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FINITE HIGHER COMMUTATORS IN ASSOCIATIVE RINGS

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Abstract

If T is any finite higher commutator in an associative ring R, for example, $T = [[R, R], [R, R]]$, and if T has minimal cardinality so that the ideal generated by T is infinite, then T is in the centre of R and $T^2 = 0$. Also, if T is any finite, higher commutator containing no nonzero nilpotent element then T generates a finite ideal.

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1. Introduction

In this paper the term 'ring' will mean an associative ring, but not necessarily containing an identity element. There are a number of papers in the literature that study when certain finite subsets of a ring generate finite ideals (for example, [1, 2, 4, 5, 7]). These led to [9] and [10], which also considered subsets with infinite cardinality. Motivated by these sorts of results, we showed in [8] that a finite higher commutator in a semiprime ring must generate a finite ideal in that ring. A simple commutator in a ring R is any $xy - yx = [x, y]$ for $x, y \in R$. A higher commutator, defined below, can be thought of as an additive subgroup of $(R, +)$ generated by a fixed succession of commutators. Simple examples are $[R, R]$, the additive subgroup generated by $\{[x, y] \mid x, y \in R\}$, and similarly $[[[R, R], R], R]$, or $[[[R, R], R], [R, R]]$. For general rings it is difficult to characterise these sets in other ways, to understand the relations between them, or to determine the ideals they generate. The purpose of this paper is to generalise [8]: must any higher commutator in R that has only finitely many elements generate a finite ideal in R? We know no example of an infinite ring having a finite higher commutator that generates an infinite ideal.

We cannot prove a complete generalisation of [8], but we are able to prove that if T is a finite higher commutator that generates an infinite ideal, and if the cardinality of such a T is minimal, then T must be a central subring with trivial multiplication. If there is an example of a finite T in a ring R generating an infinite ideal, then the direct sum of R with suitable matrix rings over finite fields would have a finite higher commutator generating an infinite ideal but neither central nor nilpotent. Thus our assumption that T has minimal cardinality excludes this direct sum possibility and enables the use of inductive arguments. We also prove a result showing that any finite higher commutator that generates an infinite ideal must contain nonzero nilpotent elements. Our approach here does not extend to the case of higher commutators of infinite cardinality, which was the case in [8].

2. Definitions and preliminary results

We begin by reviewing the formal definition of higher commutators.

Definition 2.1. In any noncommutative ring R, the unique *higher commutator* of weight 1 is R and the only higher commutator of weight 2 is the additive subgroup $[R, R] = R_2$ generated by $\{[a, b] = ab - ba \mid a, b \in R\}$. A higher commutator T of weight $m > 1$, written $\mathrm{wt}(T) = m$, is the additive subgroup $[V, W]$ generated by $\{[v, w] \mid v \in V, w \in W\}$ for some higher commutators V and W with $\mathrm{wt}(V) + \mathrm{wt}(W) = m$, where m is minimal among all choices of V and W so that $T = [V, W]$.

It is clear that every higher commutator of R is the additive subgroup generated by the evaluations in R of a homogeneous and multilinear polynomial with coefficients ± 1 in $\mathbb{Z}\{X\}$ for $X = \{x_1, x_2, \ldots\}$, which is a countable set of noncommuting indeterminates over $\mathbb{Z}$. Call such a polynomial a *commutator polynomial*. Some examples of commutator polynomials are $[x_1, x_2]$ for $[R, R]$, $[[x_1, [x_2, x_3]], [x_4, x_5]]$ for $[[R, [R, R]], [R, R]]$, and $[[[x_1, x_2], [x_3, x_4]], [[x_5, x_6], [x_7, x_8]]]$ for $[[[R, R], [R, R]], [[R, R], [R, R]]]$.

It will be useful to set notation for some simple commutator polynomials.

