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REFLECTIONS AND SYMMETRIES IN COMPACT SYMMETRIC SPACES

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Point symmetries and reflections are two important transformations on a Riemannian manifold. In this article we study the interactions between point symmetries and reflections in a compact symmetric space when the reflections are global isometries.

1. INTRODUCTION

Let (M, g) be an n-dimensional smooth Riemannian manifold, m a point of M and u a unit tangent vector at m. Let γ be the geodesic $t \to \exp_m(tu)$ through $m = \gamma(0)$ with tangent vector $u = \gamma'(0)$ and arc length t. Define the map

$$s_m\colon \exp_m(tu) \to \exp_m(-tu).$$

For each m there exists a neighbourhood of m such that s_m is a local diffeomorphism. The map s_m is called the *local geodesic symmetry* at m (or simply the *point symmetry* at m). The Riemannian manifold M is called a *(global) symmetric space* if, for each point m in M, the point symmetry s_m at m is a (global) isometry of M.

More generally, suppose B is a (connected and) topologically embedded p-dimensional submanifold which is relatively compact. Let $m \in B$ and γ be a geodesic parameterised by arc length t such that $\gamma(0) = m$ and $\gamma'(0) = u \in T_m^{\perp}B$. Then $\gamma(t) = \exp_m(tu)$. The map defined by

$$\varphi_B\colon \exp_m(tu) \to \exp_m(-tu)$$

is a diffeomorphism of a sufficiently small tubular neighbourhood of B. It is called the (local) *reflection* with respect to B. The radius of the tubular neighbourhood is assumed to be smaller than the distance from B to its nearest focal points. It is clear that the reflection φ_B is involutive, that is, $(\varphi_B)^2 = id$. On the other hand, if an involutive transformation φ of M is an isometry and has some fixed points, then φ is

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the reflection with respect to each connected component of the fixed point set $F(\varphi; M)$ and these connected components are totally geodesic submanifolds [**6**].

In [**4**], the authors have obtained a necessary and sufficient condition for a local reflection φ_B to be isometric. In particular, they showed that if (M, g) is a locally symmetric space, then the reflection φ_B is an isometry if and only if (1) B is totally geodesic and (2) for each point m in B, the normal space $T_m^{\perp} B$ of B in M at m is the tangent space of a totally geodesic submanifold $\underline{B}$ of M through m.

The purpose of this article is to study the interactions between point symmetries and reflections in a compact symmetric space when the reflections are global isometries. Our results are obtained by applying the method introduced in [**2**] and developed in [**2**] and [**3**] (for a general survey on this method and its applications, see [**1**] and [**7**]). (For general results concerning symmetric spaces, see [**5**, **6**].)

2. Preliminaries

Let M be a compact symmetric space of positive dimension and o a given point in M. Denote by G the closure of the group of isometries generated by point symmetries in the compact-open topology and by K the isotropy subgroup of G at o. Then K is compact and $M = G/K$.

In the following, by a *circle* we mean a closed smooth geodesic (circles always exist in M). Let p be an antipodal point of o on some circle. The following results were obtained in [**2**].

Lemma 2.1. *If p is an antipodal point of o in a compact symmetric space M, then $s_o s_p = s_p s_o$.*

Theorem 2.2. *Let M be a compact symmetric space and $o \in M$. Then we have*

- (i) *the fixed point set $F(s_o; M) \setminus o$ is the set of all the points p which are the antipodal points of o on some circles passing through o;*
- (ii) *for each antipodal point p of o, we have*
- (ii-a) *$K(p)$ is a complete connected totally geodesic submanifold of M and $K(p)$ is exactly the connected component $M_+^o(p)$ of $F(s_o; M)$ through p;*
- (ii-b) *there is a complete connected totally geodesic submanifold $M_-^o(p)$ through p such that $T_p M_-^o(p) = T_p^{\perp} M_+^o(p)$;*
- (ii-c) *$M_-^o(p)$ is the connected component of the fixed point set $F(s_o s_p; M)$ through p;*
- (ii-d) *$rk M_-^o(p) = rk M$; and*
- (ii-e) *$M_-^o(p) = M_-^p(o)$.*

Let p be an antipodal point of o in a compact symmetric space $M = G/K$. Then

$K(p) = M_+^o(p)$ is called a *polar* of o. A polar of o in M is called a *pole* of o if the polar is a singleton. The following results were obtained in [**3**].

