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FINITELY PRESENTED CENTRE-BY-METABELIAN LIE ALGEBRAS

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To Bernhard Neumann on his ninetieth birthday

It is shown that finitely presented centre-by-metabelian Lie algebras are Abelian-by-finite-dimensional.

1. INTRODUCTION

In [**7**], the second author proved that a finitely presented centre-by-metabelian group is Abelian-by-polycyclic. The proof of this result used the fact, proved by Bieri and Strebel in [**2**], that a finitely presented soluble group with an infinite cyclic quotient is an HNN extension with finitely generated base group. In [**3**], Bieri and Strebel deduced another proof of the result of [**7**] as a corollary of their work on finitely presented soluble groups, particularly the fact that a metabelian quotient of a finitely presented soluble group is again finitely presented.

The aim of this note is to prove a similar result for Lie algebras.

THEOREM. *A finitely presented centre-by-metabelian Lie algebra is Abelian-by-finite-dimensional.*

The key tools quoted above do not seem to be available for Lie algebras. The closest result of which we are aware is one of Wasserman [**8**, Theorem 9.1] which is similar to the result quoted from [**2**]. But the consequences of this result do not appear to be sufficiently powerful to obtain results for Lie algebras analogous to those for groups. We have therefore needed to take a substantially different approach.

The authors have shown in [**6**] that a finitely presented soluble Lie algebra of characteristic 2 which satisfies the maximal condition for ideals must be of finite dimension. Because a finitely generated Abelian-by-finite-dimensional Lie algebra must satisfy the maximal condition for ideals [**1**, Corollary 11.1.8], the Theorem implies that all finitely presented centre-by-metabelian Lie algebras of characteristic 2 are of finite dimension.

The main step in the proof of the Theorem is to show that a metabelian quotient of a finitely presented centre-by-metabelian Lie algebra is again finitely presented. It would be interesting to know to what extent this can be generalised.

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QUESTION. Is it true that a metabelian quotient of a finitely presented soluble Lie algebra is again finitely presented?

An affirmative answer to the corresponding question for groups is given by [**3**, Theorem B].

2. Quotients of finitely presented Lie algebras

Throughout this paper K denotes an arbitrary field, and all tensor and exterior products are taken over K. If L is any Lie algebra over K we write $K[L]$ for the enveloping algebra of L. Also, we write L' for the subalgebra $[L, L]$ and L'' for $[L', L']$.

Let L be a finitely presented Lie algebra over K, and suppose that A and B are ideals of L such that $B \subseteq A$ and A/B is Abelian. Set $R = K[L/A]$ and $M = A/B$. Then M has a natural structure as a (right) R-module via

$$(a + B)(l + A) = [a, l] + B$$

for all $a \in A$ and $l \in L$.

The R-module structure on M carries over to an $R \otimes R$-module structure on the tensor square $M \otimes M$. There is an algebra homomorphism $\delta : R \to R \otimes R$ given by $x\delta = x \otimes 1 + 1 \otimes x$ for all $x \in L/A$. In fact, as is well known, δ is an embedding (it has right inverse $\iota \otimes \varepsilon$ where $\iota : R \to R$ is the identity map and $\varepsilon : R \to K$ is the augmentation map). We call δ the diagonal embedding. Let $\widetilde{R} = R\delta$. Thus $M \otimes M$ is an $\widetilde{R}$-module, and therefore an R-module. The action of R on $M \otimes M$ is called the diagonal action. It induces an action of R on the exterior square $M \wedge M$ given by

$$(m \wedge n)x = (mx) \wedge n + m \wedge (nx)$$

for all $m, n \in M$ and $x \in L/A$. The action of L on itself carries over to an action of R on $B/[B, A]$ via

$$\big(b + [B, A]\big)(l + A) = [b, l] + [B, A]$$

for all $b \in B$ and $l \in L$. There is a linear map $\gamma : M \wedge M \to B/[B, A]$ satisfying

$$(a_1 + B) \wedge (a_2 + B) \mapsto [a_1, a_2] + [B, A]$$

for all $a_1, a_2 \in A$, and it is easily verified (via the Jacobi identity) that γ is a homomorphism of R-modules.

LEMMA. *With the notation above, suppose that L is finitely presented and that L/A is of finite dimension. Then the kernel of γ is a finitely generated R-module.*

PROOF: It is possible to prove this by means of the spectral sequence associated to the extension $A \to L \to L/A$, but we provide an elementary proof. Let F be a finitely generated free Lie algebra such that there is an epimorphism $\pi : F \to L$. Let U, V and W denote the complete inverse images under π of A, B and $\{0\}$, respectively. Note that we can then identify L/A with F/U and hence R with $K[F/U]$.