Definition 2.2. In the free algebra $\mathbb{Z}\{X\}$ for $X = \{x_1, x_2, \ldots\}$, a countable set of noncommuting indeterminates over $\mathbb{Z}$, let $f_1 = f_1(x_1) = x_1$, $f_2 = f_2(x_1, x_2) = [x_1, x_2] = x_1x_2 - x_2x_1$, and for $n > 1$, $f_n = f_n(x_1, \ldots, x_n) = [f_{n-1}, x_n]$. Also, set $f^{(1)} = f_2$, and for $m \geq 1$ let

$$f^{(m+1)} = f^{(m+1)}(x_1, \ldots, x_{2^{m+1}}) = [f^{(m)}(x_1, \ldots, x_{2^m}), f^{(m)}(x_{2^m+1}, \ldots, x_{2^{m+1}})].$$

Let the *additive subgroup* generated by all evaluations of f_n in R be denoted by $f_n(R) = R_n$ and the subgroup generated by the evaluations of $f^{(n)}$ in R be denoted by $f^{(n)}(R) = R^{(n)}$.

Since in any ring $R, [a, b] = -[b, a]$, we have $[V, W] = [W, V]$ for any higher commutators V and W. If T is a higher commutator of R with $\mathrm{wt}(T) = m$ then there is a commutator polynomial G of degree m with $T = G(R^m)$. It can happen that $T = G(R^m) = F(R^s)$ for F another commutator polynomial with $m \leq s$ and $G \neq F$. For example, if $R = M_n(K)$ for K a field and $n > 2$, then (see Lemma 2.3 below) $f_i(R) = f_j(R)$ for any $2 \leq i \leq j$. Thus there are infinitely many commutator polynomials all of whose sets of evaluations would be R_2. If R is nilpotent of index $m > 1$, then any commutator polynomial G of degree $k \geq m$ satisfies $G(R^k) = (0)$. When T is a higher

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commutator with $\mathrm{wt}(T) = m$ and G is any commutator polynomial of degree m with $T = G(R^m)$ we denote G by F_T and then write $T = F_T(R)$ rather than $T = F_T(R^m)$. This possible ambiguity in the choice of $F_T = G$ will not cause any confusion in our arguments. Observe that any higher commutator of weight at least 2 is contained in $R_2 = R^{(1)}$.

In [8] we showed that for any finite higher commutator W in a ring R, the higher commutator $[W, R]$ generates a finite ideal, denoted by $([W, R])$. Thus the interest here is in higher commutators not of this form, although there seems to be no advantage in assuming this to start with. For any nonempty subset S of the ring R we let (S) denote *the ideal of R generated by S*, and for $y \in R$ set $(\{y\}) = (y)$. We note that even when $R^{(2)} = [[R, R], [R, R]]$ is finite, the results and techniques in [8] do not imply that $(R^{(2)})$ must be finite, unless R is a semiprime ring. Furthermore, the main results in [8] depend in an essential way on the assumption that R is semiprime so a different approach is needed here.

The higher commutators in matrix rings over fields are well known and easily found by elementary computation with matrix units. We state this result next.

Lemma 2.3. *Let $A = M_n(F)$ for $n > 1$ and F a field, and let V be any higher commutator of A with* $\mathrm{wt}(V) > 1$. *When $n = 2$ and* char $F = 2$ *then either $V = [A, A] = f_k(A)$ for any $k > 1$, $V = f^{(2)}(A) = [[A, A], [A, A]] = FI_2$, or $V = (0)$. When $n > 2$ or* char $F \neq 2$ *we have that $V = [A, A] = f_k(A) = f^{(s)}(A)$ for all $k > 1$ and $s \geq 1$.*

Two easy observations that will be useful for reference come next. The first gives a useful dichotomy for higher commutators. The second observation, whose proof is clear, shows that higher commutators behave well under homomorphic images. An additive subgroup L of a ring R is a *Lie ideal* of R when $xr - rx = [x, r] \in L$ for all $x \in L$ and all $r \in R$.

Lemma 2.4. *If V is a higher commutator of weight $m > 1$ in the ring A, then V is a Lie ideal of A and either $V \subseteq A^{(2)} = f^{(2)}(A)$ or $V = A_m = f_m(A)$. In particular, when $V \nsubseteq A^{(2)}$ we can take $F_V = f_m$.*