THEOREM 2.3. *Let o be a point in a compact symmetric space M. Then o has a pole, say $\underline{o}$, in M if and only if M is a Riemannian double covering space of a compact symmetric space $\underline{M} = M/\tau$, where τ is the Riemannian double covering transformation on M which carries o into $\underline{o}$.*

Sometime we denote the compact symmetric space M/τ by $M/\{o, \underline{o}\}$.

THEOREM 2.4. *Let o and p be two points in a compact symmetric space M. Then p is a pole of o if and only if $s_o = s_p$.*

THEOREM 2.5. *Let o and p be two points in a compact symmetric space M. Then the point symmetries s_o and s_p commute, that is $s_o s_p = s_p s_o$, if and only if one of the following two cases occurs:*

(a) *o and p are antipodal;*
(b) *o and p are not antipodal, but $s_p(o)$ is a pole of o.*

In this paper we make the following general assumption:

Assumption. M is a compact symmetric space, B a connected embedded submanifold of M and the reflection φ_B with respect to B in M is an isometry on M.

3. WHEN DOES $\varphi_B s_o = s_o \varphi_B$?

In view of Theorem 2.5 we ask the following

PROBLEM 1. *When do a reflection φ_B and a point symmetry commute?*

The following result gives a complete answer to Problem 1.

THEOREM A. *Let M be a compact symmetric space and B a submanifold of M. Then the reflection φ_B commutes with a point symmetry s_o for some point o in M, that is $\varphi_B s_o = s_o \varphi_B$, if and only if one of the following two cases occurs:*

(1) $o \in F(\varphi_B; M)$;
(2) *$\varphi_B(o)$ is a pole of o.*

PROOF: First we observe that the submanifold B is totally geodesic in M, since φ_B is an isometry.

Assume that $\varphi_B s_o = s_o \varphi_B$ for some point o in M. Then $\varphi_B s_o(o) = \varphi_B(o) = s_o \varphi_B(o)$. So, we find $\varphi_B(o) \in F(s_o; M)$.

If $\varphi_B(o) = o$, then $o \in F(\varphi_B; M)$. In this case, case (1) of Theorem A occurs.

If $\varphi_B(o) \neq o$, then $\varphi_B(o)$ is an antipodal point of o (Theorem 2.2). Hence we have

$$s_{\varphi_B(o)} = \varphi_B s_o \varphi_B^{-1} = \varphi_B s_o \varphi_B = \varphi_B^2 s_o = s_o.$$

Therefore, by applying Theorem 2.4, we see that $\underline{o} = \varphi_B(o)$ is a pole of o.

Now, we prove the converse. First, if o lies in the fixed point set $F(\varphi_B; M)$, then $\varphi_B(o) = o$. Thus, $s_o = \varphi_B s_o \varphi_B^{-1}$. This implies that φ_B and s_o commute. Next, if $\varphi_B(o)$ is a pole of o, then we have $s_o = s_{\varphi_B(o)}$ (Theorem 2.4). On the other hand, we also have

$$s_{\varphi_B(o)} = \varphi_B s_o \varphi_B^{-1}.$$

So, φ_B and s_o commute. ∎

Remark 3.1. If B is a point, Theorem A is nothing but Theorem 2.5.

Next, we consider the following

PROBLEM 2. *When is a submanifold B of a compact symmetric space M a polar?*

The following observation gives an answer to this problem *in terms of reflections.*

THEOREM B. *A complete connected submanifold B of a compact symmetric space M is a polar of some point in M if and only if the reflection φ_B has an isolated fixed point not in B.*

PROOF: Since the reflection φ_B is isometric, B is totally geodesic. Assume φ_B has an isolated fixed point, say o, not in B. Then, since φ_B is involutive and both s_o and φ_B have o as an isolated fixed point, we have $s_o = \varphi_B$. Because B is fixed by s_o, each point of B is an antipodal point of o (Theorem 2.2) and because B is a connected component of the fixed point set $F(\varphi_B; M)$, B is the polar $M_+^o(p)$, for some antipodal point p in B.

Conversely, if B is a polar of o, say $B = M_+^o(p)$ for some antipodal point p in B, then $o \notin B$. Since the reflection φ_B with respect to the polar $M_+^o(p)$ is the point symmetry s_o, o is an isolated fixed point of φ_B. ∎

Remark 3.2. In view of Theorem B, it is interesting to note that the reflection φ_B with respect to B has an isolated fixed point in B if and only if B is a singleton.

