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PRODUCTS OF ELATIONS AND HARMONIC HOMOLOGIES

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In memory of Robert Arnold Smith

ABSTRACT. The projective special linear group PSL(V) is generated by elations. Among all factorizations of $p \in$ PSL(V) into elations there will be one (or more) with the least number of factors. We determine this number, i.e. we solve the length problem for the projective special linear group. We solve a similar problem for the projective unimodular group which is generated by harmonic homologies. The projective special linear group and the projective unimodular group are the most important special cases of projective hyperreflection groups. We also solve the length problem for the general case.

1. **Introduction**. Every element in the orthogonal group of an n-dimensional regular vector space is a product of reflections; in fact, n factors suffice (Cartan-Dieudonné, cf. [1]). Peter Scherk [6] provided more detailed information in showing that (with few exceptions) every element π in the orthogonal group O can be expressed as a product of t, but not fewer than t, reflections. Here t = codim $F(\pi)$ and $F(\pi)$ is the set of vectors that are fixed under the isometry π.

Friedrich Bachmann [2] pointed out that it would be desirable to obtain similar results for other classical groups and he introduced the term "length problem" for this question. Solutions of length problems could lead to Bachmann-type characterizations of groups. Bachmann's endeavour is to characterize geometric groups. So far most length problems have been solved for subgroups of the general linear group. This can be considered as a first step towards the solution of length problems for subgroups of the projective general linear group and therefore for groups that have geometric significance.

The length problem for the projective general linear group has been solved in [4]. But for the subgroup generated by elations, the subgroup generated by harmonic homologies (= projective reflections), or more generally, for projective hyperreflection groups, [4] produces only an upper bound for the minimal number of factors needed. In other words, the length problem for projective hyperreflection groups remained unsolved. In the present paper we give a solution of this problem. In Section 2 we treat the general case, while we give more detailed information on products of elations and products of harmonic homologies in Section 3.

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2. **The length of an element in the projective hyperreflection group**. Let V be a finite-dimensional vector space over a commutative field K. We assume the dimension n of V is greater than zero. We put $K^* = K\backslash\{0\}$. For $\lambda \in K^*$ let η_λ denote the homothety $v \rightarrow \lambda v$ for all $v \in V$ and let $H = \{\eta_\lambda; \lambda \in K^*\}$.

For $\pi \in \mathrm{GL}(V)$ and $\lambda \in K^*$ we define $F_\lambda(\pi) = \{v \in V; v^\pi = \lambda v\}$. Then $F_\lambda(\pi) = \ker(\pi - \lambda) = \ker(\pi\eta_{\lambda^{-1}} - 1) = F_1(\pi\eta_{\lambda^{-1}})$. Often we write $F(\pi)$ instead of $F_1(\pi)$. We abbreviate $d_\lambda(\pi) = \operatorname{codim} F_\lambda(\pi)$. The mapping π is called simple if $d_1(\pi) = \operatorname{codim} F(\pi) = 1$.

We introduce $\overline{\pi_\lambda}: v + F_\lambda(\pi) \rightarrow v^\pi + F_\lambda(\pi)$. Clearly $\overline{\pi_\lambda}$ is well defined since $F_\lambda(\pi)$ is an invariant subspace under π. We put $\bar{\pi} = \overline{\pi_1}$ and observe that $\overline{\pi\eta_{\lambda^{-1}}} = \overline{\pi_\lambda}\eta_{\lambda^{-1}}$ since $v^\pi\eta_{\lambda^{-1}} + F_\lambda(\pi) = \lambda^{-1}v^\pi + F_\lambda(\pi)$. Obviously, $\overline{\pi\eta_{\lambda^{-1}}} = \eta_\mu$ for some $\mu \in K^*$ if and only if $\overline{\pi_\lambda} = \eta_{\lambda\mu}$.