Proof. It is well known that all higher commutators are Lie ideals. This follows from the identity $[[x, y], r] = [[x, r], y] + [x, [y, r]]$ which shows that $[A, A]$ is a Lie ideal of A; in fact, this identity shows that the commutator of any two Lie ideals is again a Lie ideal. Thus any higher commutator is a Lie ideal by induction on its weight. The second conclusion of the lemma is obvious if $2 \leq \mathrm{wt}(V) \leq 4$ since these higher commutators are A_2, A_3, A_4, and $A^{(2)}$. Assume that $\mathrm{wt}(V) = m > 4$ and that $V = [W, U]$, for higher commutators W and U. If $U = A$ then by induction on weight, either $W \subseteq A^{(2)}$ forcing $V \subseteq A^{(2)}$ since W is a Lie ideal, or $W = A_{m-1}$ so $V = [A_{m-1}, A] = A_m$. Clearly by this argument we may assume that $\mathrm{wt}(U), \mathrm{wt}(W) > 1$. However, now $W, U \subseteq A^{(1)}$ force $V \subseteq A^{(2)}$. □

Lemma 2.5. *If T is a finite higher commutator of R and I is an ideal of R then in the quotient ring R/I, $F_T(R/I) = T + I$ is a higher commutator of R/I, and $(F_T(R/I)) = (T) + I$.*

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It is natural to ask why we are considering higher commutators rather than arbitrary Lie ideals. One reason is that any finite additive subgroup A of the centre $Z(R)$ of R is a finite Lie ideal, and for many examples with R infinite, A will generate an infinite ideal, even in semiprime rings. Elementary examples are the polynomial rings $K[X]$ or free algebras $K\{X\}$ with K a finite field and X infinite, the matrix ring $M_n(L)$ for L an infinite field with char $L = p > 0$, and $\mathbb{Z}[y]\{X\}/(3y, y^2)$ with X infinite. Taking the sum of any finite Lie ideal with a finite additive subgroup of the centre will produce a finite Lie ideal that again is likely to generate an infinite ideal. Considering higher commutators excludes arbitrary central Lie ideals, but when a higher commutator is central we have more information about it to use. Further, by Lemma 2.5, higher commutators behave well when taking quotients—an important technique in using induction arguments.

For any higher commutator T we may assume that $T = F_T(R)$ for F_T a multilinear and homogeneous (commutator) polynomial, so T is a module over $Z(R)$. The proof of our main theorem requires two related and known results that we state next in the form needed here. Our statement gives parts of, or special cases of, [8, Theorems 1 and 5].

Theorem A. *Let R be any ring and T a higher commutator of R. If $[T, R]$ is finite then so is $([T, R])$, and if T is finite and R is a semiprime ring then (T) is finite.*

3. Main theorem

Our main result characterises T when it is finite with minimal cardinality so that (T) is infinite. We denote the cardinality of any set S by $|S|$.

Theorem 3.1. *If T is a finite higher commutator in the ring R so that (T) is infinite, and if among all such choices of T and R, $|T|$ is minimal, then $T \subseteq Z(R)$, the centre of R, and $T^2 = (0)$.*

Proof. We first show that $T \subseteq Z(R)$. If this were not true, then since $T = F_T(R) \nsubseteq Z(R)$, we would have $(0) \neq [T, R] \subseteq T$ since T is a Lie ideal by Lemma 2.4. From Theorem A, $I = ([T, R])$ is a finite ideal of R. Clearly, if $T \subseteq I$ we get the contradiction that (T) is finite. Otherwise in R/I, by Lemma 2.5, $1 < |F_T(R/I)| = |T + I| < |T|$. By the minimality of $|T|$ we see that in R/I, $(F_T(R/I)) = (T) + I$ is finite, using Lemma 2.5 again. But $(T) + I$ finite in R/I and I finite force the contradiction that (T) is finite. Consequently $T \subseteq Z(R)$. As we observed above, T is a $Z(R)$ module, so T is a subring of R.

