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ANALYTIC DISCS IN SYMPLECTIC SPACES

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Abstract. We develop some symplectic techniques to control the behavior under symplectic transformation of analytic discs A of $X = \mathbb{C}^n$ tangent to a real generic submanifold R and contained in a wedge with edge R.

We show that if A^* is a lift of A to T^*X and if χ is a symplectic transformation between neighborhoods of p_o and q_o, then A is orthogonal to p_o if and only if $\widetilde{A} := \pi\chi A^*$ is orthogonal to q_o. Also we give the (real) canonical form of the couples of hypersurfaces of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ whose conormal bundles have clean intersection. This generalizes [10] to general dimension of intersection.

Combining this result with the quantized action on sheaves of the "tuboidal" symplectic transformation, we show the following: If R, S are submanifolds of X with $R \subset S$ and $p_o \in T^*_S X|_R$ but $ip_o \notin T^*_R X$, then the conditions $\mathrm{cod}_{T^{\mathbb{C}}S}(T^{\mathbb{C}}R) = \mathrm{cod}_{TS}(TR)$ (resp. $\mathrm{cod}_{T^{\mathbb{C}}S}(T^{\mathbb{C}}R) = 0$) can be characterized as opposite inclusions for the couple of closed half-spaces with conormal bundles $\chi(T^*_R X)$ and $\chi(T^*_S X)$ at $\chi(p_o)$.

In §3 we give some partial applications of the above result to the analytic hypoellipticity of CR hyperfunctions on higher codimensional manifolds by the aid of discs (cf. [2], [3] as for the case of hypersurfaces).

§1. Real symplectic manifolds

Let X be a real manifold and T^*X the cotangent bundle to X, (x, ξ) symplectic coordinates, $\alpha = \xi\, dx$ the canonical one form, σ the two form, H the Hamiltonian isomorphism, ν the Euler vector field, $\chi : T^*X \to T^*X$ a real symplectic transformation.

Let D be a C^1 manifold, D^* a C^1 section of T^*X over D. Suppose

$$\begin{array}{ccc} D^* & \xrightarrow{\ \chi\ } & \chi(D^*) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ D & & \widetilde{D}, \end{array}$$

and let $p_o = (x_o, \xi_o)$, $q_o = \chi(p_o) = (\tilde{x}_o, \eta_o)$.

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PROPOSITION 1. *ξ_o is orthogonal to $T_{x_o}D$ if and only if η_o is orthogonal to $T_{\tilde{x}_o}\widetilde{D}$*

Proof. We have

$$\begin{aligned}\langle \xi_o, T_{x_o}D\rangle &= \langle \xi_o, \pi' T_{p_o}D^*\rangle = \langle \pi^*\xi_o, T_{p_o}D^*\rangle \\ &= \sigma(H\pi^*\xi_o, T_{p_o}D^*) = \sigma(-\nu(p_o), T_{p_o}D^*) \\ &= \sigma(-\chi'\nu(p_o), \chi' T_{p_o}D^*) = \sigma(-\nu(q_o), T_{q_o}(\chi D^*)) \\ &= \sigma(H\pi^*(\eta_o), T_{q_o}(\chi D^*)) = \langle \pi^*\eta_o, T_{q_o}(\chi D^*)\rangle \\ &= \langle \eta_o, \pi' T_{q_o}\chi(D^*)\rangle = \langle \eta_o, T_{\tilde{x}_o}\widetilde{D}\rangle.\end{aligned}$$

□

A Lagrangian submanifolds Λ of T^*X is a C^1 submanifold whose tangent plane $\lambda(p) = T_p\Lambda$ verifies $\lambda(p)^\perp = \lambda(p)$, $\forall\, p$ (with $\perp$ denoting the σ-orthogonal). The intersection $\Lambda_1 \cap \Lambda_2$ is said to be clean when it is a manifold and when $T(\Lambda_1 \cap \Lambda_2) = T\Lambda_1 \cap T\Lambda_2$. All manifolds will be conic i.e. invariant under $\dot{\mathbb{R}}^+$.

