)^{-2}u(\eta, \theta)E^+(0)\{U(\theta)c(\theta)v(\eta, \theta)\} - d(\eta)^{-2}v(\eta, \theta)E^+(\eta+)\{U(\theta)c(\theta)u(\eta, \theta)\}.$$

Using $U(\theta)c(\theta) = (-\infty, 0) + 1$, we see that

$$\begin{aligned} E^+(0)\{U(\theta)c(\theta)v(\eta, \theta)\} &= \mathbf{v}(\eta, 0) = d(\eta)^2, \\ E^+(\eta+)\{U(\theta)c(\theta)u(\eta, \theta)\} &= 0, \end{aligned}$$

from which the first formula of (2) follows, etc.

The following result is the basic recursion formula for the $u(\eta, \theta)$'s and $v(\eta, \theta)$'s.

THEOREM 9(b). *If $\eta_2 > \eta_1 \geqslant \zeta_4$, then*

$$(3') \qquad u(\eta_2, \theta) - u(\eta_1, \theta) = \sum_{\eta_1 < \xi \leqslant \eta_2} \boldsymbol{\alpha}(\xi)v(\xi, \theta)(\xi, \theta)$$

and

$$(3'') \qquad v(\eta_2, \theta) - v(\eta_1, \theta) = \sum_{\eta_1 < \xi \leqslant \eta_2} \boldsymbol{\beta}(\xi)u(\xi, \theta)(-\xi, \theta).$$

Proof. By Theorem 7(e) we have

$$\begin{aligned} U(\theta) &= U'(\theta) + \sum_{\xi > \zeta_4} \boldsymbol{\alpha}(\xi)d(\xi)\phi(\xi, \theta) \\ &= U'(\theta) + \sum_{\xi > \zeta_4} \boldsymbol{\alpha}(\xi)(\xi, \theta)v(\xi, \theta) \end{aligned}$$

where $U'(\theta) \in E^+(0)E^-(\zeta_4)\mathscr{A}$. Applying $S_+(\eta_1)$ and $S_+(\eta_2)$ to both sides and using Theorem 8(a) and Lemma 9(a), we obtain our desired result.

COROLLARY 9(c). *If $\eta_2 > \eta_1 \geqslant \zeta_4$, then*

$$(4) \qquad \mathbf{u}(\eta_2, 0) - \mathbf{u}(\eta_1, 0) = \sum_{\eta_1 < \xi \leqslant \eta_2} \boldsymbol{\alpha}(\xi)\boldsymbol{\beta}(\xi)\mathbf{u}(\xi, 0).$$

Proof. If in (3″) we compare coefficients of $(-\eta_2, \theta)$, we obtain (writing ξ for η_2)

$$\mathbf{v}(\xi, -\xi) = \boldsymbol{\beta}(\xi)\mathbf{u}(\xi, 0), \qquad \xi \geqslant \zeta_4.$$

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If now, using this result, we compare constant coefficients in (3′), we obtain (4).

COROLLARY 9(d). *$\alpha(\eta)\beta(\eta) \neq 1$ for $\eta > \zeta_4$.*

Proof. Suppose that for some $\eta_2 > \zeta_4$, $\alpha(\eta_2)\beta(\eta_2) = 1$. In this case (4) can be rewritten in the form

$$-\mathbf{u}(\eta_1, 0) = \sum_{\eta_1 < \xi < \eta_2} \alpha(\xi)\beta(\xi)\mathbf{u}(\xi, 0),$$

which implies that

$$\lim_{\eta_1 \to \eta_2 -} \mathbf{u}(\eta_1, 0) = 0.$$

Since

$$|\mathbf{u}(\eta_1, 0)| \geqslant D^{-2} > 0$$

for $\eta_1 \geqslant \zeta_4$, this is a contradiction.

It is evident from Theorems 9(b) and 9(c) that the $u(\eta, \theta)$'s and $v(\eta, \theta)$'s are in some sense built from the $\alpha(\xi)$'s and $\beta(\xi)$'s. It is therefore of importance to know how large these constants are.

THEOREM 9(e). *We have*

$$\sum_{\xi \geqslant \zeta_3} \alpha(\xi)(\xi, \theta) \in \mathscr{A}, \qquad \sum_{\xi \geqslant \zeta_3} \beta(\xi)(-\xi, \theta) \in \mathscr{A}.$$

Proof. This is a special case of a more general result.

We have

$$d(\xi)\alpha(\xi) = \int_\Theta U(\theta)\psi(\xi, \theta)c(\theta)\, d\theta,$$

$$\alpha(\xi) = d(\xi)^{-2}\int_\Theta U(\theta)u(\xi, \theta)(-\xi, \theta)c(\theta)\, d\theta.$$

Since

$$[u(\xi, \theta)]^\# \prec C_+(\theta),$$

it follows that

$$|\alpha(\xi)| \leqslant \mathbf{F}(\xi),$$

where $F(\theta) = D^2C_+(\theta)^\# U(\theta)^\# c(\theta)^\# \in \mathscr{A}$, etc.