Our next claim is that if T is a nilpotent ring then $T^2 = (0)$. If $T^2 \neq (0)$, let the index of nilpotence of T be $k > 2$. Clearly $(0) \neq T^{k-1} \subseteq \operatorname{ann}(T)$, the annihilator of T, which is an ideal of R since $T \subseteq Z(R)$. Also, T a subring implies $T \cap \operatorname{ann}(T) \neq (0)$. As above, from Lemma 2.5, we get from $T^2 \neq (0)$ that $1 < |F_T(R/\operatorname{ann}(T))| = |T + \operatorname{ann}(T)| < |T|$, hence the minimality of $|T|$ shows that $(T) + \operatorname{ann}(T)$ is a finite ideal of $R/\operatorname{ann}(T)$. For $t \in T$ define $g_t\colon R \to (T) + \operatorname{ann}(T) \subseteq R/\operatorname{ann}(T)$ by $g_t(r) = rt + \operatorname{ann}(T)$. It is immediate that g_t is additive with a finite image in $R/\operatorname{ann}(T)$ so $\ker g_t$ has finite (additive) index in R. Since T is finite, as is well known [11, Theorem 4.3, page 160], it follows that $K = \bigcap_T \ker g_t$ is an additive subgroup of R of finite index. Thus there are $a_1, \ldots, a_m \in R$ so that $R = \bigcup_j \{a_j + K\}$. From the definition of K we have $KT \subseteq \mathrm{ann}(T)$, so $RT = \bigcup_j \{a_j T + KT\}$ implying that $RT^2 = \bigcup_j \{a_j T^2\}$ is finite. Since T is central, $(T^2) = T^2 + RT^2$ is finite. Again by Lemma 2.5 and the minimality of $|T|$, either $(T) + (T^2)$ is finite and nonzero in $R/(T^2)$, forcing the contradiction that (T) is finite, or $T \subseteq (T^2)$. The latter possibility, together with $k > 2$ the index of nilpotence of T, yields $(0) \neq T^{k-1} \subseteq (T^2)^{k-1} = (T^{2(k-1)}) = (0)$ since $2k - 2 > k$. This contradiction establishes our claim that T nilpotent forces $T^2 = (0)$.

From above, T is a subring of $Z(R)$ and if T is nilpotent then $T^2 = (0)$, proving the theorem. Thus, we may assume that T is not nilpotent and prove the theorem by contradiction, by showing that (T) must be finite. Now T a finite ring means that its nil radical $J(T)$ is nilpotent. Hence, the finite semiprime ring $T/J(T)$ has an identity element, say $u + J(T)$. Since $u^2 - u \in J(T)$ is nilpotent it follows that T contains a nonzero idempotent e [3, Lemma 1.3.2, page 22]. Briefly, if $(u^2 - u)^k = 0$, then $u^k = u^{k+1} p(u)$ for some $p(x) \in \mathbb{Z}[x]$. This yields $u^k = u^{2k} q(u)$ with $q(x) \in \mathbb{Z}[x]$, and then $u^k q(u) = e = e^2$. Should $e = 0$ then $u^k = u^k e = 0$ so $u + J(T) \neq 1_{T/J(T)}$. Using $e \in T \subseteq Z(R)$, we may write $R = eR \oplus (1 - e)R$ as ideals, where $(1 - e)R = \{r - er \mid r \in R\}$ and $eR(1 - e)R = (0) = (1 - e)ReR$. Since F_T is multilinear and homogeneous, $T = F_T(R) = F_T(eR) + F_T((1 - e)R)$, so if $e \neq 1_R$, the identity element of R, then either $eT = T$ or by the minimality of $|T|$ we have that $(F_T(eR))$ and $(F_T((1 - e)R))$ are finite, forcing (T) to be finite. Therefore $T = eT \subseteq eR$, so (T) in R is the ideal of eR generated by T and we may replace R with eR: $T \subseteq eR$ generates an infinite ideal of eR exactly when it generates an infinite ideal of R. In particular, we may assume that $e = 1_R$. Since now $1_R \in T \subseteq Z(R)$ and T is a $Z(R)$ module, it follows that $T = Z(R)$. Note that our argument shows that the only central idempotents in T are 0 or 1_R; that is, T cannot contain proper idempotents.

Suppose that for some $0 \neq y \in T$, $y^s = 0$ for some minimal $s > 1$. Using that $y \in Z(R)$ and that R has an identity element, $(y) = Ry$. Consider the chain of ideals $R \supsetneq Ry \supsetneq Ry^2 \supsetneq \cdots \supsetneq Ry^{s-1} \supsetneq (0)$. Setting $y^0 = 1_R$, for each $0 \leq j \leq s - 1$ the quotient Ry^j/Ry^{j+1} is naturally an R/Ry module generated by $y^j + Ry^{j+1}$, so is finite if R/Ry is. Since $y \in T$ and $1_R \in T$, in the quotient R/Ry, $F_T(R/Ry) = T + Ry \neq Ry$ is a higher commutator with fewer elements than T, by Lemma 2.5. Thus the minimality of $|T|$ and $1_R \in T$ force $(T) + Ry = R/Ry$ to be finite. This implies that each Ry^j/Ry^{j+1} is finite. Consequently, R itself must be finite, and this contradiction shows that T cannot contain nonzero nilpotent elements. Since T is a finite commutative ring, it follows that T is a direct sum of finite fields. We have seen that T has no proper idempotents, so T is a finite field.

