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SEMIGROUP ENDOMORPHISMS OF RINGS

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To the memory of D. J. Kinsman-Sullivan

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Abstract

We show that rings for which every non-constant multiplicative endomorphism is additive are trivial or power rings (that is, rings R such that $R = R^2$ and $x^2 = 0 = x+x$ for all $x \in R$) and that if R is a power ring for which every multiplicative endomorphism is additive, then End (R) is a zero semigroup or a semilattice according to how the product is defined.

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1. Introduction

Steve Ligh (1978) has removed the assumption of an identity in Theorem 2 of Cresp and Sullivan (1975), so that we now have (compare Exercise 38.5 of Warner (1971)):

PROPOSITION. *The only non-trivial rings* (*other than the ring* $\{0, a\}$ *with* $a^2 = 0$) *in which every subsemigroup containing the zero is a subring are the finite fields* F *for which* F^* *has order a Mersenne prime.*

In this note we shall improve Theorem 4 of Cresp and Sullivan (1975) by describing all commutative rings which contain a non-nilpotent element and have property (ε'): any non-constant semigroup endomorphism is a ring endomorphism.

It has been shown by Cresp and Sullivan (1975) that if R is a commutative ring with the property (ε): every semigroup endomorphism of R is a ring endomorphism, then R can only be the trivial ring, the ring $\{0, a\}$ where $a^2 = 0$, or a non-trivial ring satisfying (i) $R = \{xy: x, y \in R\}$ (denoted by R^2) and (ii) $a + a = 0 = a^2$ for all $a \in R$. The existence of this last-mentioned type of ring (now called *power rings*) and the question of whether they possess (ε) was left open. We shall show that power rings (if they exist) do not always have (ε), but do have particularly simple endomorphism semigroups.

We are indebted to A. Brunner for some stimulating conversations, and to M. Newman for encouraging correspondence, on this work.

2. Endomorphisms of rings

In the following result we use Lemma 1.2 of Kist (1963).

THEOREM 1. *If R is a non-trivial commutative ring and satisfies (ε'), then either $|R| = 2$ and $R^2 = 0$, or R is the field of 2 elements, or R is a power ring.*

PROOF. Let $a \in R$ be non-nilpotent and note that $\{0\} \cap \langle a \rangle = \square$. Hence we can choose a prime semigroup ideal M in R maximal with respect to not meeting $\langle a \rangle$ and for each $n \geqslant 1$, define λ_n: $R \to R$ by setting $x\lambda_n = x^n$ for all $x \notin M$ and equal to 0 otherwise. It is easily seen that λ_n is a non-constant semigroup endomorphism of R and so by (ε') preserves addition. Hence if $a + a \in M$, we have

$$0 = (a+a)\lambda_2 = a\lambda_2 + a\lambda_2 = a^2 + a^2.$$

Since the same conclusion occurs if $a + a \notin M$, we have $a(a+a) = 0 \in M$ from which we obtain $a + a \in M$. Using λ_1, we now find $a + a = 0$.

Next we assert that $x^2y + xy^2 = 0$ for all $x, y \notin M$. For, consider $x, y \notin M$ and suppose $x + y \in M$. Using λ_1, we obtain $x + y = 0$ and the assertion follows. On the other hand, if $x + y \notin M$, then using λ_2 we find $xy + xy = 0$: this and the use of λ_3 now shows that $xy^2 + x^2y = 0$ as required. We now conclude that $a + a^2 \in M$. For, if $a + a^2 \notin M$, then

$$0 = a(a+a^2)^2 + a^2(a+a^2) = a(a^2+a^4) + (a^3+a^4).$$

Hence $a^3(a+a^2) = 0 \in M$ and so $a + a^2 \in M$, a contradiction. One final application of λ_1 shows that $a + a^2 = 0$.

We can now define a semigroup endomorphism γ: $R \to R$ by setting $x\gamma = a$ for all $x \notin M$ and equal to 0 otherwise. From (ε') we observe that if $x, y \in M$ and $x - y \notin M$, then $a = (x+y)\gamma = x\gamma + y\gamma = 0$, a contradiction. Hence, M is a ring

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ideal and if $x \in M$ then $x+a \notin M$. In this event, we have

$$x+a = (x+a)\lambda_1 = x\lambda_1 + a\lambda_1 = 0+a$$

and so $x = 0$. Thus $R = \{0, a\}$ where $a^2 = a$.

