Multiple Periodic Variable Stars
IAU Colloquium No.29, Budapest, 1975

THEORY OF MULTIPERIODIC RR LYRAE STARS

R. F. Stellingwerf

Department of Astronomy, Columbia University, New York, N. Y.

1. Introduction

The subject "Multiperiodic RR Lyrae Stars" encompasses two entirely different types of phenomena: 1) long-term modulation of amplitude known as the Blazhko-effect and 2) simultaneous excitation of several radial modes. In the first category the observational material is now quite extensive, but very little theoretical analysis has been undertaken. In the second case the results of pulsation theory are applicable, but only one such object has been found. Much progress in our understanding of RR Lyrae stars has been made in recent years, however, and many results do have a bearing on the present problem. Several excellent reviews of current pulsation theory are available (see Iben 1971c, or Cox 1974a,b). I will therefore restrict my comments to those results bearing directly on choice of mode for RR Lyrae stars.

Many suggestions have been made as to the mechanism of the Blazhko-effect; among them we have:

1) Resonance effects in radial modes (Kluyver, 1936),
2) Resonances involving nonradial modes (Ledoux, 1951),
3) Splitting of radial modes caused by nonadiabatic effects (Ledoux, 1963, p. 421),

https://doi.org/10.1017/S0252921100061959 Published online by Cambridge University Press

4) Tidal effects (Fitch, 1967, 1968),

5) Oblique rotator effects (Balázs, 1959; Preston, 1964), and

6) Stellar magnetic cycle effects (Detre and Szeidl, 1972).

Note that a stellar magnetic cycle does not necessarily imply an oblique rotator since variations during a rotation period could be caused by other non-spherical effects, such as star-spots. The resonance and splitting suggestions for radial modes can now be ruled out since detailed nonlinear nonadiabatic models show no trace of such behavior. Also, Lucy (1975) has shown that a careful analysis predicts no splitting due to low order non-adiabatic effects. Resonance with a nonradial mode is unlikely to occur over the considerable range of observed periods. Detailed theoretical tests of the remaining hypotheses have not been attempted; in view of the complexity introduced, this is likely to remain an observational problem for the near future.

2. Linear Results

Although a linear stability analysis cannot predict full amplitude behavior, this technique provides much valuable information and is subject to less computational uncertainty than the nonlinear approach. In particular, it is likely that all observed modes are unstable in the linear theory. It is clear, however, that all unstable modes are not necessarily present in the final motion.

The first detailed models of this type were computed by Baker (1965). Models with a range of mass, effective temperature and composition at luminosities near $M_{bol} = 0.5$ were computed in this investigation. Convection was included in the static envelope, but convective variations were ignored. Baker found that his periods supported the suggestion by Schwarzschild (1940) that Bailey type a and b stars are fundamental

https://doi.org/10.1017/S0252921100061959 Published online by Cambridge University Press

pulsators, while those of type c prefer the first overtone (sometimes loosely referred to as "first harmonic"). Globular cluster variables show a rather abrupt switch of modes as a function of period. Baker's stability analysis indicated, however, that both modes are unstable over a wide range of effective temperature. This is shown in Figure 1 (Baker's Figure 6) where growth rate is plotted versus effective temperature for the fundamental mode--solid line, and the first overtone--dashed line, of a typical sequence. An estimate of the location of the variables in M3 is also shown. No hint of a mode switch is indicated. A continuation of this investigation and a detailed comparison with observations has been given by van Albada and Baker (1971a).

Baker's results may be compared with those of Castor (1971), who computed purely radiative models with mass $M = 0.58\ M_{\odot}$, bolometric magnitude $M_{bol} = 0.76$, helium abundance $Y = 0.3$ and metal abundance $Z = 0.002$. In Figure 2 (Castor's Figure 5) the growth rate η --fractional change in kinetic energy per period--is shown for the first three modes. The results are similar to Baker's except that the radiative instability strip extends to cooler models. Note that the second overtone is always stable.