Fix $p_o = (x_o, \xi_o) \in \dot{T}^*X$:

PROPOSITION 2. *Let M_1, M_2 hypersurfaces, p_o a point of $T^*_{M_1}X \cap T^*_{M_2}X$ and set*

$$R = \pi(T^*_{M_1}X \cap T^*_{M_2}X).$$

Then $T^*_{M_1}X \cap T^*_{M_2}X$ is clean if and only if R is a manifold and there exist real coordinates $t = (t_1, t', t'')$ such that

$$\begin{cases} M_1 = \{t_1 = 0\}, \\ R = \{t_1 = t' = 0\}, \\ M_2 = \{t_1 = Q(t') + O(t')o(t', t'')\}, \quad Q \text{ non degenerate.} \end{cases}$$

Proof. Since $\pi|_{T^*_{M_1}X \cap T^*_{M_2}X}$ has fiber-dimension $\equiv 1$, then clearly $T^*_{M_1}X \cap T^*_{M_2}X$ is a manifold if and only if R is so. Take then real coordinates $t = (t, t', t'')$ in $\mathbb{R}^N \simeq X$ such that

$$M_1 = \{t_1 = 0\}, \quad R = \{t_1 = 0,\ t' = 0\}, \quad M_2 = \{t_1 = g(t', t'')\},$$

and $p_o = (0; dt_1)$, $g(0,0) = 0$, $dg(0,0) = 0$. We have

$$\begin{aligned} T_{p_0}T^*_{M_1}X &= \{(u; t\, dt_1)\ ;\ u \in TM_1,\ t \in \mathbb{R}\}, \\ T_{p_0}T^*_{M_2}X &= \{(u; t\, dt_1 + \mathrm{Hess}(g)u\ ;\ u \in TM_2,\ t \in \mathbb{R}\}. \end{aligned}$$

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Since $g|_R \equiv 0$ and $dg|_R \equiv 0$, then $\mathrm{Hess}(g)u = 0$ if $u'' = 0$; therefore $g = Q(t') + O(t')o(t', t'')$. Next cleanness is equivalent to the implication: "$\mathrm{Hess}(g)u' = 0$ implies $u' = 0$" which is in turn equivalent to nondegeneracy of Q. $\square$

Remark 3. When $\mathrm{cod}_{T^*_{M_1}X}(T^*_{M_1}X \cap T^*_{M_2}X) = 1$, then Q is necessarily definite (positive or negative). Hence $R = M_1 \cap M_2$ and M_1, M_2 intersect to the order 2 along R. Let M_1^+, M_2^+ denote the (closed) half-spaces with boundary M_1, M_2 (and inward conormal p). By the above remarks we must then have either $M_2^+ \subset M_1^+$ or $M_1^+ \subset M_2^+$.

§2. Complex symplectic manifolds

Let X be a complex manifold of dimension n, T^*X the cotangent bundle to X with symplectic coordinates (z, ζ), σ $(= d\zeta \wedge dz)$ the canonical 2-form on T^*X, R a real submanifold of X, $T^*_R X$ the conormal bundle to R in X, $p_o = (z_o, \zeta_o)$ a point of $T^*_R X$ with $ip_o \notin T^*_R X$. In this situation we can identify, by a choice of coordinates, $T^*_R X_{z_o}$ to a totally real plane $\mathbb{R}^l_{y'} \subset \mathbb{C}^n \simeq T^*_{z_o} X$.

For a vector $\zeta \in \mathbb{C}^n$ we shall denote by $|\zeta|$ the Euclidean norm $|\zeta| = \big(\sum_i |\zeta_i|^2\big)^{1/2}$. If $|\Im\mathrm{m}\,\zeta| < |\Re\mathrm{e}\,\zeta|$ we also define $\|\zeta\| = \big(\sum_i \zeta_i^2\big)^{1/2}$ (for the determination of the square root which is positive over $\mathbb{R}^+$). If B is a neighborhood of z_o, and Γ_z for $z \in R \cap B$ is a continuous distribution of cones in $T^*_R X_z$ such that Γ_{z_o} is conic neighborhood of ζ_0 in $T^*_R X_{z_o}$, we consider the neighborhood $\Sigma = \{(z, \Gamma_z)\ ;\ z \in R \cap B\}$ of p_0 and denote by Σ_ε its ε-trumcation.

We have an identification

$$\text{(1)} \qquad \begin{array}{ccc} \Sigma_\varepsilon & \longrightarrow & W \\ (z'; \zeta) & \longmapsto & z' + \frac{|\zeta|\zeta}{\|\zeta\|}. \end{array}$$

Here W is a wedge of X with edge R; for an identification $X \simeq \mathbb{C}^n$ (in coordinates)

$$W \supset ((R \cap B) + \Gamma) \cap B$$

with Γ a cone of $\mathbb{R}^l \subset X$. In fact we see that if ζ and ζ_1 belong to Γ_z with $\zeta \neq \zeta_1$, then $\zeta/\|\zeta\| \neq \zeta_1/\|\zeta_1\|$ because $\Gamma_z \cap i\Gamma_z = \emptyset$. On the other hand the normals issued from different points of the C^2 manifold R cannot have nontrivial intersection in a neighborhood of R; and this is still true if one replaces normal directions $\zeta/|\zeta|$ by $\zeta/\|\zeta\|$.