To finish the proof of the theorem we show that if T is a finite field with $1_R \in T$ then $R = (T)$ is finite. For N the prime (lower nil) radical of R, T a field yields $T \cap N = (0)$. Thus by Lemma 2.5, in R/N, $F_T(R/N) = T + N \cong T$, as rings. Since R/N is a semiprime ring and $1_R \in T$, Theorem A and Lemma 2.5 show that $(T) + N = R + N = R/N$ is a finite ring. Standard structure theory [3, Theorem 2.1.6, page 48 and Theorem 3.1.1, page 70] shows that R/N is a finite direct sum of matrix rings over finite fields. Now $F_T(R/N) = T + N \subseteq Z(R/N)$ is a field containing $1_R + N = 1_{R/N}$ and is a module for $Z(R/N)$, as we observed earlier, forcing $Z(R/N) = F_T(R/N) \cong T$, so R/N must be simple: $R/N \cong M_n(T)$. If $n = 1$ then R/N is commutative and $T + N = F_T(R/N) = (0)$, a contradiction. But with $n \geq 2$, Lemma 2.3 and $T \neq 0$ show that $R/N \cong M_2(T)$ with char $T = 2$.

Since char $T = 2$, and $1_R \in T$, $2R = (0)$ follows. Using the fact that N is a nil ideal and that R/N is a finite-dimensional T algebra, the Wedderburn principal theorem [6, Theorem 33, page 127] produces a T-subalgebra $S \cong R/N \cong M_2(T)$ of R so that, as T-vector spaces, $R = S \oplus N$. Note that if substitutions from R are made in F_T, with an element of N replacing some indeterminate, then that evaluation is in both T and N, so must be zero. Since F_T is multilinear and homogeneous $T = F_T(S)$ follows. In particular, $1_R \in T \subseteq S$.

Suppose that $T = [V, W]$ for higher commutators V and W of R. We may assume that the weights of V and of W are greater than 1 since otherwise $T = [R, W]$ or $T = [V, R]$ so (T) would be finite by Theorem A. Let $T = F_T(R) = [F_V(R), F_W(R)] = [F_V(R), F_W(S)]$, as just above, using $T \cap N = (0)$. If $V \subseteq R^{(2)}$ then $T = [V, F_W(S)] \subseteq [R^{(2)}, F_W(S)] = [(S + N)^{(2)}, F_W(S)] \subseteq [S^{(2)} + N, F_W(S)]$. By Lemma 2.3 $S^{(2)} = Z(S)$, so $T \subseteq [N, F_W(S)] \subseteq N$, a contradiction. Similarly, we cannot have $W \subseteq R^{(2)}$ and it follows from Lemma 2.4 that we may assume that $T = [f_k(R), f_m(R)] = [f_k(R), f_m(S)]$ for $k, m > 1$. In any evaluation of F_T in R, when $y \in N$ replaces an indeterminate in f_k then the evaluation results in $0 \in T$.