In the remaining part of this section, we give a characterisation of M_-^o *in terms of reflections.*

THEOREM C. *Let B be a complete connected submanifold of a compact symmetric space M and $o \in B$. Then $s_o\varphi_B$ (or $\varphi_B s_o$) has an isolated fixed point $p (\neq o)$ if and only if $p \in B$ and exactly one of the following cases occurs:*

(a) *o and p are antipodal and $B = M_-^o(p)$;*

(b) *$s_p(o)$ is a pole of o and B is the connected component of the fixed point set $F(s_o s_p; M)$ through o. (In this case, $B \neq M^{o}_{-}(p)$).*

PROOF: First, we observe again that B is totally geodesic in M, since φ_B is an isometry.

($\Longleftarrow$): If $p \in B$ and p is an antipodal point of o and $B = M^{o}_{-}(p)$, then the reflection φ_B is nothing but $s_o s_p$ (Theorem 2.2). Thus, $s_o \varphi_B = s_p$ which has p as an isolated fixed point.

Now, suppose that $p \in B$ and $s_p(o)$ is a pole of o such that B is the connected component of $F(s_o s_p; M)$ through o. Let $\psi = s_o s_p$. Then $\psi = s_p s_o$ according to Theorem 2.5. Thus, ψ is an involutive isometry. Since B is a connected component of $F(\psi; M)$, $\psi = \varphi_B$. Thus, $s_o \varphi_B = s_o^2 s_p = s_p$ which has p as an isolated fixed point.

($\Longrightarrow$): Assume that $s_o \varphi_B$ (or $\varphi_B s_o$) has an isolated fixed point p. Since $o \in B$, Theorem A implies that $s_o \varphi_B = \varphi_B s_o$. Thus, $s_o \varphi_B$ is an involutive isometry. Because $p (\neq o)$ is an isolated fixed point of $s_o \varphi_B$ and s_p, we have $s_o \varphi_B = s_p$. This implies that $\varphi_B = s_o s_p$. Therefore, $s_o s_p = s_p s_o$. Consequently, by applying Theorem 2.5, either p is an antipodal point of o or $s_p(o)$ is a pole of o. Suppose o and p are antipodal. Since $M^{o}_{-}(p)$ is the connected component of $F(s_o s_p; M)$ through o (Theorem 2.2) and B is also a connected component of $F(s_o s_p; M) (= F(\varphi_B; M))$ through o, we obtain $B = M^{o}_{-}(p)$.

Suppose o and p are not antipodal in M. Then $\underline{o} = s_p(o)$ is a pole of o in M. So, by applying Theorem 2.3, M is a Riemannian double covering space of a compact symmetric space $\underline{M} = M/\tau$, where τ is the double covering transformation on M which carries o into $\underline{o}$. Let $\pi\colon M \to \underline{M} = M/\tau$ be the natural projection. Since $\varphi_B = s_o s_p$, the reflection φ_B induces a reflection $\varphi_{\underline{B}}$ on $\underline{M}$ with respect to the submanifold $\underline{B} := \pi(B)$ in $\underline{M}$. Since $\varphi_B = s_o s_p$, we have $\varphi_{\underline{B}} = s_{\underline{o}} s_{\underline{p}}$. Because $\underline{o}$ and $\underline{p}$ are antipodal in $\underline{M}$, the previous case applies which yields $\underline{p} \in \underline{B}$ and $\underline{B} = \underline{M}^{\underline{o}}_{-}(\underline{p})$. Thus, $\underline{p} \in F\left(s_{\underline{o}} s_{\underline{p}}; \underline{M}\right)$. Let $\underline{c}$ be a circle in $\underline{B}$ joining $\underline{o}$ and $\underline{p}$. Then $\underline{c}$ lifts to a circle c in B. The point p and $s_p(o)$ must lie in c. So, $p \in B$. And hence B is the connected component of $F(s_o s_p; M)$ through o. ∎

As an easy application of Theorem B, we obtain the desired characterisation of M^{o}_{-}.