In the identification (1) we shall call z' the R-components of z and $|\zeta|$ the distance to R. Thus $X \setminus R$ is foliated by the surfaces of fixed distance:

$$\widetilde{R}_t = \left\{ z = z' + t\frac{\zeta}{\|\zeta\|} \, ; (z'; \zeta) \in T^*_R X \times_X B \right\}, \quad t > 0 \text{ small.} \tag{2}$$

We consider the symplectic transformation $\chi = \chi_t$ of T^*X into itself:

$$\chi : (z; \zeta) \longmapsto \left(z + t\frac{\zeta}{\|\zeta\|}; \zeta \right).$$

Let $s^{\pm}_R(p)$ denote the number of respectively positive and negative eigenvalues for the Levi form $L_R(p)$ and also set

$$\gamma_R(z) = \dim(T^*_R X_z \cap iT^*_R X_z)$$

and

$$d_R(p) = \operatorname{cod}(R) + s^-_R(p) - \gamma_R.$$

Consider now a new manifold $S \supset R$, suppose $p \in R \times_S T^*_S X$ and note that

$$d_S \leq d_R \leq d_S + \operatorname{cod}_S R. \tag{3}$$

Also notice that $T^*_S X \cap T^*_R X$ is clean. Let $\widetilde{R} = \widetilde{R}_t$, $\widetilde{S} = \widetilde{S}_t$ $(t \ll 1)$ be the subspaces defined by (2). Denote by $\widetilde{S}^+$, $\widetilde{R}^+$ the closed half spaces with boundary $\widetilde{S}$, $\widetilde{R}$ and inward conormal q_o.

Theorem 4. *Let $R \subset S \subset X$, and let $p_o \in R \times_S T^*_S X$, $ip_o \notin T^*_R X$.*
(i) *Assume*

$$\gamma_R = \gamma_S. \tag{4}$$

Then $\widetilde{R}$, $\widetilde{S}$ intersect at the order 2 along $\pi(T^*_{\widetilde{S}} X \cap T^*_{\widetilde{R}} X)$ with $\widetilde{S}^+ \subset \widetilde{R}^+$.

(ii) *Assume*

$$\gamma_R - \gamma_S = \operatorname{cod}_S R. \tag{5}$$

Then the same conclusion as in (i) *holds but with $\widetilde{S}^+ \supset \widetilde{R}^+$ instead of $\widetilde{S}^+ \subset \widetilde{R}^+$.*

Proof. Consider

$$R \subset S_1 \subset S_2 \subset \cdots \subset S_m = S, \quad \operatorname{cod}_{S_{i+1}}(S_i) = 1.$$

We have

$$\begin{aligned}(6)\qquad \mathbb{Z} &= \mu\hom(\mathbb{Z}_{S_{i+1}}, \mathbb{Z}_{S_i})_p \\ &= \mu\hom(\mathbb{Z}_{\widetilde{S}_{i+1}}, \mathbb{Z}_{\widetilde{S}_i})_q[(d_{\widetilde{S}_{i+1}} - d_{S_{i+1}}) - (d_{\widetilde{S}_i} - d_{S_i})] \\ &= R\Gamma_{\widetilde{S}^+_{i+1}}(\mathbb{Z}_{\widetilde{S}^+_i})_{\tilde{z}}[(d_{\widetilde{S}_{i+1}} - d_{S_{i+1}}) - (d_{\widetilde{S}_i} - d_{S_i})].\end{aligned}$$

Note now that

$$(7)\qquad s^-_{\widetilde{S}_i}(q) = s^-_{S_i}(p) \quad \forall\, i.$$

In fact we have

$$(8)\qquad \begin{cases} \mathrm{Ker}(L_{S_i}) \underset{\pi'}{\overset{\sim}{\longleftarrow}} T_pT^*_{S_i}X \cap iT_pT^*_{S_i}X \underset{\chi'}{\overset{\sim}{\longrightarrow}} T_qT^*_{\widetilde{S}_i}X \cap iT_qT^*_{\widetilde{S}_i}X \underset{\pi'}{\overset{\sim}{\longrightarrow}} \mathrm{Ker}(L_{\widetilde{S}_i}), \\ \dim T^{\mathbb{C}}\widetilde{S}_i - \dim T^{\mathbb{C}}S_i = \mathrm{cod}_X\, S_i - 1 - \gamma_{S_i}, \end{cases}$$

i.e.