An extension of these results to a very wide range of parameters has been reported by Iben and collaborators (Iben and Huchra 1970, 1971; Iben 1971a,b; Tuggle and Iben 1972, 1973). One result of interest here is the location of the blue edge of the instability strip in the H-R diagram. As shown in Figure 3 (Figure 19 of Iben 1971c) separate edges are found for the two lowest modes and the relative position of these edges depends on luminosity. At high luminosities (for a given mass) the fundamental mode dominates--as in W Virginis stars; at lower luminosities the first overtone growth rate is somewhat larger than the fundamental growth rate causing the first overtone blue edge to move to the left, and creating a region in

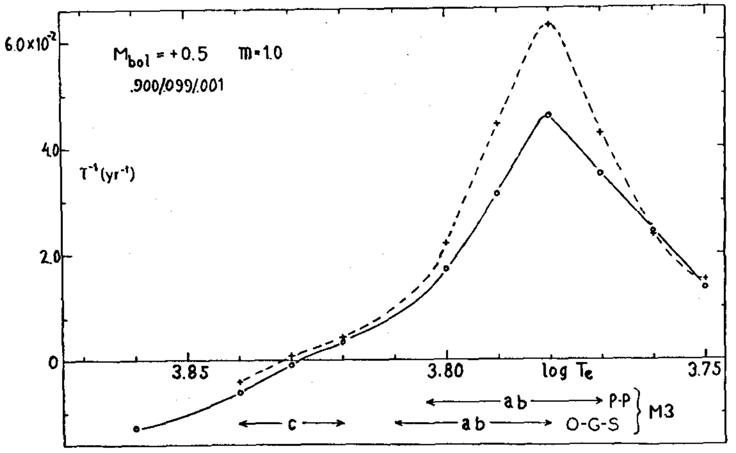

Fig. 1.

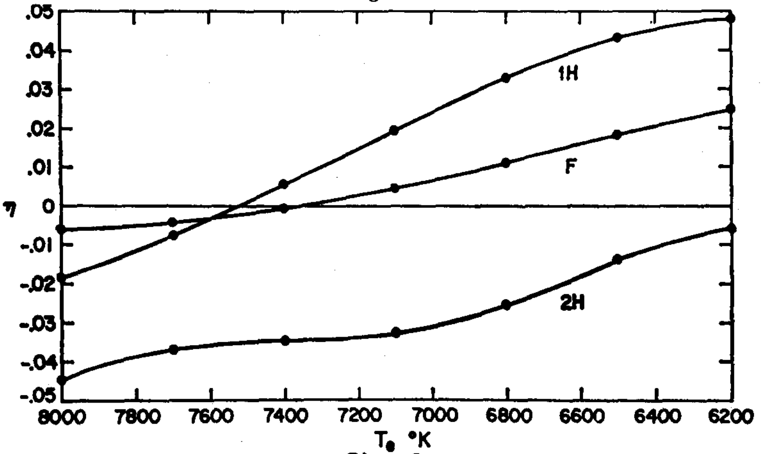

Fig. 2.

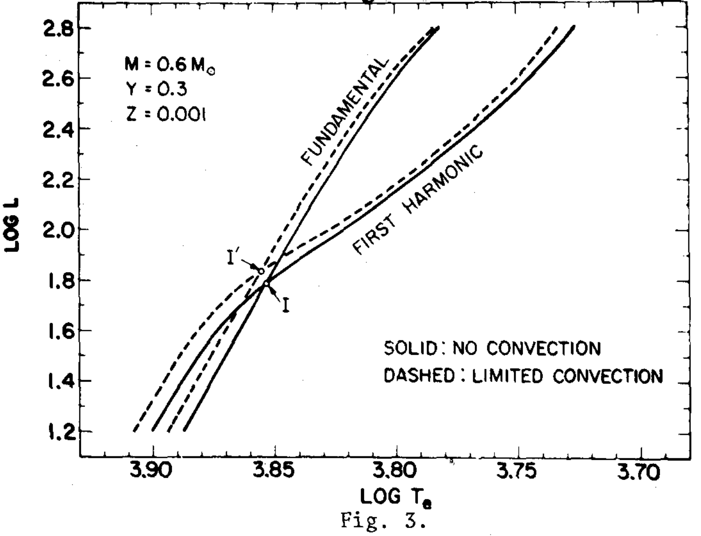

Fig. 3.