COROLLARY D. *Let o be a point in B. Then $B = M^{o}_{-}(p)$ for some antipodal point $p \in B$ of o if and only if $s_o \varphi_B$ has an isolated fixed point $p \in M$ such that o and p are antipodal.*

Combining Theorem B and Corollary D we obtain the following

COROLLARY E. *Let (B, C) be a pair of complete connected totally geodesic sub-*

manifolds of a compact symmetric space M through a point $p \in M$ such that the tangent space T_pB and T_pC are orthogonally complementary. Then (B, C) is a pair (M_+, M_-) in the sense of [**2**] *if and only if either the reflection φ_B has an isolated fixed point not in B or $s_p\varphi_C$ has an isolated fixed point which is an antipodal point of p.*

4. When does $\varphi\tau = \tau\varphi$?

Let B be a submanifold in a compact symmetric space M. As we have already mentioned in the introduction, the reflection φ_B is involutive. And on the other hand, if φ is an involutive isometry on M with non-empty fixed point set, then φ is the reflection with respect to each connected component of the fixed point set $F(\varphi; M)$. For this reason, we may also call an involutive isometry on M with non-empty fixed point set a *reflection* when no specified component B is mentioned.

Now, we consider the following

Problem 3. *Let M be a Riemannian double covering space of a compact symmetric space and B a submanifold of M. When does a reflection φ commute with the double covering transformation τ on M?*

The following result gives an answer to this problem.

Theorem F. *Let M be a compact symmetric space and φ a reflection. Then the following three statements are equivalent:*

(1) *φ carries some point $o \in M$ into a pole of o;*

(2) *M is a Riemannian double covering space of some compact symmetric space $\underline{M} = M/\tau$ and φ commutes with the double covering transformation τ, that is $\varphi\tau = \tau\varphi$;*

(3) *M is a Riemannian double covering space of some compact symmetric space $\underline{M} = M/\tau$ and the fixed point set $F(\varphi; M)$ is stable under the covering transformation τ, that is, $\tau(F(\varphi; M)) = F(\varphi; M)$.*

Proof: (2) $\Longrightarrow$ (3): Assume that $\varphi\tau = \tau\varphi$. Then, for any point p in $F(\varphi; M)$, we have $\varphi\tau(p) = \tau\varphi(p) = \tau(p)$. Thus, $\tau(p) \in F(\varphi; M)$.

(3) $\Longrightarrow$ (2): Let $p \in F(\varphi; M)$. Then $(\tau\varphi\tau)(p) = \tau(\varphi(\tau(p))) = \tau(\tau(p)) = p = \varphi(p)$. Thus, both $\tau\varphi\tau$ and φ fix the point p. Now, let B be the connected component, $F(\varphi; M)_p$, of $F(\varphi; M)$ through p. Then, for any $X \in T_pB$, we have $\tau_*(X) \in T_{\tau(p)}\tau(B)$. Since τ permutes connected components of the fixed point set $F(\varphi; M)$, τ_* carries the normal space $T_p^\perp B$ onto $T_{\tau(p)}^\perp \tau(B)$ and the totally geodesic submanifolds B and $\tau(B)$ are isometric via τ. Therefore, we have

$$(\tau\varphi\tau)_*(X) = \tau_*\varphi_*(\tau_*X) = \tau_*(\tau_*X) = X = \varphi_*X.$$

Now, for any $Y \in T_p^{\perp}B$, we have $\tau_* Y \in T_{\tau(p)}^{\perp}\tau(B)$. Thus, $\varphi_*(\tau_* Y) = -\tau_* Y$. Therefore, we find

$$(\tau\varphi\tau)_*(Y) = \tau_*\varphi_*(\tau_* Y) = -\tau_*^2 Y = -Y = \varphi_* Y.$$

Since both $\tau\varphi\tau$ and φ are isometries, $\tau\varphi\tau = \varphi$. This gives $\tau\varphi = \varphi\tau$.

(1) $\Longrightarrow$ (3): Let φ be a reflection which carries a point $o \in M$ into a pole $\underline{o} := \varphi(o)$ in M. Let B be a connected component of the fixed point set $F(\varphi; M)$. Let c be a geodesic in M joining o and $\underline{o}$ such that c meets at a point $m \in B$ orthogonally. Then the point symmetry s_m carries o into $\underline{o}$. Since $\underline{o} = \varphi(o)$ is a pole of o in M, M is a Riemannian double covering space of some compact symmetric space $\underline{M} = M/\tau$, where τ is the double covering transformation on M which interchanges o and $\underline{o}$. Let $\underline{c} = \pi(c)$. Then $\underline{c}$ is a circle in $\underline{M}$ such that $\underline{o} = \pi(o)$ is an antipodal point of $\underline{m} := \pi(m)$. Let γ be the lift of $\underline{c}$ in M. Then γ is a double covering of $\underline{c}$ which passes through o, m and $\underline{o}$. Let m' be the antipodal point of m on γ. Then m' is a pole of m. Thus, $m' = \tau(m)$. Since $m \in F(\varphi; M)$, $s_m\varphi = \varphi s_m$. Therefore, we obtain