$$(9)\qquad \mathrm{rank}(L_{\widetilde{S}_i}) = \mathrm{rank}(L_{S_i}) + (\mathrm{cod}_{\mathbb{C}^n}\, T^{\mathbb{C}}S_i - 1).$$

On the other hand it is easily seen that

$$(10)\qquad s^+_{\widetilde{S}_i} \geq s^+_{S_i} + (\mathrm{cod}_{\mathbb{C}^n}\, T^{\mathbb{C}}S_i - 1).$$

Thus (9), (10) give (7). It follows from (7):

$$(11)\qquad (d_{\widetilde{S}_{i+1}} - d_{S_{i+1}}) - (d_{\widetilde{S}_i} - d_{S_i}) = \mathrm{cod}_{T^{\mathbb{C}}S_{i+1}}(T^{\mathbb{C}}S_i).$$

(i): Assume (4). Note that

$$\begin{aligned}(12)\qquad \gamma_R = \gamma_S &\iff \gamma_{S_{i+1}} = \gamma_{S_i} \quad \forall\, i \\ &\iff \mathrm{cod}_{T^{\mathbb{C}}_{S_{i+1}}}\, T^{\mathbb{C}}S_i = 1 \quad \forall\, i.\end{aligned}$$

Thus in this case (6) gives:

$$(13)\qquad \mathbb{Z} \simeq R\Gamma_{\widetilde{S}^+_{i+1}}(\mathbb{Z}_{\widetilde{S}^+_i})_{\tilde{z}}[1].$$

We know on the other hand from Proposition 2 that $\widetilde{S}_i$ and $\widetilde{S}_{i+1}$ intersect at the order 2 along a 1-codimensional manifold (namely $\pi(T^*_{\widetilde{S}_{i+1}}X \cap T^*_{\widetilde{S}_i}X)$) with either $\widetilde{S}^+_{i+1} \subset \widetilde{S}_i{}^+$ or $\widetilde{S}^+_{i+1} \supset \widetilde{S}_i{}^+$. But (13) says that $\widetilde{S}^+_{i+1} \subset \widetilde{S}^+_i$ $\forall\, i$. Iteration of this inclusion gives the conclusion.

(ii): Assume (5). We have

$$\begin{aligned}(14)\qquad \gamma_R - \gamma_S = \operatorname{cod}_S(R) &\iff \gamma_{S_i} - \gamma_{S_{i+1}} = 1 \quad \forall\, i \\ &\iff \operatorname{cod}_{T^{\mathbb{C}}S_{i+1}}(T^{\mathbb{C}}S_i) = 0 \quad \forall\, i.\end{aligned}$$

Thus we have in this case

$$\mathbb{Z} \simeq R\Gamma_{\widetilde{S}^+_{i+1}}(\mathbb{Z}_{\widetilde{S}^+_i})_{\tilde{z}}$$

which obviously implies $\widetilde{S}^+_{i+1} \supset \widetilde{S}_i$. ∎

§3. Application to analytic discs and symplectic transformations

Let R be a real submanifold of codimension l of a complex manifold X of dimension n in a neighborhood of a point z_o. Let us choose complex coordinates such that $T^*_RX_{z_o}$ is the plane $\mathbb{C}^{\gamma}_{z_1,\dots,z_\gamma} \times i\mathbb{R}^{l-2\gamma}_{y_{\gamma+1},\dots,y_{l-\gamma}}$ and write $z = (z', z'')$, $z' = z_1, \dots, z_{l-\gamma}$. Let us introduce a new complex symplectic transformation, that we still call χ:

$$\chi : (z; \zeta) \longmapsto \left(z + \frac{\zeta'}{\|\zeta'\|}; \zeta\right)$$

from a neighborhood of a conormal $p_o = (z_o, \zeta_o)$ with $\zeta_o \in (\mathbb{C}^l \times i\mathbb{R}^{l-2\gamma}) \setminus (\mathbb{C}^l \times \{0\})$ to a neighborhood of $p_o = \chi(p_o)$. For this transformation χ all conclusions of §2 hold without modifications. In particular