If R is a non-trivial ring in which every element is nilpotent, then R certainly contains a non-zero element a such that $a^2 = 0$. Hence if, in addition, R is commutative and satisfies (ε'), then the proof of Theorem 3 of Cresp and Sullivan (1975) can be used to show that R is either $\{0, a\}$ or a power ring. To see this, first note that if θ_n is constant then $(x+y)^n = 0 = x^n$ for all $x, y \in R$. Thus, regardless of whether or not θ_2, θ_3 are constant, we have $xy+xy = 0$ and $xy^2+x^2y = 0$ for all $x, y \in R$ and so if also $R = R^2$, then R is a power ring. On the other hand, if $R \neq R^2$, then μ_a (see Cresp and Sullivan (1975), p. 175) is non-constant and we have: if $u \notin R^2$, $v \in R^2$, then $u+v \notin R^2$. If λ_a is constant, then $R = R^2 \cup c$ where $c \notin R^2$ and so for all $x, y \in R$, $xy+c$ must equal c. Thus, $R^2 = 0$ and $R = \{0, c\}$. If λ_a is non-constant, (ε') enables us to establish the above remark as in the proof of Theorem 3 of Cresp and Sullivan (1975).

We now consider power rings in general and start by noting that if $a, b \in R$, then $(a+b)^2 = 0$ implies $ab = ba$ (thanks to R. Bowshell for this simple observation). Moreover, if $x, y \in R$ satisfies $x = xy$ then $x = xy = xy^2 = 0$, and so if $x \in R$, $x \neq 0$, then there exist distinct $x_1, x_2 \in R$ such that $x = x_1 . x_2$ and neither x_i equals x. Similarly, $x_1 = x_{11} . x_{12}$ and $x_2 = x_{21} . x_{22}$ for suitable $x_{11}, x_{12}, x_{21}, x_{22}$ in R which must in fact be distinct and not equal to x. This procedure can be continued indefinitely, the number of elements rising by a multiple of 2 after each successive factorization. Power rings are therefore both commutative and infinite, and this fact will be used without further mention.

The Remark of Cresp and Sullivan (1975) raised the question of whether all power rings are (ε)-rings. We now answer this in the negative by using an idea introduced at the end of Martindale (1969). Let R be a non-trivial power ring. It is easily checked that, under component-wise addition and multiplication, $R \times R$ is also a non-trivial power ring, but it does not possess (ε). For, if it does, then the mapping

$$\lambda: R \times R \to R \times R, \quad (a, b) \to (ab, 0)$$

is not only multiplicative, but also additive. In this case, $(a+x).(0+y) = a.0+x.y$ for all $a, x, y \in R$ and we have $R^2 = 0$, a contradiction.

The set, End(R), of all semigroup endomorphisms of a commutative ring R is a semigroup under either pointwise multiplication or functional composition. The next result shows that in either case the product on End(R) is quite elementary when R is a power ring with (ε); the second case involves an ingenious argument due to A. Brunner.

THEOREM 2. *If R is a non-trivial power ring with* (ε), *then* $\alpha.\beta = 0$ *and* $\alpha \circ \alpha = \alpha$ *for all* $\alpha, \beta \in \mathrm{End}(R)$.

PROOF. Let $\alpha, \beta \in \mathrm{End}(R)$ and consider λ: $R \to R$, $x \to x\alpha . x\beta$. Since λ is multiplicative, it is also additive and consequently $x\alpha . y\beta = x\beta . y\alpha$ for all $x, y \in R$. This shows that $(xy)\lambda = 0$ for all $x, y \in R$ and since $R = R^2$ the first assertion follows.

In case β is the identity on R, we have $x\alpha . y = x . y\alpha$ for all $x, y \in R$. Let $x, y, z \in R$ and note that

$$xy . z\alpha^2 = (xy)\alpha . z\alpha = x\alpha . y\alpha . z\alpha$$
$$= x . y\alpha^2 . z\alpha = xy . z\alpha^3.$$

Since $R = R^2$, we conclude that $x . y\alpha^2 = x . y\alpha^3$ for all $x, y \in R$. From this we have

$$(xy)\alpha^2 = x\alpha^2 . y\alpha^2 = x\alpha . y\alpha^3 = x\alpha . y\alpha^2$$
$$= x . y\alpha^3 = x . y\alpha^2 = x\alpha . y\alpha = (xy)\alpha$$

and $R = R^2$ now implies $\alpha = \alpha^2$.

To conclude we note that when R is a power ring with (ε), the identity $x\alpha . y\beta = x\beta . y\alpha$ implies that $\mathrm{End}(R)$ is closed under pointwise addition and so $\alpha \circ \beta = \beta \circ \alpha$ for all $\alpha, \beta \in \mathrm{End}(R)$.

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