$$m' = s_m\varphi(m') = \varphi s_m(m') = \varphi(m').$$

Consequently, we get $m' = \tau(m) \in F(\varphi; M)$. Let q be any other point in B. Then there exists a geodesic c' in B joining m and q. Let b be the middle point on c' between m and q. Then $s_b(m) = q$. Since $s_b(B) = B$, s_b carries the normal space $T_m^{\perp}B$ onto $T_q^{\perp}B$. Moreover, $s_b(\underline{o})$ is a pole of $s_b(o)$ and $s_b(\underline{o}) = \varphi(s_b(o))$. Applying the same argument to q, we obtain $\tau(q) \in F(\varphi; M)$.

Let B' be another component of $F(\varphi; M)$. Then the reflection with respect to B' is exactly the reflection with respect to B. So, the reflection with respect to B' also carries o into the pole $\underline{o}$ of o. The same argument applies to this case. Thus, we obtain the same result for B'. Consequently, we obtain $\tau(F(\varphi; M)) = F(\varphi; M)$.

(2) $\Longrightarrow$ (1): Let B be a connected component of $F(\varphi; M)$. Assume that M is a Riemannian double covering space of a compact symmetric space $\underline{M} = M/\tau$ for some covering transformation τ on M such that $\varphi\tau = \tau\varphi$. Then φ gives rise to a reflection $\underline{\varphi}$ in $\underline{M}$ with respect to $\underline{B} = \pi(B)$. Let $m' \in \underline{B}$ and let $\underline{C}$ be the connected component of $F(\varphi s_{m'}; M)$ through m'. Then $\underline{C}$ is a complete totally geodesic submanifold of $\underline{M}$. Let o' be an antipodal point of m' in $\underline{C}$. Then $\underline{\varphi}(o') = o'$.

Let o and $\underline{o}$ be the preimages of o', that is, $\pi(o) = \pi(\underline{o}) = o'$. Then $\underline{o}$ is a pole of o and $\varphi(o) = \underline{o}$. This proves statement (1). ∎

Remark 4.1. It is easy to see that two reflections φ and ψ commute if and only if their product $\varphi\psi$ is again a reflection.

Remark 4.2. According to Theorem 2.5, the product $s_o s_p$ of two point symmetries s_o and s_p, $o, p \in M$, is a reflection if and only if either o and p are antipodal or $s_p(o)$ is a pole of o.

5. When does $\varphi\psi = s_o$?

In view of Remark 4.2, we consider the following

Problem 4. *When is the product of two reflections a point symmetry?*

The following result gives a solution to this problem.

Theorem G. *Let M be a compact symmetric space and φ and ψ two reflections on M. Then the product $\varphi\psi$ is a point symmetry s_o for some $o \in M$ if and only if exactly one of the following two cases occurs:*

(a) *$o \in F(\varphi; M) \cap F(\psi; M)$ and the connected components B and D of $F(\varphi; M)$ and $F(\psi; M)$, respectively, through o are orthogonally complementary, that is,*

$$(*) \qquad \dim B + \dim D = \dim M \text{ and } T_oB \perp T_oD;$$

(b) *$o \notin F(\varphi; M) \cup F(\psi; M)$ and $\varphi(o) = \psi(o)(:= \underline{o})$ is a pole of o such that if τ is the Riemannian double covering transformation corresponding to the pair of poles, $\{o, \underline{o}\}$, then $o \in F(\tau\varphi; M) \cap F(\tau\psi; M)$ and the connected components E and F of $F(\tau\varphi; M)$ and $F(\tau\psi; M)$ through o, respectively, satisfy condition (*).*

Proof: ($\Longleftarrow$): *Case (a).* If there is a point $o \in F(\varphi; M) \cap F(\psi; M)$ such that the connected components B and D through o satisfy condition (*), then $\varphi(o) = o$, $\psi(o) = o$ and, for any vectors $X \in T_oB$ and $Y \in T_oD$, we have

$$(\varphi\psi)_*(X) = \varphi_*\psi_*(X) = \varphi_*(-X) = -X,$$

and

$$(\varphi\psi)_*(Y) = \varphi_*\psi_*(Y) = \varphi_*Y = -Y.$$

Because $\varphi\psi$ is isometric, these imply $\varphi\psi = s_o$.