$$\widetilde{R} := \pi\chi(T^*_RX) \text{ is a hypersurface.}$$

We shall deal with analytic discs in X and denote $A = \{A(\tau)\ ;\ \tau \in \Delta\}$ (where Δ is the unit disc in $\mathbb{C}$). We shall say that A is "attached" to R if $\partial A \subset R$. The transformation above defined has the great advantage of giving a rule to interchange analytic discs "attached" to R and $\widetilde{R}$ respectively. Assume that R is defined by a system of equations $r = 0$ ($r = r_1, \dots, r_{l-\gamma}$) with $\partial r_j|_{z_o} = dz_j$, $j = 1, \dots, \gamma$, $\partial r_j|_{z_o} = -i\, dy_j$, $j = \gamma + 1, \dots, l - \gamma$ and that $\zeta_o = (\dots, 0, -i, 0, \dots)$ where $-i$ is in the $(l-\gamma)$-th position. We write $z = (z', z'')$, $z' = (z_1, \dots, z_{l-\gamma})$; we similarly write $\zeta = (\zeta', \zeta'')$, $\partial = (\partial', \partial'')$ and so on. Let A be a "small" analytic disc attached to R with $A(1) = z_o$. It is easy to prove existence of an $(l-\gamma) \times (l-\gamma)$ matrix G, real on $\partial\Delta$ with $G(z_o) = id$ such that

$$G\partial' r \text{ extends holomorphically from } \partial\Delta \text{ to } \Delta.$$

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To this end it is enough to solve the Bishop equation

$$G\,\Im\mathrm{m}\,\partial' r - T_1(G\,\Re\mathrm{e}\,\partial' r) = id_{l-\gamma\times l-\gamma} \quad \text{on } \partial\Delta \tag{15}$$

where T_1 is the Hilbert transform with $T_1(\,\cdot\,)|_1 = 0$. Note that (15) is solvable, in suitable Banach spaces, by the implicit function theorem, due to $|\,\Re\mathrm{e}\,\partial' r| \ll 1$. Let $\lambda = (\dots, 0, 1, 0, \dots)G$ and define

$$A^* = (A(\tau); \lambda\partial' r|_{A(\tau)}), \quad \widetilde{A} = \left\{A(\tau) + \lambda\frac{\partial' r(A(\tau))}{\|\lambda\partial' r(A(\tau))\|}\right\}.$$

It is clear that, if $\pi : T^*X \to X$ is the canonical projection, then

$$\widetilde{A} = \pi\chi A^*. \tag{16}$$

It is also obvious that A^*, and hence $\widetilde{A}$ are holomorphic discs and that

$$\partial\widetilde{A} \subset \widetilde{R}$$

due to $\lambda\partial r|_{\partial A} \hookrightarrow T^*_R X|_{\partial A}$. If we apply Proposition 1 to $A^* \subset T^*X^{\mathbb{R}}$ we get $\Re\mathrm{e}\langle\partial_\tau\widetilde{A},\, \zeta_o\rangle = 0$; if we apply it to $iA^* \hookrightarrow T^*X^{\mathbb{R}}$ we get $\Im\mathrm{m}\langle\partial_\tau\widetilde{A},\, \zeta_o\rangle = 0$ which implies $\partial_\tau\widetilde{A} \in T^{\mathbb{C}}\widetilde{R}$.

Let W be a "wedge" with edge R (cf. [8]). For an open cone $\Gamma \subset (T_S X)_{z_o}$ the so called "profile" of W, in an identification by coordinates $X \simeq \mathbb{C}^n = T_{z_o}R \oplus (T_R X)_{z_o}$, and for a neighborhood B of z_o, W has the form

$$W = ((B \cap R) + \Gamma) \cap B.$$

Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. Let S be a submanifold of X which contains R and which has ζ_o among its conormals at z_o. Let $\mathcal{C}_{S|X}$, $\mathcal{B}_{S|X}$ be the complexes of CR microfunctions and CR hyperfunctions along S respectively. Let $sp : H^0(\pi^{-1}\mathcal{B}_{S|X}) \to H^0(\mathcal{C}_{S|X})$ be the spectral morphism, and define

$$SS(u) = \mathrm{supp}\, sp(u), \quad u \in \mathcal{B}_{S|X}.$$

Note that $SS(u)_{z_o} = \{0\}$ if and only if u is a holomorphic function in a neighborhood of z_o. Let $\zeta_o \in T^*_S X_{z_o}$, take $\Gamma \subset \{\Re\mathrm{e}\langle z, \zeta_o\rangle > 0\}$, and set $W^{\pm} = ((B \cap R) \pm \Gamma) \cap B$.