which the first overtone is the only possible mode. At still lower luminosities higher modes can also become unstable (Castor, 1970). If the mass is increased, the point of intersection of the two edges shifts to higher luminosities, while an increase in helium abundance causes a shift to higher temperatures. It has been shown by Cox, Castor and King (1972) that the reduction in excitation of higher modes at high luminosities is caused by the proximity of the outermost node to the driving region. The movement of the node appears to be a function of the parameter M/R, where R is the radius. The first overtone excitation is reduced when, in solar units, $M/R < 0.1$, and the period ratio P_1/P_0 is also reduced (Cox, King and Stellingwerf, 1972). For the second overtone both the growth rate and the period ratio P_2/P_1 (as well as P_2/P_0) decrease rapidly when $M/R < 0.15$. We may therefore define the "dominant linear mode" as that mode with the largest growth rate per unit time at small amplitudes. Figure 4 shows the dominant mode on the H-R diagram for $M = 0.58\ M_\odot$, $Y = 0.3$, $Z = 0.001$ (taken from the results of Stellingwerf, 1975 and unpublished results). Also shown are the regions in which only one mode is excited, and approximate luminosities and temperatures for the variables in M3. Note that for a horizontal branch at $M_{bol} > 1.0$, this figure predicts second overtone pulsation toward the blue. This additional mode does appear in observations of the cluster M68 (van Albada and Baker, 1973).

The relationship of the linear results to the observed modal behavior is discussed by Iben (1971b) and Cox, Castor and King (1972), but in general these predictions have not been born out by the nonlinear calculations. In particular, the dominant linear mode is not invariably the final preferred mode. In the case of stars showing a mixture of modes, however, we may be witnessing the process of approach to the preferred mode (Fitch, 1970) and in this case the dominant linear mode will temporarily be seen.

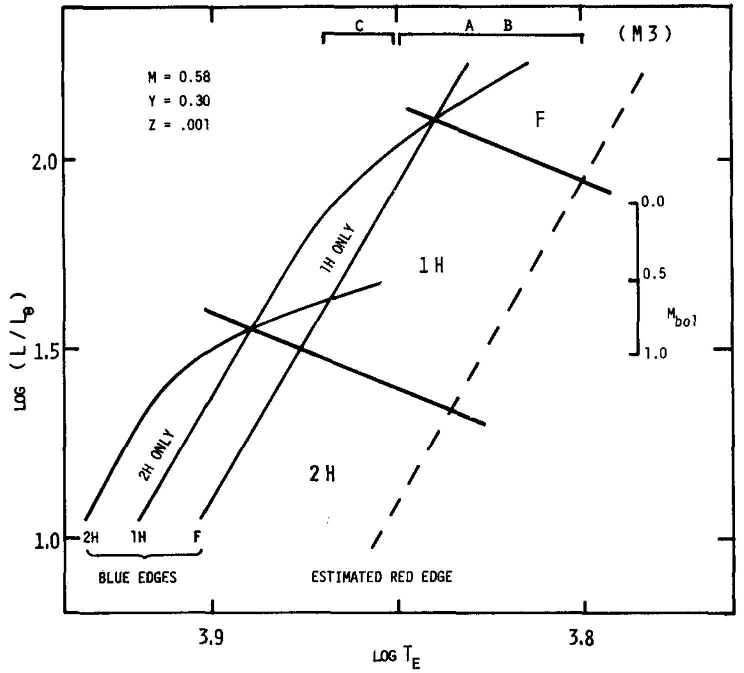

Fig. 4.