Case (b). Assume that there is a point o such that (1) $\varphi(o) = \psi(o)$ is a pole of o, (2) $o \notin F(\varphi; M) \cup F(\psi; M)$, and (3) the corresponding covering transformation τ satisfies the condition given in the statement (b). Then, by Theorem F, we see that the double covering transformation τ commutes with φ and ψ. Thus, we have $(\tau\varphi)^2 = (\tau\psi)^2 = id$, that is $\tau\varphi$ and $\tau\psi$ are involutive. Since o lies in $F(\tau\varphi; M)$ and in $F(\tau\psi; M)$, $\tau\varphi$ and $\tau\psi$ are reflections on M. So, by applying case (a) to this case, we have $(\tau\varphi)(\tau\psi) = s_o$ which implies $\varphi\psi = s_o$.

($\Longrightarrow$): Assume that $\varphi\psi = s_o$ for some point o in M. Then we have $\varphi\psi = \psi\varphi$. Thus,

$$\psi = \varphi\varphi\psi = \varphi s_o \qquad \text{and} \qquad s_o\varphi = \varphi\psi\varphi = \varphi^2\psi = \psi.$$

From these we obtain $\varphi s_o = s_o\varphi$. Similarly, we also have $\psi s_o = s_o\psi$. Therefore, by applying Theorem A, we see that either $o \in F(\varphi; M)$ or $\varphi(o)$ is a pole of o. Similarly, either $o \in F(\psi; M)$ of $\psi(o)$ is a pole of o.

If $o \in F(\varphi; M)$, then $\psi(o) = \varphi s_o(o) = \varphi(o)$. Hence, we have $o \in F(\varphi; M) \cap F(\psi; M)$. If $\varphi(o)$ is a pole of o, then $\varphi(o) = s_o(\varphi(o)) = \varphi\psi\varphi(o) = \psi(o)$. Since $\varphi(o) \neq o$, this implies $o \notin F(\varphi; M) \cup F(\psi; M)$. We consider these two cases separately.

Case (i). $o \in F(\varphi; M) \cap F(\psi; M)$. In this case, if we denote by B and D the connected components of the fixed point sets $F(\varphi; M)$ and $F(\psi; M)$ through o, respectively, then B and D are totally geodesic in M. If $T_oB \cap T_oD \neq \{0\}$, then there exists a nonzero vector X in $T_oB \cap T_oD$ and $(\varphi\psi)_*(X) = X$. Hence, we have $\varphi\psi \neq s_o$ which is a contradiction. Consequently, we get

$$T_oB \cap T_oD = \{0\}. \tag{5.1}$$

Now, for any vectors $X \in T_oB$ and $Y \in T_oD$, we have

$$g(X, Y) = g(X, \varphi_* Y) = g(s_{o*}X, s_{o*}\varphi_* Y) = -g(X, \psi_* Y) = -g(X, Y)$$

which implies that B and D meet orthogonally at o.

Now, we claim that $\dim B + \dim D = \dim M$. If not, there exists a nonzero vector Z in T_oM such that Z is perpendicular to B and D at o. Hence, we have

$$-Z = s_{o*}(Z) = \varphi_*\psi_*(Z) = \varphi_*(-Z) = Z$$

which is a contradiction. Consequently, Case (a) of the theorem occurs.

Case (ii). $o \notin F(\varphi : M) \cup F(\psi; M)$ and $\underline{o} := \varphi(o) = \psi(o)$ is a pole of o. In this case, there exists a Riemannian double covering transformation $\tau\colon M \to M$ such that $\tau(o) = \underline{o}$ (Theorem 2.3). Hence, we have $\tau\varphi(o) = \tau(\underline{o}) = o = \tau\psi(o)$ and $o \in F(\tau\varphi; M) \cap F(\tau\psi; M)$. Furthermore, by Theorem F, the double covering transformation τ commutes with φ and ψ. Therefore, we also have $(\tau\varphi)(\tau\psi) = \varphi\psi = s_o$. Consequently, case (b) of the theorem occurs. ∎

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