The correspondence given by

$$(u_1\pi + B) \wedge (u_2\pi + B) \mapsto [u_1, u_2] + [V, U],$$

for all $u_1, u_2 \in U$, leads to an explicit R-module isomorphism between $M \wedge M$ and $U'/[V,U]$. For details, see [**6**, Section 2.2]. The epimorphism π also induces an isomorphism of $B/[B,A]$ with $V/\big([V,U]+W\big)$, and this is again an R-module isomorphism. We can thus identify γ with the map

$$U'/[V,U] \to V/\big([V,U]+W\big)$$

induced from the inclusion of U' into V. Therefore

$$\begin{aligned}\ker\gamma &\cong \Big(U' \cap \big([V,U]+W\big)\Big)/[V,U] = \Big([V,U] + (U' \cap W)\Big)/[V,U] \\ &\cong (U' \cap W)/\Big([V,U] \cap (U' \cap W)\Big) = (U' \cap W)/\big([V,U] \cap W\big).\end{aligned}$$

We must show that this section of F is finitely generated as an R-module.

Since L is finitely presented, W is finitely generated as an ideal of F. Hence $W/[W,U]$ is finitely generated as an R-module. But $(U' \cap W)/\big([V,U] \cap W\big)$ is isomorphic to an R-section of $W/[W,U]$. Since L/A is of finite dimension, R is Noetherian (see, for example [**5**, Proposition 6 of I.2.6]) and so this section is also finitely generated, as required. ∎

Observe that this lemma, although technical in nature, has some important consequences in special cases. For example, if $B/[B,A]$ is finite dimensional, then we may deduce that $M \wedge M$ is finitely generated as an R-module and so, using [**6**, Theorem A], that L/B is also finitely presented. The following is another special case where we can deduce that L/B is finitely presented.

PROPOSITION. *Let L be a finitely presented centre-by-metabelian Lie algebra over the field K. Let A and B be ideals of L with $B \subseteq A$ such that L/A and A/B are Abelian and B is central; and write $M = A/B$. Then $M \wedge M$ is finitely generated as a $K[L/A]$-module. As a consequence, L/B is finitely presented (so, taking $B = L''$, we have that L/L'' is finitely presented).*

PROOF: Observe that the last sentence of the Proposition follows from [**6**, Theorem A]. Write $R = K[L/A]$ and let I denote the augmentation ideal of R (that is, the ideal of R generated by the elements of L/A). By the Lemma, the kernel of $\gamma : M \wedge M \to B$

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is finitely generated as an R-module. Since B is central in L it is trivial as an R-module (by which we mean that each element of L/A has zero action on B). Thus the R-module $(M \wedge M)I$ is contained in the kernel of γ. Therefore, by the Lemma, it is finitely generated. We shall use this to show that $M \wedge M$ is finitely generated as an R-module.

By [**6**, Lemmas 2.1 and 2.2], we may assume that K is algebraically closed. We shall use arguments similar to those of [**6**, Proposition 2.4]. By [**4**, Theorem 1 of IV.1.4], M has a finite series of submodules

$$\{0\} = M_0 \leqslant M_1 \leqslant M_2 \leqslant \ldots \leqslant M_k = M,$$

where each quotient M_i/M_{i-1} is isomorphic to an R-module of the form R/P_i where P_i is a prime ideal of R. Further, by [**4**, Theorem 2 of IV.1.4], each P_i contains a prime ideal Q_i of R which is associated to M.

It will clearly suffice to show that $M \otimes M$ is finitely generated as an R-module under the diagonal action. But the series above for M yields a finite series of R-submodules of $M \otimes M$ in which each quotient is of the form $R/P_i \otimes R/P_j$ (here, of course, R acts via the diagonal embedding of R into $R \otimes R$). Since $R/P_i \otimes R/P_j$ is a quotient of $R/Q_i \otimes R/Q_j$, it suffices to prove that each $R/Q_i \otimes R/Q_j$ is finitely generated as an R-module.

Suppose firstly that $R/Q_i \otimes R/Q_j$ is trivial as an R-module. Then, for each $x \in L/A$,

$$0 = \big((1 + Q_i) \otimes (1 + Q_j)\big)x = (x + Q_i) \otimes (1 + Q_j) + (1 + Q_i) \otimes (x + Q_j).$$

But this implies that $x + Q_i \in K + Q_i$ and $x + Q_j \in K + Q_j$ for each $x \in L/A$, so that R/Q_i and R/Q_j have dimension 1. It is then clear that $R/Q_i \otimes R/Q_j$ is finitely generated as an R-module.

Thus we can assume that $R/Q_i \otimes R/Q_j$ is not trivial as an R-module. Choose an element x of L/A which has non-zero action on $R/Q_i \otimes R/Q_j$. We observe for future reference that, because K is assumed algebraically closed and because R/Q_i and R/Q_j are integral domains, $R/Q_i \otimes R/Q_j$ is also an integral domain (see [**9**, Corollary 1 to Theorem 40 of Chapter III]). Thus multiplication in $R/Q_i \otimes R/Q_j$ by the image of x is a monomorphism of R-modules.