THEOREM 5. *Assume*

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(i) $A \subset R \cup W^-$ (*resp.* $A \subset R \cup W^+$),

(ii) $\gamma_S = \gamma_R$ (*resp.* $\gamma_R - \gamma_S = \mathrm{cod}_S\, R$),

(iii) $T_{z_o} A \perp \zeta_o$,

(iv) $A \not\subset R$ *in any neighborhood of* z_o.

Then for $f \in (\mathcal{B}_{S|X})_{z_o}$ *we have* $p_o \notin SS(b(f))$ (*resp.* $-p_o \notin SS(b(f))$).

Remark 6. It is not necessary to assume $A \not\subset R$ in order to get an analytic disc $\widetilde{A} \subset \widetilde{S}^{\mp} \setminus \widetilde{S}$ which is the only fact we really need in the proof. Here again $\widetilde{S}^{\mp}$ are the closed half spaces with boundary $\widetilde{S}$ and inward conormal $\mp\zeta_o$. Thus let $S : r' = 0$, $R : r' = 0, r'' = 0$. Assume for instance there is an analytic "lift" A^* i.e. a holomorphic section of T^*X over A such that:

$$A^*|_{\partial A} \subset T^*_R X \setminus T^*_S X$$

i.e. $A^* = (A; \theta\partial r)$ with $\theta\partial r$ extending holomorphically, θ real over ∂A, $\theta'' \not\equiv 0$. Then

$$\partial\widetilde{A} \subset \widetilde{R} \subset \widetilde{S}^- \quad \text{but} \quad \partial\widetilde{A} \not\subset \widetilde{S}.$$

Proof. Let $\{B_r\}$ (resp. $\{\widetilde{B}_r\}$) be the family of spheres with center z_o (resp. $\widetilde{z}_o$) and radius r. We can find a sequence of subdiscs A_ν such that

$$A_\nu \subset A \cap B_{r_\nu}, \quad \partial A_\nu \not\subset R$$

(for a sequence $r_\nu \to 0$). Suppose we are proving the statement for p_o. By the discussion above, these are interchanged to analytic discs $\widetilde{A}_\nu$ such that

$$\partial\widetilde{A}_\nu \subset (\widetilde{R}^- \cap \widetilde{B}_{s_\nu}) \subset (\widetilde{S}^- \cap \widetilde{B}_{s_\nu}) \quad \text{but} \quad \partial\widetilde{A}_\nu \not\subset \widetilde{S},$$

(since $\widetilde{R}^- \subset \widetilde{S}^-$ due to $\gamma_R = \gamma_S$) for a new sequence $s_\nu \to 0$). By Proposition 1 we also have

$$T_{\tilde{z}_o}\widetilde{A}_\nu \subset T^{\mathbb{C}}_{\tilde{z}_o}\widetilde{S}. \tag{17}$$

We then enter [3, Theorem 1] and conclude that holomorphic functions $\tilde{f}$ in $\overset{\circ}{\widetilde{S}}{}^- \cap \widetilde{B}_\nu$ extend to a full neighborhood of $\tilde{z}_o$; thus germs of holomorphic functions on $\overset{\circ}{\widetilde{S}}{}^-$ extend to $\mathbb{C}^n$ at $\tilde{z}_o$. Now we introduce a quantization ϕ_K of χ by a kernel K. This induces a "microlocal" transformation of $\mathcal{O}_X$. CR hyperfunctions u at z_o are transformed into sums of boundary values $b(\tilde{f}^+)+b(\tilde{f}^-)$ on $\widetilde{S}$ of germs $\tilde{f}^\pm \in \mathcal{O}_X\big(\overset{\circ}{\widetilde{S}}{}^\pm\big)_{\tilde{z}_o}$ in such a way that $p_o \notin SSb(f)$ if and only if $\tilde{f}^-$ extends at $\tilde{z}_o$. The proof is complete. ∎

If we take $R = S$ in Theorem 5 and consider wedges $W^{\pm}$ with edge S we regain [2, Proposition 7] by a new method of "reduction to a hypersurface". If moreover we assume that A is orthogonal to any conormal $\zeta \in T^*_S X_{z_o}$ (instead of the only ζ_o) we get:

COROLLARY 7. *Assume*

(i) $A \subset W^{\mp} \cup S$ *but* $A \not\subset S$ *in any neighborhood of* z_o,

(ii) $T_{z_o} A \subset T^{\mathbb{C}}_{z_o} S$.