The adjustment time as given by the linear models is about 2 months for a 0.6 $M_{\odot}$ model. If the mass is increased to 3.0 $M_{\odot}$--appropriate for a star in the first crossing of the instability strip--growth time for the fundamental increases to 5 years only, and is even smaller for higher modes (Stellingwerf, 1975c). This suggests that the observed mixture of modes represents a stable nonlinear configuration rather than a transient coexistance of quasilinear modes.

3. Nonlinear Results

An investigation of modal behavior in nonlinear pulsation models requires an extensive survey over several parameters. The definitive work of this type is due to Christy (1962, 1964, 1966). Christy's approach involves integrating the nonlinear set of equations until the motion appears to be approximately periodic--usually about fifty periods. If the final motion is reasonably stable, Christy argued, it should represent the long-term preferred behavior of the model. Christy was also able to outline the limits of the instability strip by following models at moderate (half maximum) amplitude for only a few periods to determine whether the model showed amplitude growth.

Three types of modal behavior were encountered, 1) stable fundamental pulsation, 2) stable first overtone pulsation, and 3) stable pulsation in either the fundamental or the first overtone ("either-or" behavior). On the H-R diagram the fundamental region occurred at lower effective temperatures than the first overtone region, with the either-or behavior at the fundamental/first overtone boundary. Furthermore, at high luminosity the fundamental behavior is preferred as opposed to first overtone pulsation at low luminosities. The period at the center of the either-or region satisfies the relation

$$P_{tr} = 0.057 \ (L/L_{\odot})^{0.6} \quad . \tag{1}$$

This relation, as projected on the H-R diagram, will be referred to as the "Christy transition line." The width of the either-or region was found to be between 300 K and 600 K in effective temperature (note Christy's seq. 5). This behavior is qualitatively similar to the linear results shown in Figure 4, but the slope of Christy's transition line on this diagram is five times larger than the slope of the R = constant lines dividing the regions of different dominant linear mode.

It should be noted that uncertainties exist with this approach that may not be predictable. The final behavior of a model depends on initialization and total integration time. In fact, the long-term continuation of a calculated mode must always be regarded as problematical.

Spangenberg (1974) has repeated Christy's calculation for $M = 0.6\ M_\odot$, $Y = 0.3$, $Z = 0.001$ using models with improved outer boundary condition and opacity. The uncertainties mentioned above were reduced by extending the integration time to several hundred periods. These results resemble Christy's with the exception of a much wider either-or region, including nearly the entire instability strip at some luminosities. Actual mode transition will occur at the edges of this region, so Christy's transition line--which describes the center of this zone--should be used with caution.

Since the nonlinear models reproduce observed parameters very accurately, it would appear that the problems associated with mode of pulsation are due primarily to uncertainties in the mathematical approach rather than uncertainties in the models themselves. One major difficulty, identification and characterization of the nonlinear limit cycles, can be avoided by using a scheme devised by Baker and von Sengbusch (1969; see also von Sengbusch, 1973; Stellingwerf 1974). This scheme is essentially a generalized Newton-Raphson relaxation and allows the calculation of exact periodic limit cycles. Stability coefficients for perturbations toward other limit cycles are also obtained through application of the Floquet theorem. This technique has been applied to RR Lyrae stars by Stellingwerf (1975a). The relaxation technique was first applied to models similar to those of Christy and a very wide either-or region was found--in agreement with Spangenberg. The survey was then repeated with improved opacity and treatment of shocks, with the results shown in Figure 5 (Figure 17 of Stellingwerf, 1975a). Both linear and nonlinear analyses were performed. The solid lines are the linear blue edges, the dashed lines are modal transition lines, and the short dashed line is the estimated red edge. In this case the either-or region is only 300 K wide for reasonable luminosities. The actual temperature at which the change of mode occurs depends