In the sequel to the present paper we shall show that Theorem 9(b) implies the product formula (3) of §2 and that Corollary 9(c) implies the formula

$$\mathbf{u}(\eta_1, 0) = \Big(\prod_{\eta_1 < \xi \leqslant \eta_2} [1 - \alpha(\xi)\beta(\xi)]\Big)\mathbf{u}(\eta_2, 0).$$

10. Concluding remarks. In **(5)**, I obtained for $\Theta = T$ and $\Xi = I$ a theory for a class $\mathfrak{B}$ of Banach algebras which is largely parallel to the theory of the present paper specialized to this case.

DEFINITION 10(a). *Let $\mathscr{A}$ be a Banach algebra of complex functions $f(\theta)$ on T with norm $||\cdot||$. $\mathscr{A}$ will be said to be of type $\mathfrak{B}$ if the following conditions are satisfied:*

(i) $\mathscr{A}_0 \supset \mathscr{A}, \sum_{-\infty}^{\infty} |\mathbf{f}(k)| \leqslant ||f||$ *for all* $f \in \mathscr{A}$;

(ii) $\exp(2\pi ik\theta) \in \mathscr{A}$ *for* $k = 0, \pm 1, \pm 2, \ldots$;

(iii) $E^+(n)$, $E^-(n)$ *are bounded operators on* $\mathscr{A}$ *for each* $n = 0, \pm 1, \pm 2, \ldots$;

(iv) *for every* $f \in \mathscr{A}$,

$$\lim_{n\to\infty} E^+(n)f\cdot(\theta) = 0 \quad \textit{and} \quad \lim_{n\to-\infty} E^-(n)f\cdot(\theta) = 0;$$

(v) *for every* $f \in \mathscr{A}$,

$$\lim_{n\to+\infty} \exp(-2\pi in\theta)E^+(n)f\cdot(\theta) = 0 \quad \textit{and} \quad \lim_{n\to-\infty} \exp(-2\pi in\theta)E^-(n)f\cdot(\theta) = 0.$$

We shall now show by an example that $\mathfrak{B}$ is not a subset of $\mathfrak{S}$. For f in $\mathscr{A}_0$ let

$$\mathscr{N}_1[f] = \sum_{-\infty}^{\infty} |\mathbf{f}(k)|, \qquad \mathscr{N}_2[f] = \underset{n_1, n_2}{\text{l.u.b.}} \left| \sum_{k=n_1}^{n_2} \mathbf{f}(k)k \right|.$$

Let $\mathscr{A}$ be the set of all $f \in \mathscr{A}_0$ for which $\mathscr{N}_1[f]$ and $\mathscr{N}_2[f]$ are finite and for which in addition

$$\lim_{n_1, n_2\to+\infty} \sum_{k=n_1}^{n_2} k\mathbf{f}(k) = 0, \qquad \lim_{n_1, n_2\to-\infty} \sum_{k=n_1}^{n_2} k\mathbf{f}(k) = 0.$$

For $f \in \mathscr{A}$, we set

$$||f|| = \mathscr{N}_1[f] + \mathscr{N}_2[f].$$

We assert, leaving the verification to the reader, that $\mathscr{A}$ is of type $\mathfrak{B}$. To see that it is not of type $\mathfrak{S}$, let

$$f(\theta) = \sum_{-\infty}^{\infty} (-1)^k(1 + k^2)^{-1}\exp(2\pi ik\theta).$$

Then $f \in \mathscr{A}$, but $||f^{\#}|| = \infty$!

REFERENCES

1. G. Baxter, *Polynomials defined by a difference system*, J. Math. Anal. Appl., *2* (1961), 223–263.
2. ——— *A convergence equivalence related to polynomials on the unit circle*, Trans. Amer. Math. Soc., *99* (1961), 471–487.
3. ——— *A norm inequality for a "finite section" Wiener–Hopf equation*, Illinois J. Math. *7*, (1963), 97–103.
4. G. Baxter and I. I. Hirschman, Jr., *An explicit inversion formula for finite section Wiener–Hopf operators*, Bull. Amer. Math. Soc., *70* (1964), 820–823.
5. I. I. Hirschman, *Finite sections of Wiener–Hopf equations and Szegö polynomials*, J. Math. Anal. Appl., *11* (1965), 290–320.
6. ——— *On a theorem of Szegö, Kac, and Baxter*, J. Analyse Math., *14* (1965), 225–234.
7. ——— *Finite section Wiener–Hopf equations on a compact group with ordered dual*, Bull. Amer. Math. Soc., *70* (1964), 508–510.
8. M. Shinbrot, *A class of difference kernels*, Proc. Amer. Math. Soc., *13* (1962), 399–406.

Stanford University and Washington University