The group $G_m = \{\pi \in \mathrm{GL}(V); (\det\pi)^m = 1\}$ is called a hyperreflection group. Let ϵ be a fixed primitive mth root of unity and $S_m = \{\rho \in \mathrm{GL}(V); \rho$ is simple, $\det\rho = \epsilon\}$. An element in S_m is called a hyperreflection. The set S_m is a generating set for G_m, i.e. for every $\pi \in G_m$ there are $\rho_i \in S_m$ such that $\pi = \rho_1 \cdots \rho_t$. The smallest possible t is called the length of π. It will be denoted by $l_m(\pi)$.

We shall be interested in the projective counterparts of hyperreflection groups, the projective hyperreflection groups $\mathrm{PG}_m \cong G_m/(H \cap G_m)$ generated by the set PS_m. The length of $p \in \mathrm{PG}_m$ is similarly defined as that of $\pi \in G_m$; it will be denoted by $l_m(p)$.

THEOREM 1. *Let $p \in \mathrm{PG}_m$ and $\pi \in G_m$ such that $P(\pi) = p$. Then* $l_m(p) = \min\{l_m(\pi\eta_{\lambda^{-1}}); \lambda \in K^*, \lambda^{nm} = 1\}$.

PROOF. Clearly $\pi\eta_{\lambda^{-1}} \in G_m$ and $P(\pi\eta_{\lambda^{-1}}) = p$ for all $\lambda = K^*$ such that $\lambda^{nm} = 1$. Therefore every factorization of $\pi\eta_{\lambda^{-1}}$ into hyperreflections results in a factorization of p into the same number of projective hyperreflections.

Conversely, if $p = c_1 \cdots c_t$, where c_i are projective hyperreflections, then there are hyperreflections ρ_i, $i = 1, \ldots, t$, such that $c_i = P(\rho_i)$. Thus $p = c_1 \cdots c_t = P(\rho_1) \cdots P(\rho_t) = P(\rho_1 \cdots \rho_t)$ and $\rho_1 \cdots \rho_t \in G_m$. Therefore $\rho_1 \cdots \rho_t = \pi\eta_{\lambda^{-1}}$ for some $\lambda \in K^*$ such that $\lambda^{nm} = 1$.

Clearly, there are at most nm elements $\lambda \in K^*$ satisfying the equation $\lambda^{nm} = 1$. We shall see now that it suffices to calculate $l_m(\pi\eta_{\lambda^{-1}})$ only for those λ for which $d_\lambda(\pi)$ is small. Namely, let $\lambda_0 \in K^*$ with $\lambda_0^{nm} = 1$ and $d_{\lambda_0}(\pi) = \min\{d_\lambda(\pi); \lambda \in K^*, \lambda^{nm} = 1\}$. Then $l_m(p) \leq l_m(\pi\eta_{\lambda_0^{-1}}) \leq d_{\lambda_0}(\pi) + m$ by [3], Theorem 7. On the other hand, $d_{\lambda_0}(\pi) + m$ is the minimal number of factors that may result from a factorization of $\pi\eta_{\lambda^{-1}}$ for which $d_\lambda(\pi) > d_{\lambda_0}(\pi) + m - 1$.

It is easy to see that $d_{\lambda_0}(\pi)$ depends only on p, i.e. it is the same for every π such that $P(\pi) = p$. We shall denote this number by $d(p, m)$. With this we can strengthen the assertion of Theorem 1:

$$(*) \qquad l_m(p) = \min\{l_m(\pi\eta_{\lambda^{-1}}); \lambda \in K^*, \lambda^{nm} = 1, d_\lambda(\pi) < d(p, m) + m\}.$$

3. **Products of elations and of harmonic homologies**. The group G_1 is the special

linear group and the group G_2 the unimodular group. Both groups are well known. It will be interesting to obtain more detailed information for PG_1 and PG_2. We start making a technical observation. We shall denote the spectrum of π by spec π.

LEMMA 2. *If* $\pi \in GL(V)$ *has more than two distinct characteristic values, i.e. if* $|\text{spec}\,\pi| > 2$, *then* $\overline{\pi\eta_{\lambda^{-1}}}$ *is not a homothety for any* $\lambda \in K^*$.

