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A duality theorem for a nondifferentiable nonlinear fractional programming problem

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A duality theorem, and a converse duality theorem, are proved for a nonlinear fractional program, where the numerator of the objective function involves a concave function, not necessarily differentiable, and also the support function of a convex set, and the denominator involves a convex function, and the support function of a convex set. Various known results are deduced as special cases.

Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$, and $h : \mathbb{R}^n \to \mathbb{R}^m$ be continuous functions, with $-f$ and g convex. Let $S \subset \mathbb{R}^m$ be a closed convex cone, which may in particular be the nonnegative orthant $\mathbb{R}^m_+$; let the function h be S-convex [6]. Let C_1 and C_2 be closed convex sets in $\mathbb{R}^n$. Consider the nonlinear fractional programming problem

$$(1) \qquad (\text{P}): \ \underset{x \in X_0}{\text{maximize}} \ \frac{f(x)-s\left(x|C_1\right)}{g(x)+s\left(x|C_2\right)} \ \text{subject to} \ -h(x) \in S ,$$

in which X_0 is an open convex set in $\mathbb{R}^n$, $s\left(\cdot|C_i\right)$ is the support function of the set C_i $(i = 1, 2)$, and it is assumed that

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(2) $$x \in X_0 \ (\text{or } -h(x) \in S) \Rightarrow g(x) + s(x|C_2) > 0 .$$

Associate to (P) the problem

(D): $\underset{u\in X_0, y, z, v, w}{\text{minimize}} \ z$ subject to $z \geq 0$, $y \in S^*$, $v \in C_1$, $w \in C_2$,

(3) $$0 \in \partial(-f+zg)(u) + \partial(y^T h)(u) + (v+zw) ,$$

(4) $$-f(u) + zg(u) + y^T h(u) + (v+zw)^T u \geq 0 .$$

In (D), S^* is the dual cone to S [6], and ∂ denotes subdifferential [15].

Under suitable hypothesis, (D) will be shown to be a dual problem to (P); under somewhat different assumptions, (P) will be shown to be a dual problem to (D).

The constraint $-h(x) \in S$ is *locally solvable* [4], [5] at x_0 if $-h(x_0) \in S$ and, for some $\delta > 0$, whenever the direction d satisfies $-h(x_0) - h'(x_0; d) \in S$ and $\|d\| < \delta$ (where $h'(x_0; d)$ denotes directional derivative in direction d), there exists a solution $x = x_0 + \alpha d + o(\alpha)$ to $-h(x) \in S$, valid for sufficiently small $\alpha > 0$. (This requirement reduces to the Kuhn-Tucker constraint qualification for a constraint system $h_i(x) \leq 0$ $(i = 1, 2, \ldots, m)$.) The problem (P) will be said to satisfy a *constraint qualification* at x_0 if $-h(x_0) \in S$, and *either*

(a) Slater's constraint qualification holds, namely $-h(x) \in \text{int } S$ for some $x \in X_0$, *or*

(b) $-h(x) \in S$ is locally solvable at $x_0 \in X_0$, and the set

(5) $$\bigcup_{s\in S^*} \{sh(x_0)\} \times \partial(sh)(x_0)$$

is closed in $\mathbb{R} \times \mathbb{R}^m$.

(The latter is automatic if S is a polyhedral cone and h is differentiable at x_0 [9].)

Duality theorem

The assumptions stated in the Introduction will be assumed throughout.

THEOREM 1. *Weak duality holds for* (P) *and* (D), *namely* $\sup(P) \le \inf(D)$. *If* (P) *reaches a maximum at* $x = x_0 \in X_0$, *if* $\max(P) \ge 0$, *and if a constraint qualification holds for* (P), *then* (D) *reaches a minimum at some* (u, z, y, v, w) *with* $u = x_0$, *and* $\max(P) = \min(D)$.

Thus (D) *is a* strong dual [7], [8] *to* (P).