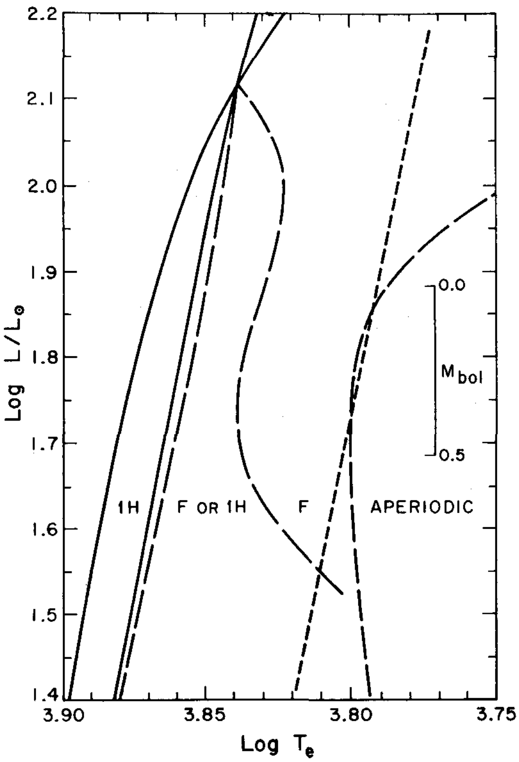

Fig. 5.

upon direction of evolution, and there is rather strong observational evidence that the resulting hysteresis effect on evolving stars is responsible for the Oosterhof dichotomy of globular clusters (van Albada and Baker 1971b, 1973). The mode switching rates depend on the rate of evolution; they appear to be on the order of 100 years for these stars. Another result seen in Figure 5 is a region toward the red in which stable mixed-mode pulsation was found, behavior very similar to that seen in the star AC And, the mixed mode Cepheids, and the dwarf Cepheids. Convection was not included in these models, so the redder models must be regarded as tenative.

AC And is the only observed RR Lyrae star showing simultaneous excitation of several radial modes. The fundamental, first overtone and second overtone modes have been detected with periods of 0.711, 0.525 and 0.421 days (Fitch and Szeidl, 1975). Stellingwerf (1975a) showed that the fundamental and first overtone periods were consistent with red horizontal branch models. Fitch and Szeidl, however, using the new second overtone data and the metal rich Population I models of Cogan (1970), found that the periods could also be fit by a 3 $M_{\odot}$ "first crossing" model with luminosity $L = 100\ L_{\odot}$ and effective temperature $T_e = 5500$ K. This is a rather cool temperature, especially since the instability strip moves to the blue for higher masses, so convection could be a factor here as well. The first-crossing hypothesis does account for the uniqueness of this object : this evolutionary phase is relatively short.

We may conclude that there exist three possibilities for the mixed-mode stars:

1) Stable, unique, nonlinear mixture of modes,
2) A process of switching between stable pure modes, or
3) A state of initial growth of pulsation, showing a mixture of quasi-linear modes.

Of these only the first can produce stable oscillations over a long period of time; the others will result in strong progressive changes in the amplitudes on a timescale of "years." This implies that continued observation could very well distinguish between these alternatives. Although the theoretical treatment of these objects is still rather sketchy, we can now make some statments about the mode-switching timescales, transition points, and overtone periods. Also, new mathematical techniques are becoming available that should make possible a more detailed comparison with observed stars.

https://doi.org/10.1017/S0252921100061959 Published online by Cambridge University Press

Acknowledgements

The author's research is supported by National Science Foundation Grant GP-30927 through Columbia University. Special thanks are extended to Dr. L. B. Lucy for transmitting this communication.

References

Albada, T. S. van 1969, Bull. Am. astr. Soc., 1, 366.

Albada, T. S. van and Baker, N. H. 1971a, Astrophys. J., 169, 311.

Albada, T. S. van and Baker, N. H. 1971b, Bull. Am. astr. Soc., 3, 241.