PROOF. Clearly, $\overline{\pi\eta_{\lambda^{-1}}}$ is a homothety if and only if $\overline{\pi_\lambda}$ is a homothety.

Since π has at least three distinct characteristic values, there are distinct $\alpha, \beta \in K^*\backslash\{\lambda\}$ such that $F_\alpha(\pi) \neq \{0\}$ and $F_\beta(\pi) \neq \{0\}$. Let $u \in F_\alpha(\pi)\backslash\{0\}$ and $w \in F_\beta(\pi)\backslash\{0\}$. Then $(u + F_\lambda(\pi))^{\overline{\pi_\lambda}} = \alpha u + F_\lambda(\pi)$ and $(w + F_\lambda(\pi))^{\overline{\pi_\lambda}} = \beta w + F_\lambda(\pi)$. Now if $\overline{\pi_\lambda}$ is a homothety on $V/F_\lambda(\pi)$, then these equations imply $\eta_\alpha = \overline{\pi_\lambda} = \eta_\beta$ and therefore $\alpha = \beta$ which is a contradiction.

In the context at hand, we are mainly concerned with characteristic values λ of $\pi \in GL(V)$ for which $\lambda^{nm} = 1$. We shall denote the set of these characteristic values by $\text{spec}_m \pi$. Clearly, if $\pi, \pi' \in GL(V)$ such that $P(\pi) = P(\pi') = p \in PG_m$, then $|\text{spec}_m \pi| = |\text{spec}_m \pi'|$. This enables us to define $|\text{spec}_m \pi| = s_m(p)$.

First we shall consider the case $m = 1$. The group G_1 i.e. the special linear group $SL(V)$ is generated by transvections τ. Accordingly $PSL(V) \cong SL(V)/(H \cap SL(V))$ is generated by projective transvections $P(\tau)$ which are generally called elations.

We intend to apply Theorem 1 to the case $m = 1$, and we shall see that we can derive more specific information on the factorization of $p \in PSL(V)$ into elations. In the sequel we assume $\pi \in SL(V)$ such that $P(\pi) = p$. The number of characteristic elements for π, or more specifically the integer $s_1(p)$ will play a role in the determination of the length of p.

(1) *If* $s_1(p) = 0$, *then* $l_1(p) = n$.

Namely, $F_\lambda(\pi) = \{0\}$ for all $\lambda \in K^*$ such that $\lambda^n = 1$. Therefore $\overline{\pi_\lambda} = \pi$ is not a homothety. Corollary 8 in [3] implies that $l_1(\pi\eta_{\lambda^{-1}}) = n$ for all such λ. Finally, (*) yields (1).

(2) *If* $s_1(p) = 1$, *then there is exactly one characteristic value* $\mu \in K^*$ *such that* $\mu^n = 1$ *and* $l_1(p) = l_1(\pi\eta_{\mu^{-1}})$.

Since $F_\mu(\pi) \neq \{0\}$, we have $d_\mu(\pi) < n$ and therefore $d(p, m) < n$. For all $\lambda \notin \text{spec}_1 \pi$ we have $d_\lambda = n \geq d(p, 1) + 1$. Again (*) yields our result (2).

(3) *If* $s_1(p) = 2$, *then there are exactly two characteristic values* μ, ν *for* π, *i.e.* $\text{spec}_1 \pi = \{\mu, \nu\}$ *and* $l_1(p) = \min\{l_1(\pi\eta_{\mu^{-1}}), l_1(\pi\eta_{\nu^{-1}})\}$.

As in (2) we see that $d_\lambda(\pi) = n \geq d(p, 1) + 1$ for all $\lambda \notin \text{spec}_1 \pi$.

(4) *If* $|\text{spec}\,\pi| \geq 3$, *then* $l_1(p) = d(p, 1)$.