Proof. Let x be feasible for (P), and let (u, z, y, v, w) be feasible for (D). From a constraint for (D), $\theta + \psi + (v+zw) = 0$ for some $\theta \in \partial\varphi(u)$ and some $\psi \in \partial\left(y^T h\right)(u)$, where $\varphi = -f + zg$. Since $z \ge 0$ and $-f$ and g are convex, φ is convex. Then

$$\begin{aligned}
&\left[f(x)-s\left(x|C_1\right)\right] - z\left[g(x)+s\left(x|C_2\right)\right] \\
&\quad = f(x) - zg(x) - (v+zw)^T x \\
&\quad = -\varphi(x) - (v+zw)^T x \\
&\quad \le -\varphi(u) - \theta^T(x-u) - (v+zw)^T(x-u) - (v+zw)^T u \quad \text{since } \theta \in \partial\varphi(u) \\
&\quad = -\varphi(u) - (v+zw)^T u + \psi^T(x-u) \quad \text{by a constraint for (D)} \\
&\quad \le y^T h(u) + \psi^T(x-u) \quad \text{by a constraint for (D)} \\
&\quad \le y^T h(x) \quad \text{since } y^T h \text{ is a convex function} \\
&\quad \le 0 \quad \text{since } -h(x) \in S \text{ and } y \in S^* .
\end{aligned}$$

By hypothesis, $g(x) + s\left(x|C_2\right) > 0$. Dividing by it,

$$\left[f(x)-s\left(x|C_1\right)\right]/\left[g(x)+s\left(x|C_2\right)\right] \le z .$$

Hence $\sup(P) \le \inf(D)$.

Now assume that (P) is maximized at x_0 , with $m = \max(P) \ge 0$, and also the constraint qualification. Then x_0 also maximizes

$$\left[f(x)-s\left(x|C_1\right)\right] - m\left[g(x)+s\left(x|C_2\right)\right] \tag{6}$$

subject to $-h(x) \in S$. Applying the appropriate nondifferentiable version of the Kuhn-Tucker Theorem, assuming constraint qualification (a) or (b)

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(see [9], Theorem 4),

$$0 \in \partial(-f+mg)(x_0) + v + w + \partial(y^T h)(x_0) \;,\quad y^T h(x_0) = 0 \;,$$

holds for some $y \in S^*$, $v \in \partial s(x_0|C_1)$, and $w \in \partial s(x_0|C_2)$. Then [15], $v \in C_1$, $v^T x_0 = s(x_0|C_1)$, $w \in C_2$, $w^T x_0 = s(x_0|C_2)$; therefore all constraints of (D) except (4) hold for $u = x_0$, $z = m, y, v, w$, and (4) holds also from $[f(x_0)-s(x_0|C_1)]/[g(x_0)+s(x_0|C_2)] = m$ and $y^T h(x_0) = 0$. □

Converse duality theorem

Again assume all assumptions of the Introduction, including (2). Denote by $F(x)$ the objective function for (P).

Suppose that (3) and (4) are satisfied for

$$(u, z, y, v, w) = (u_0, z_0, y_0, v_0, w_0) \;,$$

where $y_0 \in S^*$, $v_0 \in C_1$, $w_0 \in C_2$, and $g(u_0) + w_0^T u_0 > 0$. The system

(3) $$0 \in \partial(-f+zg)(u) + \partial(y^T h)(u) + (v+zw) \;,$$

together with

(7) $$z = [f(u)-v^T u-y^T h(u)]/[g(u)+w^T u] \;,$$

will be called *solvable near* u_0 if, whenever $y = y_0 + \beta\tilde{y} \in S^*$, $v = v_0 + \beta\tilde{v} \in C_1$, and $w = w_0 + \beta\tilde{w} \in C_2$, for $0 \le \beta \le 1$, then the system (3) and (7) has a solution $u = u_0 + \tilde{u}(\beta)$ for all sufficiently small positive β, satisfying $\tilde{u}(\beta) \to 0$ as $\beta \downarrow 0$.