Albada, T. S. van and Baker, N. H. 1973, Astrophys. J., 185, 477.

Baker, N. H. 1965, Kleine Veröff. Remeis-Sternw., No. 40, p122.

Baker, N. H. and Sengbusch, K. von 1969, Mitt. astr. Gesellschaft, 32, 228.

Balázs, J. 1959, Kleine Veröff. Remeis-Sternw., No. 27, p26.

Castor, J. I. 1970, Bull. Am. astr. Soc., 2, 302.

Castor, J. I. 1971, Astrophys. J., 166, 109.

Christy, R. F. 1962, Astrophys. J., 136, 887.

Christy, R. F. 1964, Rev. mod. Phys., 36, 555.

Christy, R. F. 1966, Astrophys. J., 144, 108.

Cogan, B. C. 1970, Astrophys. J., 162, 139.

Cox, J. P. 1974a, Reports on Progress in Physics, 37, 563.

Cox, J. P. 1974b, Introductory Report, Liege Astrophysical Colloquium 19.

Cox, J. P., Castor, J. I. and King, D. S. 1972, Astrophys. J., 172, 423.

Cox, J. P., King, D. S. and Stellingwerf, R. F. 1972, Astrophys. J., 171, 93.

Detre, L. and Szeidl, B. 1973, in Variable Stars in Globular Clusters and Related Systems, Ed. J. D. Fernie, D. Reidel, Dordrecht, p31.

Fitch, W. S. 1967, Astrophys. J., 148, 481.

Fitch, W. S. 1968, in Non-Periodic Phenomena in Variable Stars, Academic Press, Budapest, p287.

Fitch, W. S. 1970, Astrophys. J., 161, 669.

Fitch, W. S. and Szeidl, B. 1975, preprint.

Iben, I., Jr. 1971a, Astrophys. J., 166, 131.

Iben, I., Jr. 1971b, Astrophys. J., 168, 225.

Iben, I., Jr. 1971c, Pub. astr. Soc. Pacific, 83, 697.

Iben, I., Jr. and Huchra, J. 1970, Astrophys. J., 162, L43 .

Iben, I., Jr. and Huchra, J. 1971, Astr. and Astrophys., 14, 293.

Kluyver, M. 1936, Bull. astr. Inst. Nederland, 7, 313.

Ledoux, P. 1951, Astrophys. J., 114, 373.

Ledoux, P. 1963, in Star Evolution, Ed. L. Gratton, Academic Press, New York, p394.

Lucy, L. B. 1975, private communication.

Preston, G. W. 1964, A. Rev. Astr. Astrophys., 2, 46.

Schwarzschild, M. 1940, Harvard Circ., No. 437.

Sengbusch, K. von 1973, Mitt. astr. Gesellschaft, No. 32, p228.

Spangenberg, W. H. 1974, Ph.D. Dissertation, Univ. of Colorado.

Stellingwerf, R. F. 1974, Astrophys. J., 192, 139.

Stellingwerf, R. F. 1975a, Astrophys. J., 195, 441.

Stellingwerf, R. F. 1975b, Astrophys. J., in press.

Stellingwerf, R. F. 1975c, in preparation.

Tuggle, R. S. and Iben, I., Jr. 1972, Astrophys. J., 178, 455.

Tuggle, R. S. and Iben, I., Jr. 1973, Astrophys. J., 186, 593.

Discussion to the paper of STELLINGWERF

COX: Has Stellingwerf any further ideas about the mass of AC And?

(STELLINGWERF): The "first crossing" ($3M_{\odot}$) hypothesis is attractive for these reasons: (1) Fitch and Szeidl have obtained fits with Population I metal abundances, (2) this explains the uniqueness since the evolutionary timescale would be rather short, and (3) the mixed-mode behavior suggests similarity to the δ Scuti stars. Period fitting, however, can evidently accomodate a wide range of masses, while the other arguments are subjective in nature, so I think further data is required on this problem.