Namely, there is some $\lambda \in \text{spec}_1 \pi$ such that $d_\lambda(\pi) = d(p, 1)$. By Lemma 2, $\overline{\pi\eta_{\lambda^{-1}}}$ is not a homothety. Thus $l_1(p) = l_1(\pi\eta_{\lambda^{-1}}) = d(p, 1)$.

Second we deal with the case $m = 2$ and of course we assume now that char $K \neq$

2. The unimodular group G_2 is generated by reflections ρ. The group $PG_2 \cong G_2/(H \cap G_2)$ is generated by projective reflections $P(\rho)$ which are called harmonic homologies. Now we assume $p \in PG_2$ and $\pi \in G_2$ such that $P(\pi) = p$.

(5) *If* $s_2(p) = 0$, *then* $l_2(p) = n$ *if* $\det \pi = (-\lambda)^n$ *for some* $\lambda \in K^*$, $l_2(p) = n + 1$ *otherwise.*

Clearly $F_\lambda(\pi) = \{0\}$ for all $\lambda \in K^*$ such that $\lambda^{2n} = 1$. Thus $d(p, 2) = n$ and $\overline{\pi_\lambda} = \pi$ is not a homothety. Since $F_\lambda(\pi) = F(\pi\eta_{\lambda-1})$, by [3], Corollary 9, we get $l(\pi\eta_{\lambda^{-1}}) = d(p, 2) = n$ if $\det\pi = (-\lambda)^n$ for some $\lambda \in K^*$ such that $\lambda^{2n} = 1$, and $l(\pi\eta_{\lambda^{-1}}) = d(p, 2) + 1 = n + 1$ otherwise. Now (*) yields the result.

(6) *If* $s_2(p) = 1$, *then there is exactly one characteristic value* $\mu \in K^*$ *such that* $\mu^{2n} = 1$. *Then* $l_2(p) = l_2(\pi\eta_{\mu^{-1}})$ *if* $d_\mu(\pi) = d(p, 2) < n - 1$ *and* $l_2(p) = \min\{l_2(\pi\eta_{\lambda^{-1}}); \lambda \in K^*, \lambda^{2n} = 1\}$ *if* $d_\mu(\pi) = d(p, 2) = n - 1$.

This and the subsequent statement (7) follow immediately from (*).

(7) *If* $s_2(p) = 2$, *then there are exactly two characteristic values* $\mu, \nu \in K^*$ *for* π *such that* $\mu^{2n} = \nu^{2n} = 1$, *i.e.* $\operatorname{spec}_2\pi = \{\mu, \nu\}$. *Then* $l_2(p) = \min\{l_2(\pi\eta_{\mu^{-1}}), l_2(\pi\eta_{\nu^{-1}})\}$ *if* $d_\mu(\pi) < n - 1$ *or* $d_\nu(\pi) < n - 1$; *and* $l_2(p) = \min\{l_2(\pi\eta_{\lambda^{-1}}); \lambda \in K^*, \lambda^{2n} = 1\}$ *if* $d_\mu(\pi) = d_\nu(\pi) = n - 1$.

(8) *If* $|\operatorname{spec}\pi| \geq 3$, *then* $l_2(p) = d(p, 2)$ *if* $d_\lambda(\pi) = d(p, 2)$ *and* $\det\pi = (-1)^{d(p,2)}\lambda^n$ *for some* $\lambda \in K^*$ *such that* $\lambda^{2n} = 1$; $l_2(p) = d(p, 2) + 1$ *otherwise.*

Let $\lambda \in K^*$ such that $d_\lambda(\pi) = d(p, 2)$ and $\lambda^{2n} = 1$. Then $F(\pi\eta_{\lambda^{-1}}) = F_\lambda(\pi)$ and $\overline{\pi\eta_{\lambda^{-1}}}$ is not a homothety by Lemma 2. Therefore $l(\pi\eta_{\lambda^{-1}}) = d(p, 2)$ if $\det\pi = (-1)^{d(p,2)}\lambda^n$, and $l(\pi\eta_{\lambda^{-1}}) = d(p, 2) + 1$ otherwise. Again (*) finishes the proof.

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