This property holds, in particular, if $-f + z_0 g + y_0^T h$ has a non-singular hessian matrix, at $u = u_0$, in consequence of the implicit function theorem. However, the following converse duality theorem does not assume any differentiability of the functions f, g, h.

THEOREM 2. *Let* (D) *reach a minimum at*

$$(u, z, y, v, w) = (u_0, z_0, y_0, v_0, w_0) \;,$$

where $u_0 \in X_0$. *If* $z_0 = 0$, *assume that* $F(\hat{x}) \geq 0$ *for some* $\hat{x} \in X_0$ *satisfying* $-h(\hat{x}) \in S$. *If* $z_0 > 0$, *assume that the system* (3) *and* (7) *is solvable near* u_0 . *Then* (P) *reaches a maximum, and* $\max(P) = \min(D)$. *Hence* (P) *is a strong dual to* (D).

Proof. If $z_0 = 0$, then $F(\hat{x}) \leq z_0 = 0$ by weak duality, also $F(\hat{x}) \geq 0$ by assumption. Hence $F(\hat{x}) = z_0$, and weak duality implies that $\hat{x}$ is optimal for (P), so that $\max(P) = \min(D)$. (Weak duality is available from Theorem 1.)

Suppose now that $z_0 \neq 0$; since $z_0 \geq 0$ for (D), $z_0 > 0$. Choose any $\tilde{y}, \tilde{v}, \tilde{w}$ so that $y_0 + \tilde{y} \in S^*$, $v_0 + \tilde{v} \in C_1$, $w_0 + \tilde{w} \in C_2$. Since S^*, C_1 , and C_2 are convex sets, $y = y_0 + \beta\tilde{y} \in S^*$, $v = v_0 + \beta\tilde{v} \in C_1$, and $w = w_0 + \beta\tilde{w} \in C_2$, whenever $0 \leq \beta \leq 1$. By assumption $g(u_0) + w_0^T u_0 > 0$, and then (3) and (7) have a solution $u = u_0 + \tilde{u}(\beta)$ for sufficiently small $\beta > 0$. By continuity, $g(u) + w^T u > 0$ for sufficiently small $\beta > 0$; hence (7) implies (4). Hence this point (u, z, y, v, w) , with $u = u_0 + \tilde{u}(\beta)$ and z given by (7), is feasible for (D), for sufficiently small $\beta > 0$.

Since $(u_0, z_0, y_0, v_0, w_0)$ minimizes (D),

$$(8) \qquad z_0 \leq [f(u)-v^T u-y^T h(u)]/[g(u)+w^T u] \equiv p/q ,$$

using (7), where

$$(9) \quad p = f(u_0) + \beta f'(u_0; \tilde{u}) - v_0^T u_0 - \beta v_0^T \tilde{u} - \beta \tilde{v}^T u_0 - y_0^T h(u_0) - \beta y_0^T h'(u_0; \tilde{u}) - \beta \tilde{y}^T h(u_0) + o(\beta) ,$$

$$(10) \quad q = g(u_0) + \beta g'(u_0; \tilde{u}) + w_0^T u_0 + \beta w_0^T \tilde{u} + \beta \tilde{w}^T u_0 + o(\beta) .$$

Combining these terms shows that

$$(11) \qquad (p_0 - z_0 q_0) + \beta R - \beta\left[\tilde{y}^T h(u_0) + (\tilde{v} + z_0 \tilde{w})^T u_0\right] + o(\beta) \geq 0 ,$$

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where $p_0 = f(u_0) + y_0^T h(u_0) + v_0^T u_0$, $q_0 = g(u_0) + w_0^T u_0$, and

(12) $$R = f'(u_0;\, \tilde{u}) - z_0 g'(u_0;\, \tilde{u}) - \left(y_0^T h\right)'(u_0;\, \tilde{u}) - v_0^T \tilde{u} - z_0 w_0^T \tilde{u} .$$