Let $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ be the standard matrix units in $S \cong M_2(T)$. We proceed with some computation using these and elements of N. Since char $R = 2$, $-r = r$ for all $r \in R$. Now, using $[e_{12}, e_{21}] = e_{11} + e_{22} = 1_S = 1_T = 1_R$, we have $[S, S] = Te_{12} + Te_{21} + T1_S = f_n(S)$ for all $n \geq 2$ by Lemma 2.3, so $e_{12}, e_{21} \in f_n(S)$ for all $n \geq 1$. Given $y \in N$, $f_k(y, e_{11}, \ldots, e_{11}) = ye_{11} + e_{11}y$ since $[y, e_{11}] = ye_{11} + e_{11}y$ and then $[[y, e_{11}], e_{11}] = [y, e_{11}]$, using $2R = (0)$. Therefore, from $T \cap N = (0)$ and $e_{21} \in f_m(S)$, it follows that $0 = [ye_{11} + e_{11}y, e_{21}] = e_{11}ye_{21} + e_{21}ye_{11} + e_{21}y$. Left multiply by e_{11} to get $e_{11}ye_{21} = 0$. Clearly, for any $i, j \in \{1, 2\}$ we have $e_{i1}ye_{2j} = 0$. Interchanging the roles of 1 and 2 in this computation shows that $e_{i2}ye_{1j} = 0$ for all $i, j \in \{1, 2\}$. Hence $y = 1_R y 1_R = (e_{11} + e_{22})y(e_{11} + e_{22})$ so our computations show that $y = e_{11}ye_{11} + e_{22}ye_{22}$.

We may write $f_k(R) = [f_{k-1}(R), R] = [R, f_{k-1}(R)]$, so in $[[R, f_{k-1}(S)], S_m] \subseteq T$ we have, for our $y \in N$, that $0 = [[y, e_{12}], e_{21}] = ye_{11} + e_{12}ye_{21} + e_{21}ye_{12} + e_{22}y$; note that $e_{12} \in f_{k-1}(S)$. Thus using $y = e_{11}ye_{11} + e_{22}ye_{22} = ye_{11} + e_{22}y$ we get $y = e_{12}ye_{21} + e_{21}ye_{12}$ from which it follows that $e_{11}ye_{11} = e_{12}ye_{21}$ and $e_{22}ye_{22} = e_{21}ye_{12}$. Similarly, we have $0 = [[ye_{21}, e_{12}], e_{21}]$. But $[ye_{21}, e_{12}] = ye_{22} + e_{12}ye_{21} = ye_{22} + e_{11}ye_{11} = y$, as above, so $0 = [[ye_{21}, e_{12}], e_{21}] = [y, e_{21}]$. By interchanging 1 and 2 in these computations, we obtain $[y, e_{12}] = 0$. Finally, $y = e_{11}ye_{11} + e_{22}ye_{22}$ implies that y commutes with e_{11} and e_{22}. The result is that N must centralise S. We have seen that $T = [f_k(S), f_m(S)]$. Since F_T is multilinear, for any $y \in N$, $t \in T$ and some $\{s_j\} \subseteq S$, $yt = y[f_k(s_1, \ldots, s_k), f_m(\ldots, s_i, \ldots)] = [f_k(ys_1, s_2, \ldots, s_k), f_m(\ldots, s_i, \ldots)] = 0$ using the fact that N centralises S and $S \cap N = (0)$. Therefore $NT = (0)$ so $1_R \in T$ means

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that $N = (0)$ or, equivalently, $R = S$ is finite, contradicting our basic assumption that (T) is infinite, and proving the theorem. □

If T is any finite higher commutator of a ring R then T is a subring of R [8, Theorem 8, page 52]. This is easy to see, since $([T, R]) = I$ is finite from Theorem A, so it suffices to show that $F_T(R/I) = T + I \subseteq R/I$ is a finite subring: the ring $T + I \subseteq R$ would be finite and contains T. But $F_T(R/I)$ is central in R/I and, as we have seen, $F_T(R/I)$ is a $Z(R/I)$ submodule, so it is a finite subring of R/I.

Additional assumptions on T can replace the minimal cardinality condition in Theorem 3.1, and so enable us to prove that any such finite, higher commutator generates a finite ideal. For example, if T is finite and the ring T contains a multiplicative identity element, then (T) is finite by following the proof of Theorem 3.1. A consequence of this is that if T is finite and contains no nonzero nilpotent element then again (T) is finite.

Theorem 3.2. *If T is a finite higher commutator of the ring R, and if the ring T contains an identity element with respect to multiplication, then (T) is a finite ideal of R.*

Corollary. *If T is a finite higher commutator in a ring R so that T contains no nonzero nilpotent element, then (T) is finite.*

Proof. As we observed above, any finite higher commutator T is a subring. Now by standard structure theory, as in the proof of Theorem 3.1, any finite ring with no nonzero nilpotent element is a direct sum of finite fields. Therefore T must have an identity element, so (T) is finite by Theorem 3.2. □

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