Then any $f \in \mathcal{O}_X(W^{\mp})_{z_o}$ *extends holomorphically to a full neighborhood of* z_o.

Proof. We apply Theorem 5 to all $p \in \pm\Gamma^*$ and conclude that $\pm\Gamma^* \cap SSb(f) = \{0\}$. On the other hand recall that there is an elementary estimate of microsupport; for $f \in \mathcal{O}_X(W^{\mp})_{z_o}$ we have $SSb(f)_{z_o} \subset \pm\Gamma^*$. Hence we can conclude $SSb(f)_{z_o} = \{0\}$. $\square$

EXAMPLE 8. In $\mathbb{C}^4$ let

$$\begin{aligned} &S = \{y_3 = z_1 z_2 + \bar{z}_1 \bar{z}_2,\, y_4 = 0\}, \quad \mp p = \mp dy_3 + \lambda\, dy_4, \\ &R = \{y_2 = 0,\, y_3 = z_1 z_2 + \bar{z}_1 \bar{z}_2,\, y_4 = 0\}, \\ &A = \mathbb{C}_{z_1} \times \{0\} \times \{0\} \times \{0\}. \end{aligned}$$

We can find a section $\lambda \partial r \in (T^*_R X \backslash T^*_S X)|_{\partial A}$ which extends holomorphically. For that just notice that the tangent direction $u = (1, 0, \ldots)$ to A verifies $u \in \mathrm{Ker}(L_R)(\lambda \partial r)$. Hence Remark 6 applies and yields $\pm p \notin WF(f)$.

§4. Appendix. Positivity of Lagrangians (cf. [4])

We shall further exploit here the techniques of §2 to give an extension of the results of [4].

Let X be a complex manifold, R and S real submanifolds of X with $R \subset S$. Recall that $T^*_R X \cap T^*_S X$ is clean and that (3) of §2 holds. Let $p \in R \times_S T^*_S X$, $ip \notin T^*_R X$.

THEOREM 9. (i) *Suppose*

$$d_R - d_S = \mathrm{cod}_S\, R. \tag{18}$$

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Then there exists a germ of a homogeneus complex symplectic transformation χ of T^*X from a neighborhood of p_o to a neighborhood of $q_o = \chi(p_o)$ which interchanges

$$T^*_R X \xrightarrow{\sim} T^*_{\widetilde{R}} X, \quad T^*_S X \xrightarrow{\sim} T^*_{\widetilde{S}} X,$$

for a pair of hypersurfaces $\widetilde{R}$, $\widetilde{S}$ with $s^-_{\widetilde{R}}(q_o) = 0$, $s^-_{\widetilde{S}}(q_o) = 0$ and such that $\widetilde{R}$, $\widetilde{S}$ intersect at the order 2 along $\pi(T^*_{\widetilde{R}} X \cap T^*_{\widetilde{S}} X)$ with $\widetilde{R}^+ \supset \widetilde{S}^+$.

(ii) *Suppose*

$$d_R = d_S. \tag{19}$$

Then there exists χ such that the same conclusion as in (i) *holds but with $\widetilde{S}^+ \supset \widetilde{R}^+$ instead of $\widetilde{S}^+ \subset \widetilde{R}^+$.*

Remark 10. Generally, the transformation χ of §2 does not suffice for the conclusion of Theorem 9.

Proof. Consider

$$R = S_1 \subset S_2 \subset \cdots \subset S_m = S, \quad \operatorname{cod}_{S_{i+1}} S_i = 1.$$

Put $\tilde{a}_i = d_{\widetilde{S}_i} - d_{\widetilde{S}_{i+1}}$, $a_i = d_{S_i} - d_{S_{i+1}}$. By the result of §2 we have

$$\mathbb{Z} \simeq R\Gamma_{\widetilde{S}^+_{i+1}}(\mathbb{Z}_{\widetilde{S}^+_i})_{\tilde{z}}[a_i - \tilde{a}_i].$$

Recall that $0 \leq a_i \leq 1$. Thus (18) and (19) are equivalent to $a_i = 1 \ \forall\, i$ and $a_i = 1 \ \forall\, i$ respectively.