Then $p_0 - z_0 q_0 \geq 0$; but $p_0 - z_0 q_0 \leq 0$ by (4), so $p_0 - z_0 q_0 = 0$. Dividing (11) by β and letting $\beta \downarrow 0$, then shows that

(13) $$R - \tilde{y}^T h(u_0) - (\tilde{v} + z_0 \tilde{w})^T u_0 \geq 0 .$$

Denote $\psi = -f + z_0 g + y^T h$. From (3), $\theta + v_0 + z_0 w_0 = 0$ for some $\theta \in \partial\psi(u_0)$. Then $\psi'(u_0;\, \tilde{u}) \geq \theta^T \tilde{u}$. Since $R = \psi'(u_0;\, \tilde{u}) - (v_0 + z_0 w_0)^T \tilde{u}$, it follows that $R \leq -(\theta + v_0 + z_0 w_0)^T \tilde{u} = 0$. Hence

(14) $$\tilde{y}^T h(u_0) + (\tilde{v} + z_0 \tilde{w})^T u_0 \leq 0 .$$

Setting $\tilde{y} = 0$ and $\tilde{w} = 0$, $\tilde{v}^T u_0 \leq 0$ whenever $v_0 + \tilde{v} \in C_1$. Hence $v^T u_0 \leq v_0^T u_0$ for each $v \in C_1$. Therefore $s(u_0|C_1) \leq v_0^T u_0$. Since $v_0 \in C_1$, by a constraint of (D), $s(u_0|C_1) \geq v_0^T u_0$. Hence $v_0^T u_0 = s(u_0|C_1)$. Since $z_0 > 0$, a similar argument applied to $\tilde{w}^T u_0$ shows that $w_0^T u_0 = s(u_0|C_2)$. Now let $\tilde{v} = 0$ and $\tilde{w} = 0$. Then $\tilde{y}^T h(u_0) \leq 0$ whenever $\tilde{y} \in S^*$; since S is a closed convex cone, it follows that $-h(u_0) \in S$. Setting $\tilde{y} = -\tfrac{1}{2} y_0$, $y_0 + \tilde{y} \in S^*$, and then $(-\tfrac{1}{2} y_0)^T h(u_0) \leq 0$. But also $y_0^T h(u_0) \leq 0$ since $y_0 \in S^*$, and $-h(u_0) \in S$ has just been proved. Therefore $y_0^T h(u_0) = 0$.

Thus u_0 is feasible for (P), and the optimal objective function for (D) equals

(15) $$z_0 = [f(u_0) - s(u_0|C_1) - 0] / [g(u_0) + s(u_0|C_2)] = F(u_0) .$$

Using weak duality, it follows that u_0 is optimal for (P). Hence

$\max(P) = \min(D)$. □

Discussion and examples

If f, g , and h are differentiable functions, then (3) reduces to

$$0 = \left(-f+zg+y^T h\right)'(u) + (v+zw) . \tag{16}$$

For nondifferentiable functions, an equivalent to (3) is (see [15])

$$(\text{for all } t)\ \ \varphi'(u;\, t) + (v+zw)t \geq 0 , \tag{17}$$

where $\varphi = -f + zg + y^T h$.

The technique of proof for Theorem 2 is adapted from that of [9], Theorem 6, and [5], Theorem 4.8.1. Since the "solvable near u_0 " requirement in Theorem 2 does not demand a *unique* solution for u , the usual implicit function theorem is assuming too much. A general verifiable solvability criterion for the convex nondifferentiable case has yet to be found. An inclusion of the form $\rho \in \partial\Phi(u)$ must be solved (nonuniquely) for u , given $\rho_0 \in \partial\Phi\left(u_0\right)$. For this it suffices if $\partial\Phi(\cdot)$ maps a neighbourhood of u_0 onto some neighbourhood of ρ_0 . As a simple example let $\Phi(u) = \left(u^T u\right)^{\frac{1}{2}} + \frac{1}{2}\varepsilon u^T u$, for $u \in \mathbf{R}^n$ and ε a positive constant; set $u_0 = 0$. Denote $B(r) = \left\{\xi \in \mathbf{R}^n : \|\xi\| \leq r\right\}$. Then $\partial\Phi\left(u_0\right) = B(1)$, and $\partial\Phi(\cdot)$ maps $B(\delta)$ onto $B(1+\varepsilon\delta)$, so that the sufficient requirement is fulfilled, provided that $\|\rho_0\| \leq 1$, for this nondifferentiable function Φ . This is not so if $\varepsilon = 0$.