Suppose that $Q_i \neq Q_j$. Because Q_i and Q_j are associated prime ideals of M, there are elements m_i and m_j of M such that the submodules m_iR and m_jR are isomorphic to R/Q_i and R/Q_j, respectively. Further, because Q_i and Q_j are distinct, these submodules intersect trivially, and so $m_iR + m_jR \cong R/Q_i \oplus R/Q_j$. Since $R/Q_i \otimes R/Q_j$ is isomorphic to a submodule of $\wedge^2 (R/Q_i \oplus R/Q_j)$, it follows that $R/Q_i \otimes R/Q_j$ is isomorphic to a submodule of $M \wedge M$. Therefore $(R/Q_i \otimes R/Q_j)x$ is isomorphic to a submodule of $(M \wedge M)I$ and is finitely generated. But

$$R/Q_i \otimes R/Q_j \cong (R/Q_i \otimes R/Q_j)x.$$

Thus $R/Q_i \otimes R/Q_j$ is finitely generated.

Suppose now that $Q_i = Q_j$. Because Q_i is an associated prime ideal of M, there is an isomorphic copy of R/Q_i in M. Thus $(R/Q_i \wedge R/Q_i)x$ is isomorphic to a submodule of $(M \wedge M)I$ and is finitely generated. It is standard, and easily verified, that the linear map induced by $a \wedge b \mapsto a \otimes b - b \otimes a$ (for all $a, b \in R/Q_i$) yields an R-monomorphism from $R/Q_i \wedge R/Q_i$ to $R/Q_i \otimes R/Q_i$. Thus multiplication in $R/Q_i \wedge R/Q_i$ by the image of x is a monomorphism of R-modules. Therefore $R/Q_i \wedge R/Q_i$ is isomorphic to $(R/Q_i \wedge R/Q_i)x$ and is finitely generated. It follows, by [**6**, Theorem A], that $R/Q_i \otimes R/Q_i$ is finitely generated as an R-module, which completes the proof of the Proposition. ∎

3. Proof of the Theorem

We use the notation preceding the statement of the Lemma with $A = L'$ and $B = L''$. Here L'' is central in L. It will sometimes be convenient to consider $M \wedge M$ and $M \otimes M$ as $\widetilde{R}$-modules rather than R-modules (recall that $\widetilde{R} = R\delta \subseteq R \otimes R$). By the Proposition, $M \wedge M$ is finitely generated as an $\widetilde{R}$-module. Hence, by [**6**, Theorem A], $M \otimes M$ is also finitely generated as an $\widetilde{R}$-module.

Let $\{w_1, \dots, w_k\}$ be a finite generating set for $M \otimes M$ as an $R \otimes R$-module and, for $i = 1, \dots, k$, let J_i be the annihilator of w_i in $R \otimes R$. Further, let J be the annihilator of $M \otimes M$. Thus $J = J_1 \cap \dots \cap J_k$ and

$$(R \otimes R)/J_i \cong w_i(R \otimes R) \leqslant M \otimes M.$$

Since $M \otimes M$ is finitely generated as an $\widetilde{R}$-module, so is $(R \otimes R)/J_i$. Thus $(R \otimes R)/J$ is also finitely generated as an $\widetilde{R}$-module.

Let $\widetilde{I}$ be the augmentation ideal of $\widetilde{R}$ and let $\widehat{I}$ be the ideal of $R \otimes R$ generated by $\widetilde{I}$. Then $(R \otimes R)/(\widehat{I} + J)$ is both finitely generated and trivial as an $\widetilde{R}$-module and so is of finite dimension. Let $T = \{t \in R : t \otimes 1 \in \widehat{I} + J\}$. Then T is an ideal of R such that R/T is of finite dimension.

Let $\sigma : M \otimes M \to L''$ be the homomorphism of R-modules satisfying

$$(a_1 + L'') \otimes (a_2 + L'') \mapsto [a_1, a_2]$$

for all $a_1, a_2 \in L'$. Since L'' is a trivial R-module, $(M \otimes M)\widetilde{I}$ is contained in the kernel of σ. But

$$MT \otimes M \subseteq (M \otimes M)(\widehat{I} + J) = (M \otimes M)\widetilde{I}.$$

Thus $(MT \otimes M)\sigma = \{0\}$.

Let H be the subspace of L such that $L'' \leqslant H \leqslant L'$ and $H/L'' = MT$. Since T is an ideal of R, H is an ideal of L. From the definition of σ we find $(MT \otimes M)\sigma = [H, L']$. Thus $[H, L'] = \{0\}$ and, since $H \leqslant L'$, it follows that H is Abelian. Since T is of finite co-dimension in R and M is a finitely generated R-module, MT is of finite co-dimension in M. Thus H is of finite co-dimension in L' and so also in L. Therefore L is Abelian-by-finite-dimensional, which completes the proof of the Theorem. ∎

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