We recall that if a submanifold $\Lambda \subset T^*X$ is $\mathbb{R}$ Lagrangian (i.e. Lagrangian for $\sigma^{\mathbb{R}}$ the real part of σ) and verifies

$$\dim(T_{p_o}\Lambda \cap \mathbb{C}H(\zeta_o\, dz)) = 1, \tag{20}$$

then Λ is symplectically equivalent to the conormal bundle to a hypersurface. (Note here that if $\Lambda = T^*_R X$, then (20) is equivalent to $ip_o \notin T^*_R X$, hence this latter condition characterizes the higher codimensional manifolds R which are "symplectically equivalent" to a hypersurface.) In particular for any family of Lagrangian manifolds Λ_i, $i = 1, \dots, m$ which satisfy (20) we can find χ such that

$$\Lambda_i \underset{\chi}{\xrightarrow{\sim}} T^*_{M_i} X, \quad \operatorname{cod}(M_i) = 1 \quad \forall\, i. \tag{21}$$

Also we can arrange (cf. [6]) that

$$s^-_{M_i}(q_o) = 0 \quad \text{for at least one } i. \tag{22}$$

We shall apply the above remarks for $\Lambda_i = T^*_{S_i}X$.

(i): We take in this case χ such that

$$T^*_{S_i}X \xrightarrow{\sim} T^*_{\widetilde{S}_i}X, \quad \text{cod}(\widetilde{S}_i) = 1 \quad \forall\, i, \quad s^-_{\widetilde{R}}(q_o) = 0$$

Assume $s^-_{\widetilde{S}_i}(q_o) = 0$; we show that

$$\begin{cases} \widetilde{S}^+_{i+1} \subset \widetilde{S}_i{}^+, \\ s^-_{\widetilde{S}_{i+1}}(q_o) = 0 \quad \forall\, i. \end{cases} \tag{23}$$

In fact we are in the situation

$$\begin{cases} \tilde{a}_i = -s^-_{\widetilde{S}_{i+1}}(q_o), \\ a_i = 1, \end{cases}$$

whence $a_i - \tilde{a}_i = s^-_{\widetilde{S}_{i+1}}(p_o) + 1$ and

$$\mathbb{Z} = R\Gamma_{\widetilde{S}_{i+1}}(\mathbb{Z}_{\widetilde{S}_i})_{\tilde{z}}\big[1 + s^-_{\widetilde{S}_{i+1}}\big]. \tag{24}$$

But since we know from Proposition 2 that $\widetilde{S}_i$, $\widetilde{S}_{i+1}$ intersect at the order 2 along a 1-codimensional submanifold with either of the inclusions $\widetilde{S}^+_{i+1} \subset \widetilde{S}^+_i$ or $\widetilde{S}^+_{i+1} \supset \widetilde{S}^+_i$, then (24) implies (23).

Hence induction applies and gives the conclusion

$$\begin{cases} \widetilde{S}^+ \subset \widetilde{R}^+, \\ s^-_{\widetilde{S}}(q_o) = 0. \end{cases}$$

(ii): We take now χ:

$$\chi : T^*_{S_i}X \xrightarrow{\sim} T^*_{\widetilde{S}_i}X, \quad \text{cod}(\widetilde{S}_i) = 1 \quad \forall\, i, \quad s^-_{\widetilde{S}}(q_o) = 0.$$

Assume $s^-_{\widetilde{S}_{i+1}}(q_o) = 0$; we show that

$$\begin{cases} s^-_{\widetilde{S}_i}(q_o) = 0, \\ \widetilde{S}_i{}^+ \subset \widetilde{S}^+_{i+1}. \end{cases} \tag{25}$$

In fact we have

$$\begin{cases} \tilde{a}_i = +s^-_{\widetilde{S}_i}(q_o), \\ a_i = 0 \end{cases}$$

Thus $a_i - \tilde{a}_i = -s^-_{\widetilde{S}_i}$ and therefore

$$\mathbb{Z} = R\Gamma_{\widetilde{S}^+_{i+1}}(\mathbb{Z}_{\widetilde{S}^+_i})_{\tilde{z}}[-s^-_{\widetilde{S}_i}],$$

which implies (25). The conclusion will follow again by induction. □

Remark 11. Recall the semiorder relation of positivity "$\succcurlyeq$" between Lagrangians in the sense of [5]. Thus we have in fact proved that $T^*_R X \succcurlyeq T^*_S X$ in case (i) (resp. $T^*_S X \succcurlyeq T^*_R X$ in case (ii)).

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