Problem (P) includes various special cases. If f, g , and h are affine, then (P) reduces to that considered by Mond and Schechter [13], [14]. As noted in [13], if B is a positive semidefinite matrix, and Q is the compact set $\left\{Bv : v^T Bv \leq 1\right\}$, then

$$s(x|Q) = \left(x^T Bx\right)^{\frac{1}{2}} . \tag{18}$$

Thus, if f, g , and h are differentiable functions, and $s\left(\cdot|C_1\right)$ and $s\left(\cdot|C_2\right)$ are defined as in (18) by positive semidefinite matrices B and D , problem (P) becomes the nondifferentiable fractional programming problem considered in [11]. If also f, g, and h are affine, the results of Chandra and Gulati [3] are obtained. If $g(x) \equiv 1$, and $C_2 = \{0\}$, then $s(x|C_2) = 0$, so that a nonfractional nondifferentiable objective function $F(x) = f(x) - s(x|C_1)$ is recovered.

If S is a $k \times n$ matrix, then [13]

(19) $$\|Sx\|_p = s(x|Q)$$

for $p \geq 1$, where $Q = \left\{S^T u : \|u\|_q \leq 1\right\}$, and $p^{-1} + q^{-1} = 1$, and $q = \infty$ if $p = 1$. Here $\|x\|_p = \left[\sum |x_i|^p\right]^{1/p}$ if $p < \infty$, and $\|x\|_\infty = \sup\{|x_i| : i = 1, 2, \ldots\}$. The set Q, as defined, is convex and compact.

Thus if f, g, and h are differentiable, $s(\cdot|C_1)$ is defined as in (19) by a matrix S_1 and a scalar p_1, and similarly $s(\cdot|C_2)$ by a matrix S_2 and a scalar p_2, then problems (P) and (D) become respectively

$$\text{(P')}: \quad \underset{x \in X_0}{\text{maximize}} \ \left[f(x) - \|S_1 x\|_{p_1}\right] / \left[g(x) + \|S_2 x\|_{p_2}\right] \ \text{subject to} \ h(x) \leq 0 ,$$

$$\text{(D')}: \quad \underset{u \in X_0, z, y, v, w}{\text{minimize}} \ z \ \text{subject to} \ z \geq 0, \ y \geq 0, \ \|v\|_{q_1} \leq 1, \ \|w\|_{q_2} \leq 1 ,$$

$$\nabla\left(y^T h - f + zg\right)(u) + S_1^T v + S_2^T w = 0 ,$$

$$-f(u) + zg(u) + y^T h(u) + u^T\left(S_1^T v + S_2^T w\right) \geq 0 .$$

If, in particular, f, g, and h are affine functions, then some of the problems discussed in [13] are obtained.

If f, g, and h are differentiable, and C_2 consists only of the zero vector in $\mathbb{R}^n$, and $s(\cdot|C_1)$ is defined, as in (18), by a positive semidefinite matrix B, then (P) and (D) reduce to the problems considered by Aggarwal and Saxena [1], [2]. If also $g(x) \equiv 1$, one obtains the (nonfractional) problems discussed in [10].

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If f, g, and h are differentiable, $C_2 = \{0\}$, $g(x) \equiv 1$, and $s(\cdot|C_1)$ is defined as in (19) by a matrix S, then the present results yield those of Mond and Schechter [12].

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